A Tight Approximation for a Continuous-Review Inventory Model with Supply Disruptions

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Abstract

We consider a continuous-review inventory model for a retailer who faces constant, deterministic demand but whose supplier is unreliable. The supplier experiences "wet" and "dry" (operational and disrupted) periods whose durations are exponentially distributed. The retailer follows an EOQ-like policy during wet periods but may not place orders during dry periods; any demands occurring during dry periods are lost if the retailer does not have sufficient inventory to meet them.

This paper introduces a simple but tight approximation for such a model that maintains the tractability of the classical EOQ and permits analysis similar to that typically performed on the EOQ. We provide analytical and empirical bounds on the approximation error in both the cost function and the optimal order quantity. We prove that our cost function behaves similarly to the classical EOQ cost function in several important ways and derive an expression for the increase in cost if a sub-optimal solution is used that is similar to the classical EOQ sensitivity analysis result. We examine power-of-two policies under our cost function and prove a worst-case bound of 6% for such a policy. Finally, we show that using the classical EOQ model when supply uncertainty exists can be quite costly.

Keywords: inventory, EOQ, supply disruptions, power-of-two policies

1 Introduction

Despite the careful attention paid to inventory planning in a supply chain, supply disruptions are inevitable. Inventory managers who ignore the risk of supply disruptions will encounter excess costs when such disruptions occur, in the form of stockout costs, expediting costs, and loss of goodwill. On the other hand, disruptions are typically infrequent and unpredictable, so holding too much extra inventory is costly, as well. An effective inventory policy should strike a balance between protecting against stockouts during disruptions and maintaining low inventory levels.

Supply disruptions may come from a variety of sources including labor actions, machine breakdowns, and natural or man-made disasters. For example, a labor strike at two General Motors parts plants in 1998 halted the auto maker's North American production operations, causing a 37% drop in sales due to a lack of inventory (White 1998). Similarly, a fire in 2001 at a plant owned by Ericsson's primary supplier of semiconductors brought Ericsson's production lines to a standstill, costing the cell phone company an estimated $400 million in lost revenue (Latour 2001). Moreover, effective inventory planning can mitigate the negative effects of such disruptions: despite relying on the same semiconductor supplier, Nokia's superior planning allowed it to weather the supply disruption and even to steal a substantial portion of Ericsson's market share.

In this paper, we examine a model for setting order quantities in a continuous-review inventory system with deterministic demand and random supply disruptions. The durations of the supplier's "wet" and "dry" (operational and disrupted) periods are exponentially distributed. Orders cannot be placed during dry periods, and demands occurring during dry periods are lost if the retailer does not have sufficient inventory to meet them. We refer to this problem as the *economic order quantity with disruptions* (EOQD). The introduction of supply disruptions into classical inventory models typically destroys the tractability of those models and the analytical results that follow from them, and the EOQD is no exception. However, we introduce a tight approximation for the

EOQD that is tractable and lends itself to familiar analysis often performed for the classical EOQ.

The EOQD was first examined by Parlar and Berkin (1991), whose model was shown by Berk and Arreola-Risa (1994) to be incorrect. Berk and Arreola-Risa accurately formulate the cost of the system, but their cost function cannot be minimized in closed form, nor is it known to be convex. Moreover, the complexity of the exact cost function makes it difficult to embed into more complex models or to derive analytical results about the behavior of the optimal solution.

We present a cost function that closely approximates that of Berk and Arreola-Risa and show that:

- our cost function is convex
- our cost function yields a closed-form expression for the optimal order quantity
- the optimal solution to our model is always greater than that of the classical EOQ model, as is the optimal cost
- our cost function approaches the classical EOQ cost function in the limit as wet periods become long relative to dry periods

We also prove analytical bounds on the error introduced by our approximation. In particular:

- we prove that our approximate cost function is greater than the exact cost function for "reasonable" values of Q, including the optimal value, and derive analytical bounds on the percentage difference between the two cost functions
- we prove that the optimal order quantity for the approximate cost function is greater than that for the exact cost function and derive analytical bounds on the percentage difference between the two order quantities given a particular assumption about the exact function
- we derive an analytical bound on the percentage difference between the cost of the exact and approximate solutions under the exact cost function—that is, the error that results from implementing the approximate solution when the exact cost function prevails

Having established the validity of our approximation, we show that it lends itself to analysis often performed on the EOQ model:

- we show that the optimal cost is equal to the optimal order quantity times the holding cost
- we show that the optimal cost is a concave function of the demand rate
- we derive an expression for the increase in cost if a sub-optimal order quantity is used that is similar in form and magnitude to the classical EOQ sensitivity analysis result
- we examine power-of-two policies under our cost function and prove that the cost of the optimal power-of-two policy is no more than 6% worse than that of the overall optimal policy (and we demonstrate empirically that the actual increase in cost is often much smaller)

The remainder of this paper is structured as follows. In Section 2, we provide a review of the literature on inventory models with supply disruptions. In Section 3, we introduce the model and our approximate cost function. In Section 4, we derive the optimal solution to our cost function. In Section 5, we prove analytical bounds on the approximation error in the cost function and the optimal solution. In Section 6, we compare our model analytically to that of the EOQ model. In Section 7, we discuss sensitivity analysis and power-of-two policies. In Section 8, we provide computational results comparing the actual approximation errors to the analytical bounds, comparing the approximate EOQD solution to the EOQ solution, and evaluating optimal power-of-two policies. Finally, in Section 9, we draw conclusions from our analysis and suggest future research directions. Proofs of all lemmas, theorems, etc. are provided in the Appendix.

2 Literature Review

Supply uncertainty takes the form of either *yield uncertainty*, in which supply is always available but the quantity delivered is a random variable (see, e.g., Yano and Lee 1995), or *disruptions*, in which the supplier experiences failures during which it cannot provide any product. This paper is concerned with supply disruptions.

The earliest paper to consider supply disruptions is probably that of Meyer, Rothkopf and Smith (1979), who consider a production facility facing constant, deterministic demand. The facility has a capacitated storage buffer, and the production process is subject to stochastic failures and repairs. The goal of the paper is not to optimize the system but to compute the percentage of time that demands are met.

Parlar and Berkin (1991) introduce the first of a series of models that incorporate supply disruptions into classical inventory models. They study the EOQD: an EOQ-like system in which the supplier experiences intermittent failures. Demands are lost if the retailer has insufficient inventory to meet them during supplier failures. The retailer follows a zero-inventory ordering (ZIO) policy. Their cost function was shown to be incorrect in two respects by Berk and Arreola-Risa (1994), who propose a corrected cost function. It is their function that we approximate in this paper.

Parlar and Perry (1995) extend the EOQD in three ways. First, they relax the ZIO assumption and allow the reorder point to be a decision variable. Second, they assume that the retailer incurs a cost each time it ascertains the state of the supplier, so that the waiting time between order attempts is also a decision variable. Third, they consider both random and deterministic yields (i.e., the amount actually delivered may be less than the amount ordered). (The ZIO assumption was also considered by Bielecki and Kumar (1988), who found that, under certain modeling assumptions, a ZIO policy may be optimal even in the face of supply disruptions, countering the common view that if any uncertainty exists, it is optimal to hold some safety stock to buffer against it.) Moinzadeh and Aggarwal (1997) consider an unreliable production system; their model is like the economic production quantity (EPQ) problem with disruptions and a fixed cost to initiate production. They suggest a continuous-review (s, S) policy (the inventory level may fall strictly below the reorder point during a failure).

Parlar and Perry (1996) consider the EOQD with one, two, or multiple suppliers. They also relax the ZIO assumption, though they do not include a cost for attempting to place an order as do Parlar and Perry (1995). They allow the order quantity to depend on the states of the suppliers and show that if the number of suppliers is large, the problem reduces to the classical EOQ. The suppliers may be heterogeneous in the sense that their failure and repair processes may be described by different parameters, but they are homogeneous with respect to price, so as long as at least one supplier is active, the retailer does not care which one it orders from. (In contrast, Tomlin (2005), discussed below, allows suppliers to compete on both price and reliability.) Gürlur and Parlar (1997) generalize the two-supplier model by allowing more general failure and repair processes. They present asymptotic results for large order quantities.

Given the complexities introduced by supply disruptions, only a few papers have considered stochastic demand, as well. Gupta (1996) formulates a (Q, R)-type model with Poisson demand and exponential wet and dry periods. Parlar (1997) studies a similar but more general model than Gupta—for example, allowing for stochastic lead times—but formulates an approximate cost function. Chao (1987) and Chao, Chapel, Clark, Morris, Sandling and Grimes (1989) consider stochastic demand for electric utilities with market disruptions and solve the problem using stochastic dynamic programming.

Periodic-review inventory models with supply disruptions have received considerably less attention in the literature than their continuous-review counterparts. Arreola-Risa and DeCroix (1998) consider (s, S) models with supplier disruptions. They develop exact expressions for the expected cost as a function of the system parameters but use numerical optimization since analytical solutions cannot be obtained. Song and Zipkin (1996) present a model in which the availability of the supplier, while random, is partially known to the decision maker. The inventory model is formulated as a dynamic program whose optimal policy (for linear order costs) is a base-stock policy in which the optimal order-up-to level depends on the state of the supply process. Tomlin (2005) presents a dual-sourcing model in which orders may be placed with either a cheap but unreliable supplier or an expensive but reliable supplier. The reliable supplier may not be able to provide additional units instantaneously when the unreliable supplier fails but experiences a "ramp up" period during a disruption. Tomlin derives explicit optimal base-stock levels for certain special cases and explores the circumstances under which it is optimal for the firm to single or dual source.

