After-Tax Economic Replacement Analysis

Joseph C. Hartman
Lehigh University

Raymond V. Hartman
Triton College

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Joseph C. Hartman[1] and Raymond V. Hartman[2]

[1]Department of Industrial and Manufacturing Systems Engineering
Lehigh University

[2]School of Business and Technology
Triton College

Abstract

This paper examines after-tax serial replacement analysis under current United States tax law. The law explicitly defines the difference between an asset disposal (retirement) and a like-for-like exchange (replacement). A gain or loss is only realized when an asset is retired while a replacement leads to the transfer of any residual book value balance to the acquired asset. This transferal greatly complicates analysis and leads to non-stationary solutions, even with time invariant costs. We analyze the effect of this movement in book value for assets on replacement decisions. Furthermore, a dynamic programming formulation is presented with a state space defined by asset age and initial book value as current replacement models cannot correctly capture the after-tax cash flow implications of this balance transfer. The new model is compared to traditional after-tax replacement models which assume that a gain or loss is realized at each asset sale over the horizon. Examples illustrate that this assumption can lead to widely different solutions, especially in the cases where gains or losses from asset sales are large.

1 Introduction

Depreciation and taxes and their effect on cash flows represent one of the most important aspects of investment analysis (Park and Sharp-Bette [16]). This is especially true in equipment replacement decisions as capital costs include depreciation expenses which reduce tax liabilities. While the incorporation of savings from depreciation is explicitly included in various economic equipment replacement models (see Chisholm [6], Christer and Waller [7], Leung and Tanchoco [13] and Oakford, Lohmann and Salazar [15] for example), current United States tax law defines specific rules for the replacement of an asset with respect to its book value that complicate analysis.

To be more exact, a number definitions must be clear. A *retirement* is when an asset is salvaged and the service that it rendered is discontinued. This is often referred to as a disposal (Internal Revenue Code, Sections 1245 and 1231 [9]). At the time of disposal, a gain or loss is realized, defining a tax liability or credit. The gain or loss is defined as the difference between the salvage value revenue and book value at the time of sale with current law taxing the difference at the appropriate income tax rate. A *replacement* is when an asset is retired and another asset is acquired in its place to continue service, often termed a like-for-like exchange (Section 1031 [9]). In this situation, no gain or loss is realized and no tax is paid or credit is received on the exchange. Rather, the unadjusted cost basis or initial book value of the acquired asset is equal to its purchase price, minus the salvage value of the retired asset, plus any remaining book value of the retired asset. Realizing a gain or loss on the sale of an asset in an exchange situation is not an option under current tax law (Section 1031 [9]).

Note that the term gain or loss is used without the term "capital" gain or loss. Under current law, a capital gain is defined as the difference between the purchase price of an asset and its eventual sales price (Section 1231 [9]). Thus, if an asset appreciates in value, a capital gain is realized and a capital gains tax is paid. This is different from a gain on the sale of an asset which is the difference between the current book value of an asset and its selling price. The gain in this situation is treated as ordinary income and taxed at the appropriate income tax rate, which is the marginal rate if the sale does not change an entity's tax bracket. In this paper, we consider gains and losses from the sale of an asset, but not capital gains in that it is assumed that all salvage values are less than or equal to the purchase price of the asset.

Despite considerable research in serial replacement analysis, which is concerned with determining replacement schedules for a single, independent asset, there is apparently no model which properly addresses the issue of transferring residual book values from a retired asset to its replacement. While the Tax Reform Act of 1986 defined the Modified Accelerated Cost Recovery System (MACRS) and an alternate system (AMACRS) as the only two allowable methods of depreciation for capital assets placed in service after December 31, 1986, the transferal of book values has been in tax law for some fifty years [9]. (The reader is referred to the Internal Revenue Code [9] and to the work of Fleischer, Mason and Leung [8] for complete details on the Tax Reform Act of 1986 and current United States tax law.)

Both the recognition of a gain or loss and the transferal of an asset's book value are critical parameters when developing an after-tax economic replacement model. Assume an asset of age n with book value B_n is sold for a salvage value of S_n. If the asset is replaced with a similar asset that performs a similar service, this is considered a like-for-like exchange. Assuming a purchase price of P, the initial book value of the acquired asset is:

$$B_0 = P + B_n - S_n.$$

This transferal of the book value complicates the solution procedure of previous dynamic programming formulations for the serial replacement problem. Consider a dynamic programming model in which the age of the asset is the lone state variable (i.e. Bellman [4]) and the stage variable of time may also be used in cost calculations. In this model, it is assumed that if an asset is sold when it is one or two-years old, it returns to the state of a new asset. However, under current tax law, these assets would generally not be equivalent because they have different initial book values which lead to different depreciation charges over the service life of the asset. In an alternate formulation, Wagner [20] uses the time period as the state variable, denoted as nodes in a network, with decisions representing the length of time an asset is retained (arcs connecting the nodes in the network). As with the age based formulation, this assumes that the resulting states from purchasing an asset and retaining it for two years is equivalent to purchasing two consecutive assets and retaining each for one year. Again, according to current tax law, the resulting asset state is not the same due to the transference of book values.

As two assets of the same age at the same time period may have different initial book values, a formulation that only considers asset age and the time period in analysis is not adequate. In this paper, we examine the movement of initial book values over time and its effect on replacement decisions and the economic life of an asset. Additionally, we present a dynamic programming formulation to correctly solve the after-tax serial replacement problem and compare it with traditional after-tax approaches which incorrectly charge a tax (credit) on the gain (loss) resulting from the replacement of an asset.

The use of dynamic programming to solve serial replacement problems is not new. As decisions are made periodically based on characteristics of the asset, i.e. its age, the problem is a natural setting for the state and stage definitions of dynamic programming. Developed solutions include those over finite (Bellman [4], Wagner [20], Oakford, Lohmann and Salazar [15]) and infinite horizons (Bean, Lohmann and Smith [1] [2], Chand and Sethi [5], Hopp and Nair [10] [11], Nair and Hopp [14] and Sethi and Chand [18]). However, these models and those mentioned earlier do not explicitly model the changing initial book value, which leads to incorrect depreciation charges in after-tax analysis. To our knowledge, there is no replacement model which correctly addresses after-tax cash flows under current United States tax law. Various authors have performed explicit analyses for the optimal choice of a depreciation method (see Fleischer, Mason and Leung [8] or Schoomer [17]). However, these papers consider the retirement of an asset and not the case of replacement.

In the subsequent analysis and model derivation, we make the following general assumptions:

1. It is assumed that the company is profitable such that depreciation charges and losses from the sale of an asset may be treated as tax credits (revenues) in that they reduce tax liabilities elsewhere in the entity.

2. The sale of the asset does not affect the tax bracket of the company such that the marginal income tax rate may be used for analysis.

3. All cash flows, tax rates and the discount rate are assumed to be known with certainty at time zero.

4. Although the presented model is flexible in depreciation modeling, the choice of a depreciation method is irrevocable and the same method must be utilized for each replaced asset according to law (Section 168 [9]).

5. As rules are specific to certain property, we restrict our discussion to that of personal property which covers virtually all industrial equipment.

6. Models are developed under the depreciation rules defined in the Tax Reform Act of 1986 using the half-year convention (Section 168 [9]).

The following section examines the movement of the initial book value over time due to asset replacement for time invariant and variant cost cases. For the time variant cost case, the dynamic programming formulation is presented. In Section 3, the new model is compared to traditional dynamic programming solutions which charge a gain or loss with each asset sale. Conclusions and directions for future research are presented in the final section.

2 The Unadjusted Cost Basis and Economic Life of an Asset

To investigate the movement of the unadjusted cost basis over time, we utilize the following notation:

$P_t(k)$ = purchase cost of an asset of type k at time t;
$S_t(k,n)$ = salvage value of an n-year old set of type k at time t;
d_n = percentage depreciation of unadjusted costs basis for an asset in its nth year of operation;
N_k = maximum physical age of an asset of type k;
t_m = marginal tax rate;
i = after-tax discount rate.

Further define the book value of an asset of type k and age n at the end of time t as $B_t(k,n)$, with the unadjusted cost basis or initial book value defined at $n = 0$. The book value of the asset at any time period is the initial book value less accumulated depreciation, or:

$$B_t(k,n) = \left(1 - \sum_{j=1}^{n} d_j\right) B_{t-n}(k,0). \tag{2.1}$$

With this notation, the initial book value of an asset of type k' purchased at time t as a replacement for an asset of type k and age n is:

$$B_t(k',0) = P_t(k') - S_t(k,n) + B_t(k,n). \tag{2.2}$$

With these definitions, we examine the initial book value variable under time invariant and variant cost conditions.

2.1 Time Invariant Analysis

Although taxes are paid on actual dollar cash flows which are inflated (Thuesen and Fabrycky [19]), we analyze the initial book value under under the assumption of time invariant costs as the simplification allows for more in-depth analysis. If inflation is negligible and there is no drastic technological change in assets over time, then the following properties hold in general. As this is time invariant analysis and only one asset type is assumed, the t and k subscripts are dropped, with the exception of book value which is shown to move with time.

Theorem 1 *If $S(n) = B_n(n)$ for all n, then the initial book value $B_t(0)$ is stationary and $B_t(0) = P$ for all t.*

Proof. We show this result by induction. If the original asset is sold at age n, then the initial book value for the its replacement asset is:

$$\begin{aligned} B_n(0) &= P - S(n) + B_n(n) = P \\ &= P - S(n) + (1 - \sum_{j=1}^{n} d_j) B_0(0) = P \\ &= P - S(n) + (1 - \sum_{j=1}^{n} d_j) P = P \end{aligned}$$

The relationship holds for any replacement age $n = 1, 2, \ldots N$. Further note that the final relationship implies that:

$$-S(n) + (1 - \sum_{j=1}^{n} d_j) P = 0 \quad \forall n = 1, 2, \ldots N. \tag{2.3}$$

Assume $B_j(0) = P$ for $j = 2$ through $t-1$. For time t:

$$\begin{aligned} B_t(0) &= P - S(n) + B_{t-n}(n) \\ &= P - S(n) + (1 - \sum_{j=1}^{n} d_j) B_{t-n}(0) \end{aligned}$$

By the induction hypothesis and Equation (2.3):

$$B_t(0) = P - S(n) + (1 - \sum_{j=1}^{n} d_j) P = P \quad \forall n,$$

and it holds for all values of t. □

The following theorems explore the cases when sales always result in losses or gains.