It is worth noting that with the exception of Bielecki and Kumar (1988) and certain models in Tomlin (2005), all of the papers cited in this section propose a numerical approach for optimizing their cost functions—none of them is solved in closed form. In contrast, the approximate cost function proposed in this paper may be solved in closed form, and as a consequence, a number of analytical results may be derived for it.

3 Model Formulation

3.1 Original Model

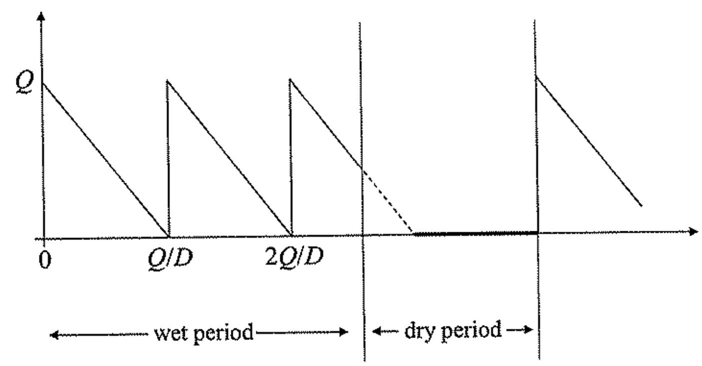
Consider an EOQ model with fixed ordering cost K, holding cost h per unit per year, and constant, deterministic demand rate D units per year. (Without loss of generality we assume that the time unit is one year.) Suppose that the supplier is not perfectly reliable—that it functions normally for a certain period (called a "wet period") and then shuts down for a certain period (a "dry period"). During dry periods, no orders can be placed, and if the retailer runs out of inventory during a dry period, all demands observed until the beginning of the next wet period are lost, with a stockout cost of p per lost sale. The durations of both wet and dry periods are exponentially distributed, wet periods with rate λ and dry periods with rate μ. Every order placed by the retailer is for the same quantity, Q, orders are only placed when the inventory level reaches 0, and orders placed during wet periods are received immediately (there is no lead time). The goal of the model is to choose Q to minimize the expected annual cost. We refer to this problem as the *economic order quantity with disruptions* (EOQD).

A typical inventory curve is pictured in Figure 1. Note that the inventory position never becomes negative since unmet demands are lost.

The EOQD was first formulated by Parlar and Berkin (1991), whose expected cost function was shown by Berk and Arreola-Risa (1994) to be incorrect in two respects. Berk and Arreola-Risa derive the following corrected expression for the expected annual cost as a function of Q:

$$g_0(Q) = \frac{K + hQ^2/2D + Dp\beta_0(Q)/\mu}{Q/D + \beta_0(Q)/\mu} \tag{1}$$

Figure 1: EOQ inventory curve with disruptions.



where

$$\beta_0(Q) = \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)Q/D}\right) \tag{2}$$

is the probability that the supplier is in a dry period when the retailer's inventory level reaches 0. We will often suppress the argument Q in $\beta_0(Q)$ when it is clear from the context.

The first-order condition $dg_0/dQ = 0$ cannot be solved in closed form because it has the form

$$\alpha_1 Q^2 + \alpha_2 Q + \alpha_3 + (\alpha_4 Q^2 + \alpha_5 Q + \alpha_6)e^{-\alpha_7 Q} = 0,$$

for suitable constants α_i, for which no closed-form solution is readily available. (The first-order condition is written out explicitly in equation (16) in the Appendix.) Moreover, Berk and Arreola-Risa prove that $g_0(Q)$ is unimodal, but it is not known whether it is convex.

The first term in β_0, $\lambda/(\lambda+\mu)$, is the steady-state probability that the supplier is in a dry period, while the second term accounts for the knowledge that when the inventory level hits 0, we were in a wet period as recently as Q/D time units ago. The essence of our approximation is to assume that the system approaches steady state quickly enough that when the inventory level hits 0, we can essentially ignore this bit of knowledge, i.e., ignore the transient nature of the system at this moment. We discuss this point further in Section 3.3.

3.2 Assumptions

Before introducing our approximation to (1), we impose three mild assumptions on the problem parameters. First, we assume that all costs and other problem parameters are non-negative. Second, we assume that $\lambda < \mu$, that is, wet periods last longer on average than dry periods.

Third, we assume that $\sqrt{2KDh} < pD$. If there were no disruptions, this model would reduce to the classical EOQ model, whose optimal annual cost is well known to equal $\sqrt{2KDh}$ (see, e.g., Nahmias 2005). Therefore $\sqrt{2KDh}$ is a lower bound on the optimal cost of the system with disruptions. One feasible solution for the EOQD is for the retailer never to place an order and instead to stock out on every demand; the annual cost of this strategy is pD. Therefore, the assumption that $\sqrt{2KDh} < pD$ is meant to prohibit the situation in which it is more expensive to serve demands than to lose them.

3.3 Approximation

We propose approximating Berk and Arreola-Risa's cost function by replacing $\beta_0(Q)$ with

$$\beta = \frac{\lambda}{\lambda + \mu}. \tag{3}$$

The resulting approximate cost function is

$$g(Q) = \frac{K + hQ^2/2D + Dp\beta/\mu}{Q/D + \beta/\mu} = \frac{h\mu Q^2/2 + KD\mu + D^2 p\beta}{Q\mu + \beta D}. \tag{4}$$

Note that the functional form of this cost function,

$$\frac{aQ^2 + b}{cQ + d}, \tag{5}$$

is similar to that of the EOQ cost function, $\frac{aQ^2+b}{cQ}$, differing in the constant added to the denominator. This similarity in structure gives rise to many of the EOQ-like properties derived in Sections 6 and 7. Indeed, many of the results in this paper hold for any cost function of the form given in (5).

As noted in Section 3.1 above, this approximation assumes that the system reaches steady-state quickly, since β differs from β_0 by omitting the term that accounts for transience. Although Berk and Arreola-Risa assume exponentially distributed wet and dry period durations, other distributions would yield similar cost functions, with the term $1-\exp(-(\lambda+\mu)Q/D)$ replaced by a distribution-specific term. Our approximation is applicable to these cases, as well, with the quality of the approximation determined by the rate with which the system approaches steady-state.

The approximation is most effective when $(\lambda+\mu)Q/D$ is large, a reasonable assumption under many realistic settings. For example, suppose wet periods last, on average, 1 year and dry periods last, on average, 1 month. Further, suppose that 4 orders are placed per year. Then $(\lambda+\mu)Q/D = (1+12)/4 = 3.25$, so $e^{-(\lambda+\mu)Q/D} = e^{-3.25} \approx 0.039$. Since $\beta_0 = \beta\left(1 - e^{-(\lambda+\mu)Q/D}\right)$, β is a close approximation for β_0, and hence $g(Q)$ is a close approximation for $g_0(Q)$. We provide theoretical evidence for the accuracy of the approximation in Section 5 and empirical evidence in Section 8.2.

One would expect that as the supplier's reliability improves, the EOQD begins to resemble the EOQ more and more closely. In particular, as λ gets small or μ gets large (so that wet periods last much longer than dry periods), g approaches the classical EOQ cost function, as Proposition 1 demonstrates. The proof is omitted; it follows from the fact that as $\lambda/\mu \to 0$, $\beta \to 0$.

Proposition 1

$$\lim_{\lambda/\mu \to 0} g(Q) = g_E(Q),$$

where $g_E(Q) = \frac{KD}{Q} + \frac{hQ}{2}$ *is the classical EOQ cost function.*

The same result holds for Berk and Arreola-Risa's g_0, though it does not hold for Parlar and Berkin's original (incorrect) cost function.

4 Optimal Solution

In this section we show that our approximate cost function g is convex, and we provide a closed-form solution for the optimal value of Q, denoted Q^*. All proofs are given in the Appendix.

Theorem 2 *(a)* $g(Q)$ *is convex in* Q

(b) The value of Q that minimizes $g(Q)$ is given by

$$Q^* = \frac{\sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)} - \beta Dh}{h\mu}. \tag{6}$$

Note that Q^* can be rewritten as

$$Q^* = \sqrt{\frac{2KD}{h} + a^2 + b} - a$$

for appropriate constants a and b, thus emphasizing the relationship between Q^* and the optimal order quantity for the classical EOQ, $\sqrt{2KD/h}$.

5 Accuracy of Approximation

5.1 Accuracy of Cost Function

In this section, we discuss the accuracy of g as an approximation for g_0. Our first result demonstrates that, for reasonable values of Q (including Q^*), $g(Q)$ overestimates $g_0(Q)$. Thus a manager using the approximate function to estimate costs will incur lower costs than expected; that is, the approximation is conservative.

Proposition 3 *(a) $g(Q) > g_0(Q)$ if and only if*

$$\frac{KD}{Q} + \frac{hQ}{2} < Dp.$$

(b) $g(Q) > g_0(Q)$ if and only if

$$\frac{Dp - \sqrt{(Dp)^2 - 2KDh}}{h} < Q < \frac{Dp + \sqrt{(Dp)^2 - 2KDh}}{h}.$$

(c) $g(Q^*) > g_0(Q^*)$.