Theorem 2 *If $S(n) \leq B_t(n)$ for all n and $B_0(0) = P$, then the initial book value $B_t(0)$ is non-decreasing in t and $S(n) \leq B_t(n)$ for all t and n.*

Proof. The proof follows the logic of Theorem 1. With the given relationship:

$$B_n(0) = P - S(n) + B_n(n) = P - S(n) + (1 - \sum_{j=1}^{n} d_j)P \geq P,$$

and Equation (2.3) is now:

$$-S(n) + (1 - \sum_{j=1}^{n} d_j)P \geq 0 \quad \forall n = 1, 2, \ldots N. \tag{2.4}$$

Thus, it can be shown by induction that $B_t(0)$ is non-decreasing. As the initial book value is non-decreasing, the current book value $B_t(n)$, defined earlier as:

$$B_t(n) = \left(1 - \sum_{j=1}^{n} d_j\right) B_{t-n}(0)$$

is also non-decreasing as the value of $(1 - \sum_{j=1}^{n} d_j)$ is fixed for a given n over time. As $B_t(n)$ is non-decreasing and $S(n)$ is stationary, then $S(n) \leq B_t(n)$ for all t. □

The meaning of Theorem 2 is that once an asset is sold for a loss, it is always sold for a loss in the time invariant case. Note that this assumes that all salvage values are below their respective book values. Similar insights can be made from sales of assets for gains.

Theorem 3 *If $S(n) \geq B_t(n)$ for all n and $B_0(0) = P$, then the initial book value $B_t(0)$ is non-increasing in t and $S(n) \geq B_t(n)$ for all t and n.*

Proof. The proof follows the logic of Theorem 2. □

As with the previous theorem, the meaning here is that if an asset is sold at a gain, it is always sold at a gain. These properties have drastic effects on the solution to the time invariant problems.

The time invariant problem is generally an easy problem to solve in replacement analysis. In the infinite horizon problem, the solution entails finding the stationary economic life of the asset, which is defined as the age that minimizes the sum of annualized capital and operating (O&M) costs (Thuesen and Fabrycky [19]). Generally, capital costs decrease with the length of ownership while operating costs increase and the economic life defines the length of time that minimizes the sum of these two functions, which are pictured in Figure 2.1a.

However, Theorems 2 and 3 greatly complicate the after-tax time invariant problem. For an owned asset with initial book value B, the annualized capital costs for keeping the asset n years is:

$$P(A/P, i, n) - S(n)(A/F, i, n) - t_m \left(\sum_{j=1}^{n} d_j (1+i)^{n-j} \right) B(A/F, i, n), \tag{2.5}$$

where $(A/P, i, n)$ and $(A/F, i, n)$ annualize present and future costs over n years, respectively, at the annual discount rate i (Thuesen and Fabrycky [19]). Specifically, Equation (2.5) annualizes the purchase cost P, salvage value $S(n)$ and the after-tax savings from the annual depreciation charges. (For a discussion of after-tax cash flow analysis, see Park and Sharp-Bette [16].) The curve of these annualized costs, based on the age n, is labeled "Capital 1" in Figures 2.1a and b.

In the traditional time invariant problem, an asset is replaced with a new asset that follows the same annualized cost curves. However, with the transfer of the residual book value at the time of sale, only the

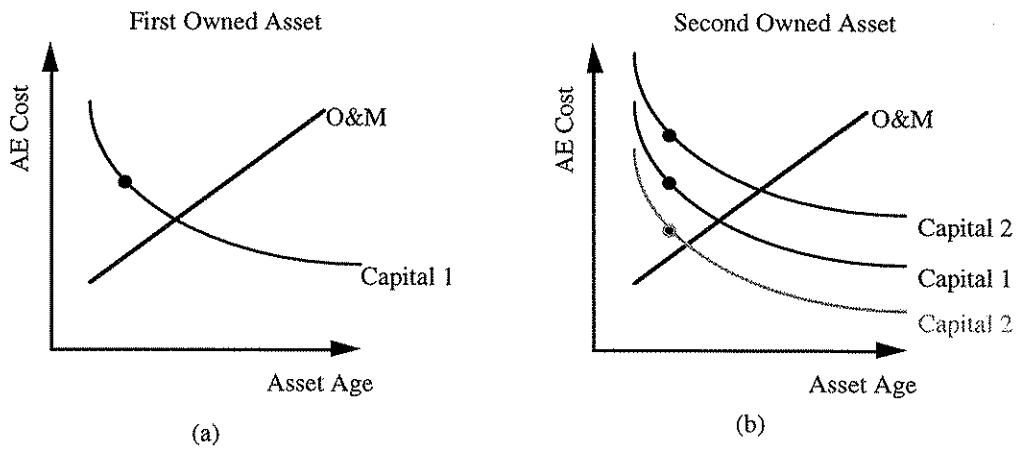

Figure 2.1: Shift in annualized capital costs due to selling asset at a gain or loss.

O&M cost curve remains stationary in the after-tax problem. The annualized capital charges for the newly acquired asset, which is kept for n years, are as follows:

$$P(A/P,i,n) - S(n)(A/F,i,n) - t_m \left(\sum_{j=1}^{n} d_j (1+i)^{n-j} \right) \bar{B}(A/F,i,n) \tag{2.6}$$

Equations (2.6) and (2.5) have different initial book values B and $\bar{B}$. Under the assumption that the first asset is sold at a loss, $\bar{B}$ is greater than B according to Theorem 2 and Equation (2.6) is less than Equation (2.5). This "downward" shift in the capital costs is depicted by the gray curve labeled "Capital 2" in Figure 2.1b. A similar "upward" shift in costs is seen under the assumption that the first asset is sold at a gain. Here, $\bar{B}$ is less than B according to Theorem 3 and Equation (2.6) is greater than Equation (2.5). This is illustrated by the black curve labeled "Capital 2" in Figure 2.1b.