The range of Q values specified in part (a) of Proposition 3 is quite wide for reasonable values of the parameters. We know that $\sqrt{2KDh} < Dp$; indeed, we would expect $\sqrt{2KDh} \ll Dp$ since $\sqrt{2KDh}$ is a lower bound on the optimal cost, while Dp is the cost of an extremely poor solution (i.e., ordering nothing). The EOQ cost function is known to be reasonably flat, which means that Q must deviate substantially from its optimum before the cost reaches Dp. Put another way, $(Dp)^2$ should dominate $2KDh$ in the condition in (b), and $\sqrt{(Dp)^2 - 2KDh} \approx Dp$. Therefore, the lower bound on Q from Proposition 3(b) is close to 0, while the upper bound is quite large. Part (b) of the theorem confirms that the optimal Q is in the critical range.

Next we show that $g(Q)$ does not overestimate $g_0(Q)$ by too much by proving a worst-case bound on the magnitude of the error. In particular, this bound holds for the special case of $Q = Q^*$, but we also prove another bound for this case.

Theorem 4 *(a) Let $g_E(Q) = \frac{KD}{Q} + \frac{hQ}{2}$ be the classical EOQ cost function. For all $Q > 0$ such that $g_E(Q) < Dp$,*

$$\frac{g(Q) - g_0(Q)}{g_0(Q)} < \frac{\beta - \beta_0(Q)}{\beta_0(Q)} \left[1 - \frac{g_E(Q)}{Dp}\right] < \frac{\beta - \beta_0(Q)}{\beta_0(Q)} = \frac{e^{-(\lambda+\mu)Q/D}}{1 - e^{-(\lambda+\mu)Q/D}}.$$

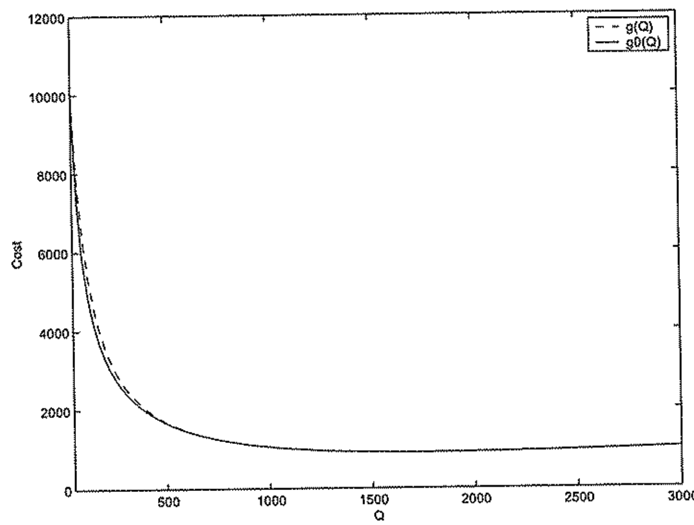
(b)

$$\frac{g(Q^*) - g_0(Q^*)}{g_0(Q^*)} < \min\left\{\frac{\beta - \beta_0(Q^*)}{\beta_0(Q^*)} \left[1 - \frac{g_E(Q^*)}{Dp}\right], \frac{\beta - \beta_0(Q^*)}{\beta}\right\} < 1.$$

The bound in Theorem 4(a) does not have a fixed worst-case value, since $\beta_0(Q) \to 0$ as $(\lambda+\mu)Q/D \to 0$. Theorem 4(b) does establish a worst-case bound of 1 on the approximation error for Q^* via the second bound of $(\beta - \beta_0(Q^*))/\beta$, but in our computational tests the first bound was smaller than the second for every instance. For reasonable values of the parameters, both bounds are much smaller than 1, as argued in Section 3.3. We provide computational results justifying this claim empirically in Section 8.2.

Typically, g approximates g_0 very tightly when $Q \approx (Dp - \sqrt{(Dp)^2 - 2KDh})/h$ (the left end-point of the range of interest from Proposition 3(b)). The approximation weakens somewhat as Q increases but tightens again quickly as Q increases. Figure 2 plots the curves g and g_0 and Figure 3 plots the approximation error $(g(Q) - g_0(Q))/g_0(Q)$ and the bound $\frac{\beta-\beta_0}{\beta_0}\left[1 - \frac{g_E(Q)}{Dp}\right]$ as functions of Q for $K = 500$, $h = 0.5$, $p = 10$, $D = 1000$, $\lambda = 1$, $\mu = 5$. In this example, $(Dp - \sqrt{(Dp)^2 - 2KDh})/h \approx 50$; as Q increases from 50, the error increases to a maximum of 0.1 (10%), then quickly decreases virtually to 0. The approximation error is 1% for $Q = 575$ and decreases thereafter. By the time $Q = Q^* = 1793$, the approximation error is 4.0×10^{-6}. At

Figure 2: g_0 (solid curve) and g (dashed curve)vs. Q.



$Q = (Dp + \sqrt{(Dp)^2 - 2KDh})/h \approx 39950$, $g(Q) - g_0(Q)$ equals 0 and then becomes very slightly negative as Q continues to increase.

5.2 Accuracy of Optimal Solution

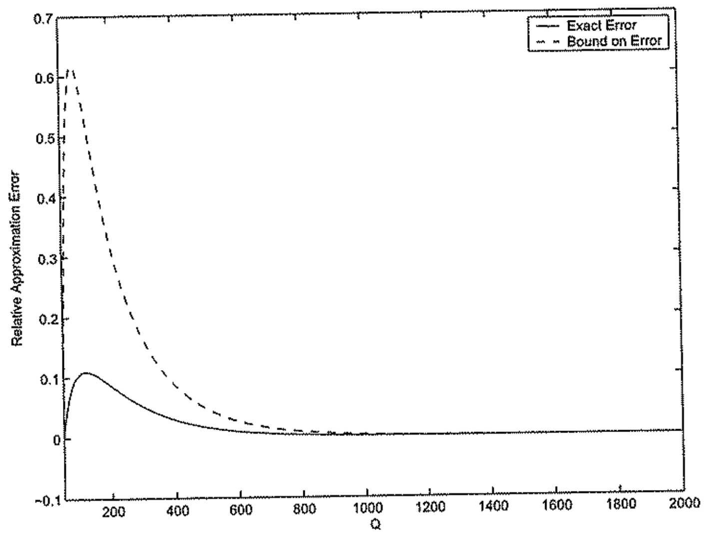
In this section we examine the gap between Q^* and the quantity Q_0 that minimizes g_0. The next proposition demonstrates that $Q^* \geq Q_0$; Theorem 6 then establishes a bound on the gap between Q^* and Q_0 under a certain condition regarding g_0.

Proposition 5 *$Q^* > Q_0$, where Q_0 is the value of Q that minimizes $g_0(Q)$.*

Theorem 6 provides an upper bound on $(Q^* - Q_0)/Q^*$, but it relies on the second derivative of g_0 being positive at Q^* and the third derivative of g_0 being negative on the range $[Q_0, Q^*]$. The sign of the second derivative is not known (since g_0 is known to be unimodal but not necessarily convex), nor is that of the third derivative. If the derivatives happen to have the correct signs, then the bound holds; otherwise the bound is likely to hold approximately, since g approximates g_0 closely in this range and the derivatives of g *do* have the correct signs: $d^2g/dQ^2 > 0$ everywhere by Theorem 2(a), and

$$\frac{d^3g}{dQ^3} = -\frac{3D\mu^2(h\beta^2 D + 2\mu^2 K + 2\mu Dp\beta)}{(Q\mu + \beta D)^4} < 0$$

Figure 3: Actual (solid curve) and bound (dashed curve) on approximation error vs. Q.



so $d^3g/dQ^3 < 0$ everywhere.

Theorem 6 *If* $\frac{d^2 g_0}{dQ^2} > 0$ *at* $Q = Q^*$ *and* $\frac{d^3 g_0}{dQ^3} < 0$ *everywhere on the range* $[Q_0, Q^*]$*, then*

(a)

$$0 \le \frac{Q^* - Q_0}{Q^*} \le \frac{g_0'(Q^*)}{Q^* g_0''(Q^*)}$$

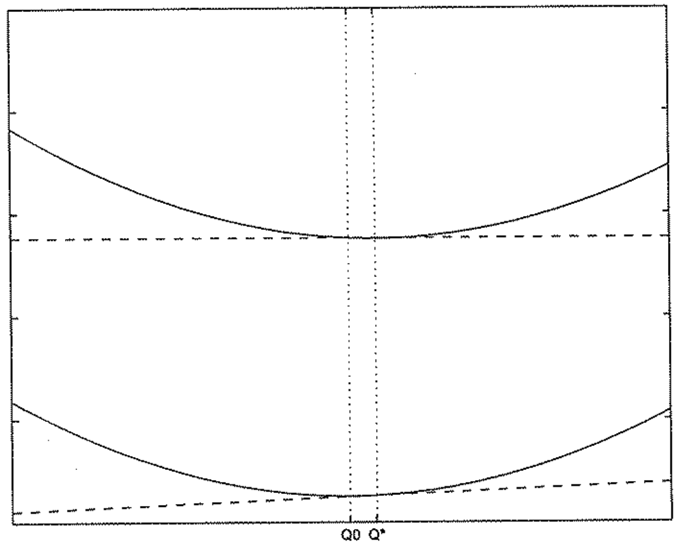
(b)

$$0 \le \frac{Q^* - Q_0}{Q_0} \le \frac{g_0'(Q^*)}{Q^* g_0''(Q^*) - g_0'(Q^*)},$$

where $g_0'(Q^*) = \left.\frac{dg_0}{dQ}\right|_{Q=Q^*}$ *and* $g_0''(Q^*) = \left.\frac{d^2 g_0}{dQ^2}\right|_{Q=Q^*}$.