As explained in Theorem 3, once an asset is sold for a gain in the time invariant case, it is always sold for a gain as the resulting book values are non-increasing over time time. Similarly, once an asset is sold for a loss, it is always sold for a loss (Theorem 2). Thus, the annualized capital cost curve will continue to shift upwards (due to gain) or downwards (due to loss) with time.

If the salvage value is equal to the book value, there is no change in the initial book value of the ensuing asset (by Theorem 1) and therefore the capital cost curve is stationary ("Capital 1" in Figures 2.1a and b). This is the only case where all costs are stationary.

Thus, the implications of Theorems 2 and 3 are severe in the time invariant cost case as the initial book values are not stationary and thus the resulting capital costs are not stationary. In the context of replacement analysis, the traditional time invariant solution which assumes a stationary policy of replacing an asset at the economic life is no longer valid.

To analyze the effect of the moving capital cost curve on the economic life of the asset, one must examine the movement of the initial book value. Given that an asset can be sold at one of n ages, there are n possible resulting capital cost curves for the ensuing asset as there are n possible initial book values.

Theorem 4 *Assume a replacement sequence for an asset such that the first asset is sold at age n_1, the second is sold at age n_2, ..., the mth is sold at age n_m and the initial book value of the first asset is P.*

Then, the initial book value of the $m+1$ asset is:

$$\sum_{i=1}^{m}(P-S(n_i))\prod_{j=i+1}^{m}(1-\sum_{k=1}^{n_j}d_k)+P\prod_{j=1}^{m}(1-\sum_{k=1}^{n_j}d_k). \tag{2.7}$$

Proof. By definition, the initial book value of the first asset is P. According to Equation (2.2), the initial book value of the second asset, given that the first asset is sold at age n_1, is:

$$\begin{aligned} B_{n_1}(0) &= P-S(n_1)+B_{n_1}(n_1) \\ &= P-S(n_1)+(1-\sum_{j=1}^{n_1}d_j)P \end{aligned}$$

For the third asset, the initial book value is calculated as follows (assuming sale of the second asset at age n_2):

$$\begin{aligned} B_{n_1+n_2}(0) &= P-S(n_2)+B_{n_1+n_2}(n_2) \\ &= P-S(n_2)+(1-\sum_{j=1}^{n_2}d_j)B_{n_1}(0) \\ &= P-S(n_2)+(1-\sum_{j=1}^{n_2}d_j)\left(P-S(n_1)+(1-\sum_{j=1}^{n_1}d_j)P\right) \\ &= P+(1-\sum_{j=1}^{n_2}d_j)P+(1-\sum_{j=1}^{n_2}d_j)(1-\sum_{j=1}^{n_1}d_j)P-S(n_2)-S(n_1)(1-\sum_{j=1}^{n_2}d_j) \\ &= P-S(n_2)+(P-S(n_1))(1-\sum_{j=1}^{n_2}d_j)+P(1-\sum_{j=1}^{n_2}d_j)(1-\sum_{j=1}^{n_1}d_j) \end{aligned}$$

Continuing in this manner, the initial book value of the $m+1$ asset, assuming the sequence of asset sales at ages n_1, n_2, $\ldots$, n_m, can be written as:

$$\begin{aligned} B_{n_1+n_2+\ldots+n_m}(0) &= P-S(n_m)+B_{n_1+n_2+\ldots+n_m}(n_m) \\ &= P-S(n_m)+(1-\sum_{j=1}^{n_m}d_j)B_{n_1+n_2+\ldots+n_{m-1}}(0) \\ &= P-S(n_m)+(1-\sum_{j=1}^{n_m}d_j)\left(P-S(n_{m-1})+(1-\sum_{j=1}^{n_{m-1}}d_j)B_{n_1+n_2+\ldots+n_{m-2}}(0)\right), \end{aligned}$$

which simplifies, through back substitution, to:

$$\sum_{i=1}^{m}(P-S(n_i))\prod_{j=i+1}^{m}(1-\sum_{k=1}^{n_j}d_k)+P\prod_{j=1}^{m}(1-\sum_{k=1}^{n_j}d_k),$$

completing the proof. □

Under the assumption of a stationary replacement policy, the equation in Theorem 4 can be more easily analyzed.

Corollary 1 *Assuming an asset starts with initial book value P and is replaced at fixed intervals of age n, the initial book value of the $m+1$ asset is:*

$$(P-S(n))\left(\frac{1-(1-\sum_{k=1}^{n}d_k)^m}{\sum_{k=1}^{n}d_k}\right)+P\left(1-\sum_{k=1}^{n}d_k\right)^m \tag{2.8}$$

Proof. Substitute the value n for n_1, n_2 through n_m into Equation (2.7). This results in:

$$\begin{aligned}
&\sum_{i=1}^{m}(P-S(n))\,(1-\sum_{k=1}^{n}d_k)^{m-i}+P(1-\sum_{k=1}^{n}d_k)^m \\
=\;& (P-S(n))(1-\sum_{k=1}^{n}d_k)^{m-1}+\ldots+(P-S(n))(1-\sum_{k=1}^{n}d_k)+(P-S(n))+P(1-\sum_{k=1}^{n}d_k)^m \\
=\;& (P-S(n))\left(1+(1-\sum_{k=1}^{n}d_k)+(1-\sum_{k=1}^{n}d_k)^2+\ldots+(1-\sum_{k=1}^{n}d_k)^{m-1}\right)+P(1-\sum_{k=1}^{n}d_k)^m
\end{aligned}$$

The value of $(1-\sum_{k=1}^{n}d_k)$ is always non-negative and less than or equal to one. Thus, the series on the left side of the equation may be reduced such that:

$$(P-S(n))\left(\frac{1-(1-\sum_{k=1}^{n}d_k)^m}{\sum_{k=1}^{n}d_k}\right)+P\left(1-\sum_{k=1}^{n}d_k\right)^m.$$

□

Although it is sub-optimal to assume that the asset should be replaced according to a stationary policy, the expression provided in Corollary 1 is much easier to evaluate and leads to numerous conclusions. First, it can be used to evaluate bounds on the movement of the initial book value with time. For instance, the largest loss that can be sustained with the sale of the asset in any period is with $S(n)=0$ for all n, assuming the salvage value cannot be negative. Substituting $S(n)=0$ into Equation (2.8) leads to:

$$\begin{aligned}
&P\left(\frac{1-(1-\sum_{k=1}^{n}d_k)^m}{\sum_{k=1}^{n}d_k}\right)+P\left(1-\sum_{k=1}^{n}d_k\right)^m \\
=\;& P\left(1+(1-\sum_{k=1}^{n}d_k)+(1-\sum_{k=1}^{n}d_k)^2+\ldots+(1-\sum_{k=1}^{n}d_k)^{m-1}+(1-\sum_{k=1}^{n}d_k)^m\right) \\
=\;& P\left(\frac{1-(1-\sum_{k=1}^{n}d_k)^{m+1}}{\sum_{k=1}^{n}d_k}\right)
\end{aligned}$$

Thus, this is the largest initial book value that can be attained after m asset replacements. At the other extreme, the largest value that an asset can be sold for, under normal assumptions, is the purchase price P. In this case, Equation (2.8) reduces to:

$$P(1-\sum_{k=1}^{n}d_k)^m.$$

Thus, if the asset is always sold for the purchase price, which translates to the largest possible gain in each period, the initial book value approaches zero with time. This is formalized as follows.

Corollary 2 *If $P\geq S(n)$ for all n and $P>0$, then the initial book value is always non-negative.*

Proof. The initial book value of any replacement asset is defined in Equation (2.7) in Theorem 4. If $P>0$ and $P\geq S(n)$ for all n, this expression is always non-negative as $(1-\sum_{k=1}^{n}d_k)\geq 0$. □

The previous theorems have shown that the unadjusted cost basis moves with time. In turn, this leads to a movement in capital costs, formally stated in the following three theorems.

Theorem 5 *If $S(n)=B_n(n)$ for all n, then capital costs are stationary for each replacement cycle.*

Proof. By Theorem 1, the initial book value of an asset is stationary. Therefore, the capital costs defined as follows:

$$P(A/P,i,n) - S(n)(A/F,i,n) - t_m \left(\sum_{j=1}^{n} d_j (1+i)^{n-j} \right) B(A/F,i,n) \tag{2.9}$$

are stationary as $B \equiv P$. □

For the cases of selling an asset for a gain or loss, the capital costs are no longer stationary.

Theorem 6 *If $S(n) \leq B_t(n)$ for all n and $B_0(0) = P$, then capital costs are non-increasing with each replacement cycle.*

Proof. By Theorem 2, the initial book value of each replacement asset is non-decreasing due to selling the asset at a loss. Thus, Equation (2.9) is non-increasing with each replacement cycle as the savings from depreciation are non-decreasing with each replacement. For the final replacement cycle, when an asset is retired at the end of the horizon, the costs are:

$$P(A/P,i,n) - S(n)(A/F,i,n) + t_m \left(S(n) - (1 - \sum_{j=1}^{n} d_j)B \right) (A/F,i,n) - t_m \left(\sum_{j=1}^{n} d_j (1+i)^{n-j} \right) B(A/F,i,n), \tag{2.10}$$

which is also non-increasing as the loss leads to a tax credit and the initial book value is non-decreasing over the previous cycle. □

Theorem 7 *If $S(n) \geq B_t(n)$ for all n and $B_0(0) = P$, then capital costs are non-decreasing with each replacement cycle.*

Proof. The proof follows the logic of the previous theorem via Theorem 3. □

As capital costs generally move with each replacement asset, the after-tax economic life of an asset is not stationary with time invariant costs. This is formally stated in Theorem 8 which is stated without proof.

Theorem 8 *The solution to after-tax serial replacement problem is non-stationary with time invariant costs.*

This analysis has explicitly examined the cases when an asset is always sold at either a loss or a gain. Obviously, there are instances when the salvage value of an asset may be greater than the book value for certain ages and less for other ages. In this situation, the book value will move with time, but not in a monotonic fashion. To solve problems of this sort, the dynamic programming formulation presented in the following section provides a solution approach.

2.2 Time Invariant Analysis: Dynamic Programming Approach

Equipment replacement decisions are often motivated by obsolescence and technological change. This situation generally requires that costs be time variant to differentiate assets over time. To explicitly analyze this situation, the following dynamic programming problem is presented. The following notation is utilized (some parameters and costs are repeated from before):

$P_t(k,n)$ = purchase cost of an n-year old asset of type k at time t;
$S_t(k,n)$ = salvage value of an n-year old set of type k at time t;
$C_t(k,n)$ = operating and maintenance cost of an n-year old asset of type k utilized from the end of time t to the end of $t+1$;
d_n = percentage depreciation of unadjusted cost basis for an asset in its nth year of operation;
N_k = maximum physical age of an asset of type k;
N_r = depreciation recovery period;
T = finite horizon time;
K = maximum number of available challengers in a given period;
t_m = marginal tax rate;
α = one period after-tax discount factor.