$g_0'(Q^*)$ and $g_0''(Q^*)$ are too cumbersome to write out explicitly here, but they can be computed simply by differentiating g_0 and plugging (6) in for Q. In general, we can expect the bounds provided by Theorem 6 to be small since $g'(Q^*) = 0$ and $g_0(Q) \approx g(Q)$ in the neighborhood near Q^*. Figure 4 depicts g (upper curve) and g_0 (lower curve) near their minima, along with tangent lines for both curves at $Q = Q^*$. Note that the tangent line to g_0 is nearly horizontal.

Figure 4: g and g_0 near their minima, with tangents at $Q = Q^*$. (Upper curve = g, lower curve = g_0.)



5.3 Use as Heuristic

It is natural to think of Q^* as a heuristic solution for the EOQD in cases for which the lack of closed-form solution for Q_0 makes it impractical to compute it exactly. Theorem 7 presents a bound on the error that results from using Q^* instead of Q_0 when the exact cost function g_0 prevails. The bound is subject to the assumption made in Theorem 6.

Theorem 7 *Let* $\theta \equiv g_0'(Q^*)/g_0''(Q^*)$. *If the assumptions of Theorem 6 hold, then*

$$\frac{g_0(Q^*) - g_0(Q_0)}{g_0(Q_0)} \leq \frac{h\mu\theta(2Q^* - \theta)/2 - D^2 p\beta_0(-\theta)\left[1 - \frac{\beta_0(Q^*)}{\beta}\right]}{h\mu(Q^* - \theta)^2/2 + KD\mu + D^2 p\beta_0(Q^* - \theta)}.$$

We argued in Section 5.2 that $\theta \approx 0$, so the numerator of the bound in Theorem 7 is small while the denominator is typically several orders of magnitude larger. Therefore, the error resulting from using Q^* as a heuristic solution is quite small.

6 Comparison to EOQ

Having established the validity of g as an approximation for g_0, we now set g_0 aside and examine properties of g itself. We first demonstrate that the optimal order quantity and cost for the (approximate) EOQD model are always greater than the classical EOQ quantity and cost, and that the differences may be arbitrarily large. Then we show that in addition to being more analytically tractable than g_0, g exhibits several properties that mirror the behavior of the classical EOQ model. In Section 7, we will show that the approximate EOQD lends itself nicely to sensitivity analysis and the analysis of power-of-two policies.

6.1 Comparison of Order Quantities

The next proposition demonstrates that Q^* $[g(Q^*)]$ is larger than the optimal EOQ solution [cost], and that the difference between them may be arbitrarily large.

Proposition 8 *Let $Q_E = \sqrt{2KD/h}$ be the optimal EOQ solution and $z_E = \sqrt{2KDh}$ its cost. Then*

(a) $Q^* > Q_E$

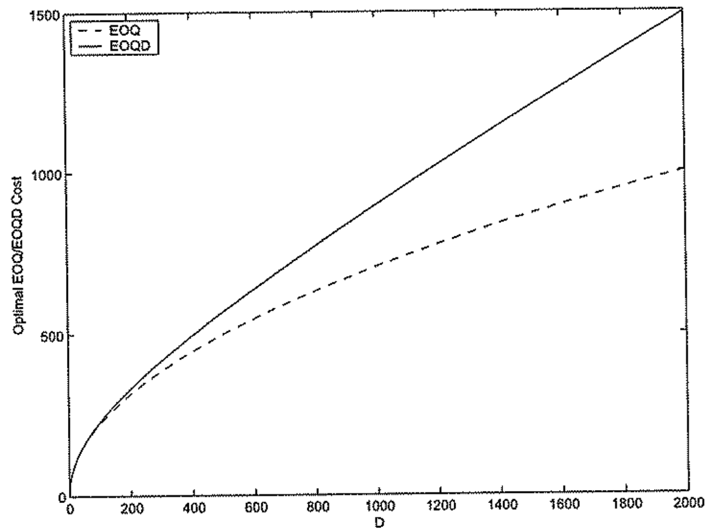
(b) For any $M \in \mathbb{R}$, there exist values of the problem parameters such that $(Q^* - Q_E)/Q_E > M$.

(c) $g(Q^*) > z_E$

(d) For any $M \in \mathbb{R}$, there exist values of the problem parameters such that $(g(Q^*) - z_E)/z_E > M$.

The implication of Proposition 8 is that ignoring disruptions in the EOQ can lead to serious errors, and the EOQ solution may perform poorly when supply is uncertain; we demonstrate this empirically in Section 8.3.

Figure 5: Optimal EOQD and EOQ costs as functions of D.



6.2 EOQ-like Properties

Recall that the optimal Q in the classical EOQ model is $\sqrt{2KD/h}$ and the corresponding cost is $\sqrt{2KDh}$; that is, the optimal cost equals h times the optimal order quantity. In addition, the optimal cost is a concave function of the demand rate. We next demonstrate that these results also hold for $g(Q)$.

Theorem 9 $g(Q^*) = hQ^*$

Proposition 10 *Let $g^*(D) \equiv g(Q^*)$ be the cost of the optimal solution under g, treated as a function of D. Then $g^*(D)$ is concave in D.*

On the other hand, the EOQD cost function is "less concave" (more linear) than that of the EOQ (see Figure 5) since we can re-write $g^*(D)$ using suitable constants as

$$g^*(D) = \sqrt{aD^2 + 2KDh} - cD \approx (\sqrt{a} - c)D.$$

The implication of this is that economies of scale are less strong in the EOQD than in the EOQ.

The concavity property in Proposition 10 is useful in several contexts, including the algorithm of Shen, Coullard and Daskin (2003) for a joint location–inventory model, which requires the inventory cost to be a concave function of the demand served. Snyder (2005) considers a location–inventory model with supply disruptions by replacing the EOQ cost function with the EOQD in the model of Shen et al. (2003). The dampening of economies of scale in the EOQD makes consolidation a less attractive strategy as supply uncertainty increases, since the benefits of consolidation are partially offset by the increased uncertainty due to the dependence on a single supplier.

7 Sensitivity Analysis and Power-of-Two Policies

In this section, we derive an expression to compare the cost of an arbitrarily chosen Q to that of the optimal Q (paralleling similar results for the EOQ model) as well as bounds on the cost of the optimal power-of-two ordering policy.

7.1 Sensitivity to Q

It is well known that if Q_E is the optimal solution to the classical EOQ model, then the ratio of the cost of an arbitrary Q to that of Q_E is given by

$$\frac{1}{2}\left(\frac{Q_E}{Q}+\frac{Q}{Q_E}\right). \tag{7}$$

We now prove a similar result for g.

Theorem 11 *Let Q^* be the order quantity that minimizes $g(Q)$ and let $Q > 0$ be any order quantity. Then*

$$\frac{g(Q)}{g(Q^*)}=\frac{1}{2}\left(\frac{Q^*}{Q}+\frac{Q}{Q^*}\right)-\frac{(Q-Q^*)^2}{2QQ^*}\cdot\frac{\beta D}{Q\mu+\beta D}. \tag{8}$$

Since the bound given in (8) is smaller than that in (7), the (approximate) EOQD cost function is flatter around its optimum than that of the classical EOQ. However, we can expect the second term in (8) to be small for the following reason. Our approximation relies on $\frac{(\lambda+\mu)Q}{D}$ being large (and hence $e^{-(\lambda+\mu)Q/D}$ being close to 0). But $\frac{(\lambda+\mu)Q}{D}=\frac{Q\lambda}{\beta D}$, and since $\lambda<\mu$, we can expect $\frac{Q\mu}{\beta D}$ to be large and $\frac{\beta D}{Q\mu+\beta D}$ to be small. Moreover, if Q is reasonably close to Q^*, then $(Q-Q^*)^2$ will be small relative to $2QQ^*$. We can therefore expect $\frac{g(Q)}{g(Q^*)}$ to be quite close to the EOQ sensitivity analysis quantity given in (7).

7.2 Power-of-Two Policies

In our analysis thus far, we have treated the order quantity, Q, as the decision variable. But we could have formulated an equivalent model in which the order interval (call it T) is the decision variable. As in the classical EOQ model, placing orders of size Q means placing orders every Q/D years (during wet periods), so $T = Q/D$. Then the expected annual cost can be expressed as a function of T as follows:

$$f(T) = g(TD) = \frac{h\mu DT^2/2 + K\mu + Dp\beta}{T\mu + \beta}.$$

It is straightforward to show that $f(T)$ is strictly convex and that the optimal value of T is given by

$$T^* = \frac{Q^*}{D} = \frac{\sqrt{(\beta h)^2 + 2h\mu\left(\frac{K\mu}{D} + p\beta\right)} - \beta h}{h\mu}. \tag{9}$$

which has cost $f(T^*) = g(Q^*) = hQ^*$.

Following Muckstadt and Roundy (1993), we define a *power-of-two* policy to be one in which the order interval is restricted to be a power-of-two multiple of some base time period T_B; that is, $T = 2^k T_B$ for some $k \in \{0, 1, 2, \ldots\}$. T_B is fixed; we assume $T_B \le T^*$.