Define the state equation as follows: $f_t(k,n,B)$ is the minimum expected net present value of costs when starting with an asset of age n, type k, and unadjusted cost basis B at time t and choosing optimal decisions through time T. (Note that we assume a minimum cost formulation here, but the model can easily be transformed to a maximization problem if revenues are to be included.)

In this finite horizon problem, the asset may be kept (if it has not reached its maximum physical age of N_k) or replaced with one of K challengers at the end of each period $t = 0, 1, 2, \ldots, T-1$, corresponding to the stages of the dynamic program. At the end of the final period T, the asset is salvaged. Purchase and salvage values are assumed to occur at the beginning of the period while O&M costs and tax savings from depreciation charges are assumed to occur at the end of the period. The half-year convention for depreciation is assumed here (under MACRS rules), although different conventions alter the formulation only slightly.

The recursive functional equation, for periods $t < T$, is written as follows:
$f_t(k,n,B) =$

$$\min \begin{cases} \text{K: } \alpha\left[(1-t_m)C_{t+1}(k,n+1) - t_m d_{n+1} B + f_{t+1}(k',n+1,B)\right], \\ \text{R: } P_t(k') - S_t(k,n) + \frac{\alpha}{2} t_m d_n B + \alpha\left[(1-t_m)C_{t+1}(k',1) - t_m d_1 \bar{B} + f_{t+1}(k',1,\bar{B})\right], \quad n \le N_r, n \ne 1 \\ \quad\; P_t(k') - S_t(k,n) + \alpha\left[(1-t_m)C_{t+1}(k',1) - t_m d_1 \bar{B} + f_{t+1}(k',1,\bar{B})\right], \quad n = 1, n > N_r \end{cases}$$

where K is the decision to keep asset k and R is the decision to replace asset k with an asset of type $k' = 1, 2, \ldots, K$. $\bar{B}$, the unadjusted cost basis of the acquired asset, is defined as follows:

$$\bar{B} = \begin{cases} P_t(k') - S_t(k,1) + (1-d_1)B, \\ P_t(k') - S_t(k,n) + \left(1 - \sum_{j=1}^{n-1} d_j - \frac{1}{2} d_n\right) B, \quad n \le N_r, n \ne 1 \\ P_t(k') - S_t(k,n), \quad n > N_r \end{cases}$$

Thus, if the asset is kept, operating and maintenance (C) costs are paid and savings from depreciation expenses (dB) are accrued (if the asset is not yet fully depreciated). The resulting state of this decision is an asset that is one-year older with no change in initial book value.

If the asset is replaced, then a salvage value (S) is received and a purchase cost (P) is paid. If the asset is replaced before the end of the recovery period, half of the depreciation from the previous year is not recovered according to the half-year convention (Section 168 [9]), thus it is added back. The new asset incurs an operating and maintenance charge in the first period and also savings from the first year's worth of depreciation. Finally, the new state is defined by an asset of age one (at the end of the period) and an initial book value equal to the purchase cost and remaining book value of the sold asset less salvage value. If the sold asset is fully depreciated, then the initial book value of the acquired asset is merely the difference in purchase price and salvage value.

As this is the finite horizon problem, the asset is sold at the end of time T for its salvage value and the corresponding gain or loss is calculated and taxed or credited. Note that according to current U.S. tax law (Sections 1245 and 1231 [9]), gains are treated as ordinary income and charged at the marginal tax rate, as follows:

$f_T(k,n,B) =$

$$\begin{cases} -(1-t_m)S_T(k,1) - t_m(1-d_1)B, \\ \frac{\alpha}{2}t_m d_n B - (1-t_m)S_T(k,n) - t_m\left(1-\sum_{j=1}^{n-1} d_j - \frac{1}{2}d_n\right)B, \ n \leq N_r, n \neq 1 \\ -(1-t_m)S_T(k,n), \quad n > N_r \end{cases} \tag{2.11}$$

As with the recursion formula, half of the depreciation charge taken in the previous period is added back due to the half-year convention if the asset is sold early. It is assumed here that the salvage value of any asset at any age is less than or equal to the purchase price. If this is not the case, an extra layer of taxes (for capital gains) must be incorporated at the time of sale.

We have specifically addressed the finite horizon problem here in that the asset is retired at the end of the final period and a gain or loss is realized. As noted earlier, several authors have examined the infinite horizon solution to the serial replacement problem with the use of dynamic programming. Although the terminology differs among work in this area, the approaches are similar in that an equivalent finite horizon problem is solved such that the time zero decision is equivalent to the infinite horizon decision. Under certain conditions, an equivalent finite horizon exists (Bean and Smith [3]). This was implemented in Bean, Lohmann and Smith [1] through an iterative procedure such that if the time zero decision did not change for N periods, where N is the maximum physical life among all challengers, the optimal decision and finite decision horizon has been found. As we may assume similar cost structures here, we can employ the same method for the infinite horizon problem, if desired.

3 Comparison of After-Tax Formulations

As noted earlier, after-tax replacement models in the literature do not incorporate the transfer of book value into their analysis. Rather, they generally charge a tax or credit on a gain or loss at each asset sale, thus maintaining a stationary initial book value for each ensuing asset. To model this situation, the terminal cost charged in the presented dynamic program would be charged at each asset sale. Additionally, the initial book value of the asset would no longer be needed as a state variable as it would equal the purchase cost at each asset sale. The recursion for this model follows:

$f_t(k,n) =$

$$\min \begin{cases} \text{K:} \ \alpha\left[(1-t_m)C_{t+1}(k,n+1) - t_m d_{n+1}P_t(k) + f_{t+1}(k',n+1)\right], \\ \text{R:} \ P_t(k') - (1-t_m)S_T(k,1) - t_m(1-d_1)P_t(k) + \alpha\left[(1-t_m)C_{t+1}(k',1) - t_m d_1 P_t(k') + f_{t+1}(k',1)\right], \\ \quad P_t(k') + \frac{\alpha}{2}t_m d_n B - (1-t_m)S_T(k,n) - t_m\left(1-\sum_{j=1}^{n-1} d_j - \frac{1}{2}d_n\right)P_t(k) + \\ \quad\quad \alpha\left[(1-t_m)C_{t+1}(k',1) - t_m d_1 P_t(k') + f_{t+1}(k',1)\right], \quad n \leq N_r, n \neq 1 \\ \quad P_t(k') - (1-t_m)S_T(k,n) + \alpha\left[(1-t_m)C_{t+1}(k',1) - t_m d_1 P_t(k') + f_{t+1}(k',1)\right], \quad n > N_r \end{cases}$$

and the terminal cost would equal that in Equation (2.11) with B equal to the appropriate purchase price.

While the economic life of an asset is not stationary under current tax law in the time invariant cost case, the traditional model has stationary costs as the unadjusted cost basis of the asset is fixed. This implies that the economic life remains fixed in time invariant economic problems.

We term this model the GAINS model and the dynamic program which incorporates the initial book value as the TAX model. Here, we compare the GAINS and TAX models and examine the implications of using the GAINS model which incorrectly models current tax law.

Theorem 9 *If $S(n) \geq B_n(n)$ for all n and $B_0(0) = P$, the GAINS model overstates capital expenses for at least the first replacement cycle.*

Proof. The sale of an asset at a gain in the GAINS model results in a charged tax equivalent to the marginal tax rate times the gain. This results in the following annualized capital costs when keeping the asset for n periods:

$$P(A/P,i,n)-S(n)(A/F,i,n)+t_m\left(S(n)-(1-\sum_{j=1}^{n}d_j)P\right)(A/F,i,n)-t_m\left(\sum_{j=1}^{n}d_j(1+i)^{n-j}\right)P(A/F,i,n) \tag{3.12}$$

As noted earlier, these costs are stationary over time.

In the TAX model, results from the sale of the asset are not apparent in the first replacement cycle annualized capital costs. They are as follows:

$$P(A/P,i,n)-S(n)(A/F,i,n)-t_m\left(\sum_{j=1}^{n}d_j(1+i)^{n-j}\right)P(A/F,i,n) \tag{3.13}$$

Under the assumption that for any value of n, the asset is sold at gain, the value of:

$$t_m\left(S(n)-(1-\sum_{j=1}^{n}d_j)P\right)(A/F,i,n)\geq 0,$$

and is the only difference between the GAINS and TAX models in the first replacement cycle. Thus Equation (3.12) is greater than (3.13) and annualized capital costs are greater for the GAINS model.

Given that the asset is sold for a gain, the annualized capital costs rise in the TAX model because the initial book value, and thus the tax savings from depreciation, decrease. For the second replacement cycle, or second owned asset, the annualized capital costs are (assuming sale of the first asset at age n and the second asset at age n_1):

$$P(A/P,i,n_1)-S(n_1)(A/F,i,n_1)-t_m\left(\sum_{j=1}^{n_1}d_j(1+i)^{n_1-j}\right)\left(P-S(n)+(1-\sum_{j=1}^{n}d_j)P\right)(A/F,i,n_1) \tag{3.14}$$

Examining the annualized capital costs for the GAINS models (3.12) and the TAX model (3.14) requires the following for the GAINS costs to be greater than the TAX costs:

$$S(n_1)-(1-\sum_{j=1}^{n_1}d_j)P>\left(\sum_{j=1}^{n_1}d_j(1+i)^{n_1-j}\right)\left(S(n)-(1-\sum_{j=1}^{n}d_j)P\right).$$

If $n = n_1$, this requires:

$$\sum_{j=1}^{n_1}d_j(1+i)^{n_1-j}<1,$$

which is not always true, as a fully depreciated asset and a positive interest rate i results in higher costs for the TAX model. Therefore, the GAINS model is guaranteed to overstate expenses in the first replacement cycle, but not thereafter. □

The results from Theorem 9 indicate that the capital costs from the GAINS model are initially overstated in early periods. However, the capital cost curve shifts upwards due to continually selling assets at a gain in the TAX model and eventually they surpass those of the GAINS model. In the final replacement cycle, where the asset is retired, the costs for the GAINS model are actually lower than the TAX model.

Theorem 10 *If $S(n) \geq B_n(n)$ for all n and $B_0(0) = P$, the GAINS model understates capital expenses for at least the last replacement cycle in a finite horizon problem with at least one replacement.*

Proof. As stated in the previous theorem, the annualized capital costs for the GAINS model are stationary as in Equation (3.12). For the TAX model, a gain or loss is realized at the end of the finite horizon as the asset is sold. For that asset, the annualized capital costs are (assuming sale at age n):

$$P(A/P,i,n)-S(n)(A/F,i,n)+t_m\left(S(n)-(1-\sum_{j=1}^{n}d_j)B\right)(A/F,i,n)-t_m\left(\sum_{j=1}^{n}d_j(1+i)^{n-j}\right)B(A/F,i,n), \tag{3.15}$$

where B is the unadjusted cost basis of the retired asset. As $S(n) \geq B_n(n)$, the asset is continually sold at a gain and the initial book value is non-increasing by Theorem 3 and $B \leq P$. Thus, Equation (3.15) cannot be less than Equation (3.12) and the GAINS model understates capital costs for the final replacement cycle. □

The following two theorems are similar with the exception that they describe the case of selling an asset for a loss. They are stated without proof as they following similar logic to those above.