Our analysis parallels the classical analysis by first deriving lower and upper bounds on the optimal $2^k T_B$ and then proving that the cost of each endpoint is less than or equal to $1.06 f(T^*)$. Since f is convex, the optimal power-of-two cost is guaranteed to be less than or equal to this value.

By the convexity of f, the optimal k is the smallest k that satisfies

$$
\begin{aligned}
& f\left(2^k T_B\right) \leq f\left(2^{k+1} T_B\right) \\
\Longleftrightarrow\ & \frac{\frac{h\mu D}{2}\left(2^k T_B\right)^2 + K\mu + Dp\beta}{2^k T_B \mu + \beta} \leq \frac{\frac{h\mu D}{2}\left(2^{k+1} T_B\right)^2 + K\mu + Dp\beta}{2^{k+1} T_B \mu + \beta} \\
\Longleftrightarrow\ & \frac{h\mu D}{2}\left(2^k T_B\right)^2 \left(\frac{1}{2^k T_B \mu + \beta} - \frac{4}{2^{k+1} T_B \mu + \beta}\right) \leq \\
& \qquad (K\mu + Dp\beta)\left(\frac{1}{2^{k+1} T_B \mu + \beta} - \frac{1}{2^k T_B \mu + \beta}\right) \\
\Longleftrightarrow\ & \frac{h\mu D}{2}\left(2^k T_B\right)^2 \left(2^{k+1} T_B \mu + 3\beta\right) \geq \mu (K\mu + Dp\beta)\left(2^k T_B\right) \\
\Longleftrightarrow\ & h\mu D\left(2^k T_B\right)^2 + \frac{3}{2}\beta h D\left(2^k T_B\right) - (K\mu + Dp\beta) \geq 0 \qquad (10)
\end{aligned}
$$

Viewed as a function of $2^k T_B$, the expression on the left-hand side of (10) has two real roots, one positive and one negative. Since $2^k T_B \geq 0$, inequality (10) holds if and only if $2^k T_B$ is greater than or equal to the positive root; that is,

$$
\begin{aligned}
\Longrightarrow\ 2^k T_B &\geq \frac{-\frac{3}{2}\beta h D + \sqrt{\left(\frac{3}{2}\beta h D\right)^2 + 4(h\mu D)(K\mu + Dp\beta)}}{2(h\mu D)} \\
&= \frac{3}{4} \cdot \frac{-\beta h + \sqrt{(\beta h)^2 + \frac{16}{9} h\mu \left(\frac{K\mu}{D} + p\beta\right)}}{h\mu}
\end{aligned}
$$

We also know that the optimal k satisfies

$$
f\left(2^{k-1} T_B\right) \geq f\left(2^k T_B\right).
$$

Using similar reasoning as above, this implies that

$$
2^k T_B \leq \frac{3}{2} \cdot \frac{-\beta h + \sqrt{(\beta h)^2 + \frac{16}{9} h\mu \left(\frac{K\mu}{D} + p\beta\right)}}{h\mu}.
$$

We have now proved the following result:

Lemma 12 *Let*

$$
\hat{T} = \frac{\sqrt{(\beta h)^2 + \frac{16}{9} h\mu \left(\frac{K\mu}{D} + p\beta\right)} - \beta h}{h\mu}. \qquad (11)
$$

The k yielding the optimal power-of-two policy satisfies

$$
\frac{3}{4}\hat{T} \leq 2^k T_B \leq \frac{3}{2}\hat{T}.
$$

By the convexity of f, the cost of the optimal power-of-two policy is no more than the maximum of the costs of the two endpoints specified in Lemma 12. In fact, the two endpoints have the same cost, and that cost is no more than 1.06 times the cost of the optimal (general) policy, as stated in the next lemma.

Lemma 13 *Let $\hat{T}$ be defined as in Lemma 12. Then*

$$\frac{f\left(\frac{3}{4}\hat{T}\right)}{f(T^*)} = \frac{f\left(\frac{3}{2}\hat{T}\right)}{f(T^*)} \leq \frac{3\sqrt{2}}{4} \approx 1.06.$$

Therefore, we have now proved:

Theorem 14 *If $2^k T_B$ is the optimal power-of-two order interval, then*

$$\frac{f\left(2^k T_B\right)}{f(T^*)} \leq \frac{3\sqrt{2}}{4} \approx 1.06.$$

It is not known whether the bound in Theorem 14 is tight. We show empirically in Section 8.4 that the optimal power-of-two policy is generally no more than a few percent more expensive than the optimal general policy.

8 Computational Results

8.1 Experimental Design

We tested our model using 10 sets of parameters h, K, p, and D, shown in Table 1. These problem instances were adapted from sample problems for the (Q, R) model found in production and inventory textbooks. For each problem, we considered 4 values for λ (0.5, 1, 2, and 5) and 4 values for μ (2λ, 4λ, 10λ, and 20λ), resulting in 160 total instances. For each instance, we computed Q^* using equation (6) and found Q_0 using MATLAB's `fminsearch` function.

Table 1: Problem parameters.

Instance	h	K	p	λ
1	0.8	30	12.96	540
2	15.0	10	40.00	14
3	6.5	175	12.50	2000
4	2.0	50	25.00	200
5	45.0	4500	440.49	2319
6	5.0	300	50.00	3000
7	0.0132	20	0.34	1000
8	5.0	28	80.00	520
9	0.005	12	0.12	3120
10	3.6	12000	65.73	8000

Table 2: Accuracy of approximation: $(\beta - \beta_0(Q^*))/\beta_0(Q^*)$.

λ	μ	Average	Max
0.5	1	0.075	0.381
0.5	2	0.068	0.339
0.5	5	0.046	0.248
0.5	10	0.021	0.127
1	2	0.026	0.156
1	4	0.023	0.139
1	10	0.011	0.077
1	20	0.003	0.020
2	4	0.007	0.049
2	8	0.005	0.040
2	20	0.001	0.011
2	40	<0.001	<0.001
5	10	<0.001	0.004
5	20	<0.001	0.002
5	50	<0.001	<0.001
5	100	<0.001	<0.001
Average		0.018	0.100

8.2 Approximation Error

We begin our analysis of the approximation error by examining $(\beta - \beta_0(Q^*))/\beta_0(Q^*)$, since our results rely on β being a good approximation for $\beta_0(Q)$, particularly at $Q = Q^*$. Table 2 provides the average and maximum values of $(\beta - \beta_0(Q^*))/\beta_0(Q^*)$, taken over the 10 problem instances in Table 1, for each value of λ and μ. These results validate our assertion in Section 3.3 that β is a good approximation for β_0, since the average error across all instances is 1.8%. As expected, the approximation is worse for smaller values of λ and μ and improves substantially as λ and μ increase. This trend persists throughout our computational study.

Table 3: Accuracy of cost function: bounds and actual.

λ	μ	Bound 1		Bound 2		Bound 3		Actual	
		Average	Max	Average	Max	Average	Max	Average	Max
0.5	1	0.048	0.212	0.060	0.276	0.048	0.212	0.025	0.116
0.5	2	0.048	0.214	0.056	0.253	0.048	0.214	0.024	0.113
0.5	5	0.036	0.179	0.040	0.199	0.036	0.179	0.013	0.070
0.5	10	0.018	0.103	0.019	0.113	0.018	0.103	0.004	0.021
1	2	0.019	0.110	0.024	0.135	0.019	0.110	0.009	0.055
1	4	0.018	0.107	0.021	0.122	0.018	0.107	0.008	0.049
1	10	0.009	0.065	0.010	0.071	0.009	0.065	0.003	0.019
1	20	0.002	0.018	0.002	0.019	0.002	0.018	<0.001	0.002
2	4	0.006	0.040	0.007	0.046	0.006	0.040	0.003	0.018
2	8	0.005	0.034	0.005	0.038	0.005	0.034	0.002	0.013
2	20	0.001	0.010	0.001	0.011	0.001	0.010	<0.001	0.002
2	40	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001
5	10	<0.001	0.004	<0.001	0.004	<0.001	0.004	<0.001	0.002
5	20	<0.001	0.002	<0.001	0.002	<0.001	0.002	<0.001	<0.001
5	50	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001
5	100	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001
Average		0.013	0.069	0.015	0.081	0.013	0.069	0.006	0.030

Table 3 provides the average and maximum approximation error in the cost function at Q^*. It lists the actual approximation error and the three theoretical bounds given in Theorem 4(b), as follows:

- Bound 1 $= \frac{\beta-\beta_0(Q^*)}{\beta_0(Q^*)}\left[1-\frac{g_E(Q^*)}{Dp}\right]$
- Bound 2 $= \frac{\beta-\beta_0(Q^*)}{\beta}$
- Bound 3 $= \min\{\text{Bound 1, Bound 2}\}$
- Actual $= (g(Q^*)-g_0(Q^*))/g_0(Q^*)$

Table 3 demonstrates that the approximation provided by g is quite tight at $Q = Q^*$. The approximate cost function differs from the exact function at Q^* by an average of 0.6%, with a theoretical bound (Bound 3) of 1.4% on average. From Table 3 it appears that Bound 1 always provides the minimum of the two bounds. Indeed, this holds for every instance in our computational set (not just in the aggregate), though we have been unable to prove that this holds in general.

Table 4 lists the actual approximation error and the theoretical bounds (from Theorem 6) for $(Q^*-Q_0)/Q^*$ and $(Q^*-Q_0)/Q_0$. The table lists the average and maximum, taken over the 10 instances, for each value of λ and μ.

Table 4: Accuracy of optimal solution: bounds and actual.

		$(Q^* - Q_0)/Q^*$				$(Q^* - Q_0)/Q_0$			
		Bound		Actual		Bound		Actual	
λ	μ	Average	Max	Average	Max	Average	Max	Average	Max
0.5	1	0.220	1.169	0.146	0.656	-0.461	1.893	0.310	1.905
0.5	2	0.179	0.969	0.120	0.526	3.212	30.924	0.195	1.109
0.5	5	0.073	0.372	0.059	0.261	0.098	0.592	0.071	0.353
0.5	10	0.021	0.088	0.020	0.079	0.022	0.097	0.021	0.086
1	2	0.071	0.437	0.059	0.343	0.111	0.777	0.081	0.523
1	4	0.052	0.331	0.043	0.257	0.070	0.494	0.053	0.346
1	10	0.014	0.094	0.013	0.084	0.016	0.103	0.014	0.092
1	20	0.002	0.012	0.002	0.012	0.002	0.012	0.002	0.012
2	4	0.018	0.125	0.016	0.113	0.020	0.143	0.018	0.128
2	8	0.011	0.082	0.010	0.075	0.011	0.089	0.011	0.081
2	20	0.001	0.012	0.001	0.011	0.001	0.012	0.001	0.011
2	40	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001
5	10	0.001	0.012	0.001	0.012	0.001	0.012	0.001	0.012
5	20	<0.001	0.004	<0.001	0.004	<0.001	0.004	<0.001	0.004
5	50	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001
5	100	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001	<0.001
Average		0.042	0.232	0.031	0.152	0.194	2.197	0.049	0.291

The average error in Q^* is 3.1% when using a denominator of Q^* and 4.9% when using a denominator of Q_0, with average theoretical bounds of 4.2% and 19.4%, respectively. The error decreases substantially as λ and μ increase. There is a single instance for which the assumptions stipulated in Theorem 6 concerning the derivatives of g_0 do not hold, resulting in a negative bound. This bound is reflected in the negative entry for the bound on $(Q^* - Q_0)/Q_0$ in the first row of Table 4.

Table 5 lists the average and maximum error (actual and bound) that results from using Q^* as a heuristic solution in place of Q_0, as discussed in Section 5.3. Clearly, Q^* is an extremely effective solution for the exact cost function, with an actual error of only 0.3% on average.

8.3 Comparison to EOQ

We proved in Proposition 8 that the optimal solution to the (approximate) EOQD, Q^*, is greater than or equal to the optimal EOQ solution, Q_E. Table 6 provides empirical evidence demonstrating the magnitude of the difference. For each value of λ and μ, the table lists the average and maximum (over the 10 problems) percentage difference

Table 5: Accuracy of Q^* as heuristic solution for g_0.

		Bound		Actual	
λ	μ	Average	Max	Average	Max
0.5	1	<0.001	1.240	0.017	0.113
0.5	2	1.561	15.012	0.013	0.091
0.5	5	0.043	0.308	0.004	0.027
0.5	10	0.006	0.039	<0.001	0.003
1	2	0.064	0.465	0.004	0.030
1	4	0.034	0.250	0.003	0.021
1	10	0.005	0.039	<0.001	0.003
1	20	<0.001	0.004	<0.001	<0.001
2	4	0.010	0.074	<0.001	0.004
2	8	0.005	0.037	<0.001	0.002
2	20	<0.001	0.004	<0.001	<0.001
2	40	<0.001	<0.001	<0.001	<0.001
5	10	<0.001	0.006	<0.001	<0.001
5	20	<0.001	0.002	<0.001	<0.001
5	50	<0.001	<0.001	<0.001	<0.001
5	100	<0.001	<0.001	<0.001	<0.001
Average		0.081	1.092	0.003	0.018

between Q^* and Q_E. By Theorem 9, this is also equal to the difference between $g(Q^*)$ and the optimal EOQ cost. The table also lists the "ignorance cost" of applying the EOQ model instead of the EOQD: the percentage increase in cost if the classical EOQ model is applied when supply uncertainty exists, computed as $(g(Q_E) - g(Q^*))/g(Q^*)$.

The EOQ and EOQD solutions can differ radically, and the cost of using the EOQ model instead of the EOQD can be quite large. On average, the EOQD order quantity is 160% larger than the EOQ order quantity, and the difference reaches 1912% for one instance. In addition, using the EOQ solution can be quite costly if supply uncertainty exists: the EOQ quantity yields a cost 39% larger than the optimal EOQD cost, on average, and reaches nearly 300% for some instances. Since the EOQD approaches the EOQ as λ decreases or μ increases (Proposition 1), the difference between the EOQ and EOQD solutions decreases as λ decreases or μ increases, as does the "ignorance cost." (In the table, to see the decrease as λ decreases, one must examine different values of λ while keeping μ fixed. For example, with $\mu = 10$, the difference in order quantities is 166% for $\lambda = 5$, 69% for $\lambda = 1$, and 43% for $\lambda = 0.5$. The corresponding ignorance costs are 49%, 18%, and 9%.)

Table 6: Comparison to EOQ solution.

λ	μ	$(Q^* - Q_E)/Q_E$ Average	Max	$(g(Q_E) - g(Q^*))/g(Q^*)$ Average	Max
0.5	1	6.031	19.121	1.116	2.777
0.5	2	3.193	10.568	0.864	2.871
0.5	5	1.119	4.170	0.317	1.496
0.5	10	0.427	1.809	0.093	0.564
1	2	4.224	13.673	1.011	2.983
1	4	2.131	7.343	0.625	2.410
1	10	0.691	2.747	0.176	0.948
1	20	0.247	1.114	0.043	0.289
2	4	2.879	9.618	0.803	2.779
2	8	1.381	5.008	0.403	1.783
2	20	0.412	1.754	0.088	0.542
2	40	0.137	0.656	0.017	0.129
5	10	1.660	5.885	0.490	2.049
5	20	0.740	2.915	0.192	1.016
5	50	0.197	0.912	0.030	0.215
5	100	0.060	0.303	0.004	0.035
Average		1.596	5.475	0.392	1.430

8.4 Power-of-Two Policies

For each instance, we computed the bound on the cost of the optimal power-of-two policy: $f\left(\frac{3}{2}\hat{T}\right)/f(T^*)$, which, by Lemma 13, equals $f\left(\frac{3}{4}\hat{T}\right)/f(T^*)$, where $\hat{T}$ is as defined in (11). We also computed the optimal power-of-two policy using $T_B = 1/52$ (1 week) by enumerating $k = 0, 1, 2, \ldots$. Table 7 lists, for each λ and μ, the average (over all 10 problems) of both theoretical bounds, as well as the value of $f(2^{k^*}T_B)/f(T^*)$, where k^* is the optimal value of k.

The optimal power-of-two policy is, on average, 2% more expensive than the optimal policy in our tests. Although $3\sqrt{2}/4$ is given as an upper bound on the *endpoint* cost in Lemma 13, this bound appears to be tight, as several of the instances produce values that are quite close to this bound. However, it is not known whether the bound is tight on the cost of the optimal power-of-two policy itself, since the maximum error found in any instance in our testing is 1.059.

Table 7: Power-of-two policies.

		$f\left(\frac{3}{2}\hat{T}\right)/f(T^*)$		$f(2^{k^*}T_B)/f(T^*)$	
λ	μ	Average	Max	Average	Max
0.5	1	1.0521	1.0559	1.0161	1.0377
0.5	2	1.0561	1.0584	1.0175	1.0567
0.5	5	1.0590	1.0601	1.0206	1.0465
0.5	10	1.0600	1.0605	1.0235	1.0484
1	2	1.0547	1.0575	1.0216	1.0426
1	4	1.0576	1.0593	1.0162	1.0426
1	10	1.0596	1.0603	1.0177	1.0408
1	20	1.0603	1.0606	1.0168	1.0542
2	4	1.0565	1.0587	1.0128	1.0339
2	8	1.0586	1.0599	1.0261	1.0550
2	20	1.0600	1.0605	1.0224	1.0489
2	40	1.0605	1.0606	1.0216	1.0573
5	10	1.0582	1.0597	1.0237	1.0587
5	20	1.0596	1.0603	1.0156	1.0374
5	50	1.0604	1.0606	1.0186	1.0557
5	100	1.0606	1.0606	1.0230	1.0593
Average		1.0584	1.0596	1.0196	1.0485

9 Conclusions

In this paper, we presented a simple approximation for an EOQ model with disruptions (EOQD). Our approximation is quite tight, especially when the order cycle time is long relative to the duration of wet and/or dry periods. We presented a closed-form solution to our model and provided theoretical and empirical bounds on the error in the cost, the optimal solution, and the optimality error resulting from using the approximate solution as a heuristic for the exact one. We then introduced a number of analytical properties of our exact model, showing that it behaves like the EOQ in several important ways and deriving sensitivity analysis and power-of-two results that mirror those for the EOQ. On the other hand, we proved that although the cost functions are similar, the EOQ solution may be a very poor substitute for the EOQD solution; thus, ignoring supply uncertainty when it exists can be very costly.