Theorem 11 *If $S(n) \leq B_n(n)$ for all n and $B_0(0) = P$, the GAINS model understates capital expenses for at least the first replacement cycle.*

Theorem 12 *If $S(n) \leq B_n(n)$ for all n and $B_0(0) = P$, the GAINS model overstates capital expenses for at least the last replacement cycle in a finite horizon problem with at least one replacement.*

It should be clear that the GAINS and TAX model can lead to different replacement decisions as they calculate different capital costs over the life of the asset. The differences in these models are exaggerated when an asset is sold at a considerable loss or gain. The following two examples illustrate these two cases and show that the replacement decisions vary widely.

Selling an Asset for Gain Example

To examine the case where an asset is sold for a gain, we utilize the following data:

$$\begin{aligned}
P &= 100,000 \\
S(n) &= 100,000(0.9)^n, \quad n = 1,2,\ldots,N \\
C(n) &= 10,000(1.15)^n, \quad n = 1,2,\ldots,N \\
T &= 50 \\
N_k &= 8 \\
N_r &= 5 \\
t_m &= 35\% \\
i &= 10\%
\end{aligned}$$

The TAX model produces net present value (NPV) costs of $145,508, with decisions generally following a four-year replacement cycle. For the GAINS model, a six-year replacement cycle is employed with a NPV cost of $159,776.

In this example, the salvage value of the asset does not decline very rapidly with time, although the operating and maintenance costs increase fairly rapidly. The GAINS model replaces the asset in a six-year cycle, thereby avoiding a gain on the sale of the asset which would be taxed immediately. However, the TAX model sells the asset earlier, due to the rising O&M costs, as the proceeds from the gain may be taxed over time through the transferal of residual book value. Although the data here is fictitious, there are a variety of real examples that follow this cost structure, including airplanes and airplane engines. These assets retain their value with time, as they are generally in high demand, although they require increased maintenance with age, often due to safety regulations.

Selling an Asset for Loss Example

The data from the previous examine is repeated here with the exception of the following costs which are designed to force the sale of an asset at a loss:

$$\begin{aligned} P &= 100,000 \\ S(n) &= 100,000(0.2)^n, \quad n = 1, 2, \ldots, N \\ C(n) &= 10,000(2.0)^n, \quad n = 1, 2, \ldots, N \end{aligned}$$

The NPV for the TAX model is $563,395, with decisions generally following a three-year replacement cycle. For the GAINS model, a two-year replacement cycle is employed with a NPV cost of $536,547.

In this model, the GAINS model profits from the tax credit on the sale of the asset at a loss. As this loss reduces future savings from depreciation charges, the TAX model retains the asset longer. Again, the data is fictitious and time invariant, but it has similar structure to assets that are technologically advanced. In this situation, it is often assumed that the salvage values of current assets drop considerably due to the availability of more advanced replacements on the market. Also, the cost to maintain technologically advanced assets can increase dramatically with time, including cost for support, such as software, and efficiency when compared to newer available models.

These two small examples illustrate the decisions that can result when employing the GAINS model which incorrectly assumes that a gain or loss is recognized at each asset sale over the horizon. Although dependent on cost data, sales of assets at a loss will tend to shorten replacement cycles motivated by tax credits while gains from sales will tend to lengthen replacement cycles to deter tax payments. While correctly modeling the transferal of residual book values complicates traditional economic replacement models, it is required to properly capture current United States tax law and all of its cash flow implications.

4 Conclusions and Directions for Future Research

This paper examines the after-tax serial replacement problem under current United States tax law. The law requires that the residual book value of an asset be transferred to the replacement asset in the case of a like-for-like exchange. We show that this transfer complicates replacement decisions, as costs are no longer stationary with time, even with time invariant purchase, operating and maintenance costs and salvage values. Costs are no longer stationary as the initial book value or unadjusted cost basis of each replacement asset is dependent on its purchase price and the salvage value and residual book value of the retired asset. Under certain conditions, the initial book value may increase or decrease monotonically, which in turn leads to depreciation charges which either grow or shrink with time. The movement of the unadjusted cost basis over time is explicitly analyzed. Also, a dynamic programming formulation, which includes asset age and the initial book value as state variables, is presented in order to solve the economic replacement problem under current United States tax law, assuming MACRS depreciation rules and the half-year convention.

Traditionally, after-tax replacement models recognize a gain or loss at each asset sale over the horizon, leading to a tax liability or credit. This over or underestimates the capital costs of assets over time. We illustrate through various examples and analysis that if the gain or loss from an asset sale is large, then the traditional models lead to widely differing solutions, both in cost and replacement decisions, than the presented model.

There remain a number of questions to be explored concerning the replacement of assets under different tax laws. This paper did not explicitly consider the question of whether MACRS or AMACRS is the optimal depreciation method, but the presented model may help define answers to that problem. Additionally, questions remain concerning taxes with multiple assets. In this situation, assets may be grouped and depreciated as one entity. As individual asset replacement decisions affect the group, this is another interesting parallel replacement problem to be considered. Parallel replacement analysis considers the replacement of assets that operate in parallel and are economically interdependent (Jones, Zydiak and Hopp [12]). In this situation, the assets would be economically linked through their depreciation groups and thus the replacement of one asset in the group is dependent on the replacement of the other assets.

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