Interest in supply chain models with supply disruptions has been growing steadily in recent years. A number of papers have appeared in the literature that incorporate supply disruptions into classical inventory models. Unfortunately, the introduction of supply uncertainty often destroys the tractability of otherwise simple models, forcing a numerical solution to the disruption models. Although these models are interesting in their own right, their impact is amplified when researchers can obtain analytical results and insights from them or embed them into more complex models (e.g., the multi-echelon supply chain design models of Qi and Shen (2005) and Snyder (2005)). The lack of closed-form solutions often makes both goals difficult to attain. We expect the formulation of approximations to other inventory and supply chain models with disruptions to be an active area of future research.

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11 Appendix: Proofs

Proof of Theorem 2. The reader can verify that

$$\frac{dg}{dQ} = \frac{\frac{h\mu^2}{2}Q^2 + \beta Dh\mu Q - (KD\mu + D^2p\beta)\mu}{(Q\mu + \beta D)^2} \tag{12}$$

$$\frac{d^2g}{dQ^2} = \frac{D\mu(h\beta^2 D + 2\mu^2 K + 2\mu Dp\beta)}{(Q\mu + \beta D)^3} \tag{13}$$

Since all terms in d^2g/dQ^2 are positive, g is convex, proving part (a). To prove part (b), note that

$$\frac{dg}{dQ} = 0 \iff \frac{h\mu^2}{2}Q^2 + \beta Dh\mu Q - (KD\mu + D^2p\beta)\mu = 0$$

$$\iff Q = \frac{-\beta Dh \pm \sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)}}{h\mu}$$

using the quadratic formula. Clearly, using the + sign in the $\pm$ yields a positive value of Q while using the $-$ sign yields a negative value. □

Before proving the remaining results, we introduce two lemmas and the proof of Theorem 9 (out of order), all of which are used in subsequent proofs.

Lemma 15 $(Q^*)^2 = \frac{2D}{h\mu}(K\mu + Dp\beta - \beta hQ^*)$

Proof.

$$(Q^*)^2 = \frac{2h\mu(KD\mu + D^2p\beta) + 2(\beta Dh)^2 - 2\beta Dh\sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)}}{(h\mu)^2}$$
$$= \frac{2D}{h\mu}(K\mu + Dp\beta - \beta hQ^*)$$

□

Proof of Theorem 9.

$$\frac{g(Q^*)}{Q^*} = \frac{KD\mu + \frac{h\mu}{2}\frac{2D}{h\mu}(K\mu + Dp\beta - \beta hQ^*) + D^2p\beta}{Q^*(Q^*\mu + \beta D)} \text{ (using Lemma 15)}$$
$$= \frac{2(KD\mu + D^2p\beta) - \beta DhQ^*}{Q^*(Q^*\mu + \beta D)}$$
$$= \frac{(Q^*)^2h\mu + \beta DhQ^*}{Q^*(Q^*\mu + \beta D)} \text{ (using Lemma 15 again)}$$
$$= \frac{Q^*h\mu + \beta Dh}{Q^*\mu + \beta D}$$
$$= h$$

□

Lemma 16 $\sqrt{2KDh} < g(Q^*) < Dp$

Proof. By assumption,

$$\sqrt{2KDh} < Dp$$
$$\implies 2\beta h\mu D\sqrt{2KDh} + 2KDh\mu^2 + (\beta Dh)^2 < 2\beta h\mu D^2p + 2KDh\mu^2 + (\beta Dh)^2$$
$$\implies \left(\mu\sqrt{2KDh} + \beta Dh\right)^2 < (\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)$$
$$\implies \sqrt{2KDh} < \frac{\sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)} - \beta Dh}{\mu}$$
$$= hQ^* = g(Q^*)$$

by Theorem 9. Similarly,

$$
\begin{aligned}
& Dp > \sqrt{2KDh} \\
\implies & (Dp\mu)^2 + 2D^2p\mu\beta h + (\beta Dh)^2 > 2KDh\mu^2 + 2D^2p\mu\beta h + (\beta Dh)^2 \\
\implies & (Dp\mu + \beta Dh)^2 > (\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta) \\
\implies & Dp > \frac{\sqrt{(\beta Dh)^2 + 2h\mu(KD\mu + D^2p\beta)} - \beta Dh}{\mu} \\
& = hQ^* = g(Q^*).
\end{aligned}
$$

□

Proof of Proposition 3.

(a)

$$
\begin{aligned}
& \frac{KD}{Q} + \frac{hQ}{2} < Dp \\
\iff & \frac{h}{2}Q^2 - DpQ + KD < 0 \\
\iff & \frac{Dp - \sqrt{(Dp)^2 - 2KDh}}{h} < Q < \frac{Dp + \sqrt{(Dp)^2 - 2KDh}}{h}
\end{aligned}
$$

The result now follows from the proof of part (b).

(b) The reader can verify that

$$
\begin{aligned}
g(Q) - g_0(Q) &= \frac{(h\mu Q^2/2 + KD\mu + D^2p\beta)(Q\mu + \beta_0 D) - (h\mu Q^2/2 + KD\mu + D^2p\beta_0)(Q\mu + \beta D)}{(Q\mu + \beta D)(Q\mu + \beta_0 D)} \\
&= \frac{(\beta - \beta_0)D\mu(DpQ - KD - hQ^2/2)}{(Q\mu + \beta D)(Q\mu + \beta_0 D)}. \qquad (14)
\end{aligned}
$$

Since $\beta - \beta_0 > 0$,

$$
\begin{aligned}
g(Q) - g_0(Q) > 0 \iff & 2DpQ - 2KD - hQ^2 > 0 \\
\iff & -\frac{1}{h}\left(hQ - Dp - \sqrt{(Dp)^2 - 2KDh}\right) \times \\
& \left(hQ - Dp + \sqrt{(Dp)^2 - 2KDh}\right) > 0 \\
\iff & hQ - Dp - \sqrt{(Dp)^2 - 2KDh} < 0 \text{ and} \\
& hQ - Dp + \sqrt{(Dp)^2 - 2KDh} > 0 \\
\iff & \frac{Dp - \sqrt{(Dp)^2 - 2KDh}}{h} < Q < \frac{Dp + \sqrt{(Dp)^2 - 2KDh}}{h}
\end{aligned}
$$

(c) It suffices to show that Q^* satisfies the condition in part (b). This condition is satisfied if and only if

$$Dp - \sqrt{(Dp)^2 - 2KDh} < Q^*h = g(Q^*) < Dp + \sqrt{(Dp)^2 - 2KDh}.$$

The equality follows from Theorem 9. The second inequality follows from Lemma 16. To prove the first inequality, we note that for any a, b, c such that $a < b$ and $c \leq a^2$, $b - \sqrt{b^2 - c} < a - \sqrt{a^2 - c}$ by the concavity of the square-root function. Since $\sqrt{2KDh} < Dp$ by assumption, we have

$$Dp - \sqrt{(Dp)^2 - 2KDh} < \sqrt{2KDh} - \sqrt{2KDh - 2KDh} = \sqrt{2KDh} < g(Q^*)$$

by Lemma 16, confirming the first inequality. □

Proof of Theorem 4.

(a)

$$g(Q) - g_0(Q) = \frac{(\beta - \beta_0)D\mu(DpQ - KD - hQ^2/2)}{(Q\mu + \beta D)(Q\mu + \beta_0 D)}$$

(see (14)). Therefore

$$\begin{aligned}
\frac{g(Q) - g_0(Q)}{g_0(Q)} &= \frac{(\beta - \beta_0)D\mu(DpQ - KD - hQ^2/2)}{(Q\mu + \beta D)(h\mu Q^2/2 + KD\mu + D^2p\beta_0)} \\
&= \frac{(\beta - \beta_0)D\mu(DpQ - KD - hQ^2/2)}{(Q\mu + \beta D)(KD\mu + h\mu Q^2/2)} \frac{KD\mu + h\mu Q^2/2}{KD\mu + h\mu Q^2/2 + D^2p\beta_0} \\
&= \frac{(\beta - \beta_0)(Dp - KD/Q - hQ/2)}{(Q\mu/D + \beta)(KD/Q + hQ/2)} \frac{KD/Q + hQ/2}{KD/Q + hQ/2 + D^2p\beta_0/\mu Q} \\
&= \frac{(\beta - \beta_0)(Dp - g_E(Q))}{(Q\mu/D + \beta)g_E(Q)} \frac{g_E(Q)}{g_E(Q) + D^2p\beta_0/\mu Q} \\
&= \frac{\beta - \beta_0}{(Q\mu/D + \beta)(1 + D^2p\beta_0/\mu Q g_E(Q))} \left[\frac{Dp}{g_E(Q)} - 1\right] \\
&< \frac{\beta - \beta_0}{(Q\mu/D)(D^2p\beta_0/\mu Q g_E(Q))} \left[\frac{Dp}{g_E(Q)} - 1\right] \quad \text{(because } Dp/g_E(Q) > 1\text{)} \\
&= \frac{\beta - \beta_0}{\beta_0} \frac{g_E(Q)}{Dp} \left[\frac{Dp}{g_E(Q)} - 1\right] \\
&= \frac{\beta - \beta_0}{\beta_0} \left[1 - \frac{g_E(Q)}{Dp}\right] \\
&< \frac{\beta - \beta_0}{\beta_0} = \frac{e^{-(\lambda+\mu)Q/D}}{1 - e^{-(\lambda+\mu)Q/D}}
\end{aligned}$$

(b) The first term in the minimization follows from (a). To prove the second:

$$
\begin{aligned}
g(Q^*) - g_0(Q^*) &= hQ^* - \frac{h\mu(Q^*)^2/2 + KD\mu + D^2p\beta_0}{Q^*\mu + \beta_0 D} \text{ (by Theorem 9)}\\
&= \frac{\frac{h\mu}{2}(Q^*)^2 + \beta_0 DhQ^* - KD\mu - D^2p\beta_0}{Q^*\mu + \beta_0 D}\\
&= \frac{D(K\mu + Dp\beta - \beta hQ^*) + \beta_0 DhQ^* - KD\mu - D^2p\beta_0}{Q^*\mu + \beta_0 D} \text{ (by Lemma 15)}\\
&= \frac{(\beta - \beta_0)(Dp - hQ^*)D}{Q^*\mu + \beta_0 D}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{g(Q^*) - g_0(Q^*)}{g_0(Q^*)} &= \frac{(\beta - \beta_0)(Dp - hQ^*)D}{Q^*\mu + \beta_0 D} \cdot \frac{Q^*\mu + \beta_0 D}{h\mu(Q^*)^2/2 + KD\mu + D^2p\beta_0}\\
&= \frac{(\beta - \beta_0)(Dp - hQ^*)D}{KD\mu + D(K\mu + Dp\beta - \beta hQ^*) + D^2p\beta_0}\\
&= \frac{(\beta - \beta_0)(Dp - hQ^*)D}{2KD\mu + \beta_0 D^2 p + \beta D(Dp - hQ^*)}\\
&= \frac{\beta - \beta_0}{\frac{2K\mu + \beta_0 Dp}{Dp - hQ^*} + \beta}\\
&\leq \frac{\beta - \beta_0}{\beta} < 1
\end{aligned}
$$

since $Dp - hQ^* > 0$ by Lemma 16. □

Proof of Proposition 5. The first derivative of g_0 is given by[1]

$$
\begin{aligned}
\frac{dg_0}{dQ} =& \frac{1}{\left[Q\mu + \beta D\left(1 - e^{-(\lambda+\mu)Q/D}\right)\right]^2}\left(\frac{h\mu^2}{2}Q^2 + \beta Dh\mu Q - KD\mu^2 - D^2p\beta\mu\right.\\
&\left. + \left[-\frac{h\lambda\mu}{2}Q^2 - \beta Dh\mu Q - KD\lambda\mu + D^2p\beta\mu + pD\lambda\mu Q\right] e^{-(\lambda+\mu)Q/D}\right) \qquad (15)
\end{aligned}
$$

The first-order condition is satisfied if the numerator is 0. The numerator can be rewritten as

$$
\begin{aligned}
&\frac{h\mu^2}{2}Q^2 + \beta Dh\mu Q - KD\mu^2 - D^2p\beta\mu && \text{(16a)}\\
&+\left[-\frac{h\mu^2}{2}Q^2 - \beta Dh\mu Q + KD\mu^2 + D^2p\beta\mu\right. && \text{(16b)}\\
&\left. + \frac{h\mu}{2}(\mu - \lambda)Q^2 - KD\mu(\mu + \lambda) + pD\lambda\mu Q\right] e^{-(\lambda+\mu)Q/D} && \text{(16c)}
\end{aligned}
$$

[1]Note: The first-order condition given in Proposition 2(c) of Berk and Arreola-Risa contains an error: the first term on the second line should read $-\frac{C_h DQ^2}{2D}$ instead of $-\frac{C_h \lambda Q^2}{2D}$. Translated into our notation, the corrected expression is the numerator of (15) above.

Now suppose that $Q = Q^*$; we will show that (16) is positive. The expression in (16a) is the first-order condition for g (see (12)) and that in (16b) is its negative, so when $Q = Q^*$, both (16a) and (16b) equal 0. Using Lemma 15, (16c) can be rewritten as

$$\left[\frac{h\mu}{2}\frac{2D}{h\mu}(K\mu + Dp\beta - \beta hQ^*)(\mu - \lambda) - KD\mu(\mu+\lambda) + pD\lambda\mu Q^*\right] e^{-(\lambda+\mu)Q^*/D}$$
$$= \left[\beta D(\mu - \lambda)(Dp - hQ^*) + \lambda\mu(Q^*Dp - 2KD)\right] e^{-(\lambda+\mu)Q^*/D}$$

The first term inside the brackets is positive since $\mu - \lambda > 0$ by assumption and $Dp - hQ^* > 0$ by Theorem 9 and Lemma 16. The second term is positive since

$$Q^*h > \sqrt{2KDh} \implies (Q^*)^2 h > 2KD$$

and

$$Dp > Q^*h \implies Q^*Dp > (Q^*)^2 h > 2KD.$$

Therefore (16), and hence dg_0/dQ, is positive when $Q = Q^*$. Since g_0 is unimodal (Proposition 2(b) in Berk and Arreola-Risa (1994)), it must attain its minimum to the left of Q^*. Therefore $Q_0 < Q^*$, as desired. □

Proof of Theorem 6.

(a) Since $\frac{d^3 g_0}{dQ^3} < 0$, g_0' is concave on $[Q_0, Q^*]$. Therefore

$$\frac{g_0'(Q^*) - g_0'(Q_0)}{Q^* - Q_0} \geq g_0''(Q^*)$$

by the concavity of g_0' (see, e.g., Bazaraa, Sherali and Shetty 1993). But $g_0'(Q_0) = 0$ since Q_0 minimizes g_0, so we have

$$\frac{g_0'(Q^*)}{Q^* - Q_0} \geq g_0''(Q^*).$$

Since $g_0''(Q^*) > 0$,

$$\frac{Q^* - Q_0}{Q^*} \leq \frac{g_0'(Q^*)}{Q^* g_0''(Q^*)},$$

as desired.

(b) By part (a), $Q_0 \geq Q^* - \frac{g_0'(Q^*)}{g_0''(Q^*)}$. Therefore,

$$\frac{Q^* - Q_0}{Q_0} \leq \frac{g_0'(Q^*)}{\left(Q^* - \frac{g_0'(Q^*)}{g_0''(Q^*)}\right) g_0''(Q^*)} = \frac{g_0'(Q^*)}{Q^* g_0''(Q^*) - g_0'(Q^*)}.$$

□

Proof of Theorem 7.

First note that

$$\frac{\beta-\beta_0(Q^*-\theta)}{\beta}=e^{-(\lambda+\mu)(Q^*-\theta)/D}=\frac{\beta-\beta_0(Q^*)}{\beta}\cdot\frac{\beta-\beta_0(-\theta)}{\beta},$$

so

$$\beta_0(Q^*-\theta)=\beta_0(Q^*)+\beta_0(-\theta)-\frac{\beta_0(Q^*)\beta_0(-\theta)}{\beta}. \tag{17}$$

Now,

$$\begin{aligned} g_0(Q_0) &= \frac{h\mu Q_0^2/2+KD\mu+D^2p\beta_0(Q_0)}{Q_0\mu+\beta_0(Q_0)D} \\ &\geq \frac{h\mu(Q^*-\theta)^2/2+KD\mu+D^2p\beta_0(Q^*-\theta)}{Q^*\mu+\beta_0(Q^*)D} \end{aligned} \tag{18}$$

The inequality follows from Theorem 6(a) and the fact that $\beta_0(Q)$ is increasing in Q. Then

$$\begin{aligned} \frac{g_0(Q^*)-g_0(Q_0)}{g_0(Q_0)} &\leq \frac{h\mu(Q^*)^2/2+KD\mu+D^2p\beta_0(Q^*)}{h\mu(Q^*-\theta)^2/2+KD\mu+D^2p\beta_0(Q^*-\theta)}-1 \\ &= \frac{h\mu\theta(2Q^*-\theta)/2-D^2p\beta_0(-\theta)\left[1-\frac{\beta_0(Q^*)}{\beta}\right]}{h\mu(Q^*-\theta)^2/2+KD\mu+D^2p\beta_0(Q^*-\theta)} \end{aligned}$$

using (17) and (18). □

Proof of Proposition 8.

(a) By Lemma 15,

$$(Q^*)^2=\frac{2KD}{h}+\frac{2D\beta}{h\mu}(Dp-hQ^*).$$

By Lemma 16, $Dp-hQ^*>0$, so $(Q^*)^2>2KD/h$, i.e., $Q^*>\sqrt{2KD/h}$.

(b) As $p\to\infty$, $Q^*\to\infty$ but Q_E stays constant.

(c),(d) Follow from Theorem 9. □

Proof of Proposition 10 The second derivative of $g^*(D)$ with respect to D is negative:

$$\frac{d^2g^*}{dD^2}=\frac{-h^2\mu^3K^2}{[(\beta Dh)^2+2h\mu(KD\mu+D^2p\beta)]^{\frac{3}{2}}}<0.$$

□

Proof of Theorem 11. By Theorem 9,

$$
\begin{aligned}
\frac{g(Q)}{g(Q^*)} &= \frac{h\mu Q^2/2 + KD\mu + D^2p\beta}{Q\mu + \beta D} \cdot \frac{1}{hQ^*} \\
&= \frac{KD\mu + D^2p\beta}{hQ^*(Q\mu + \beta D)} + \frac{\mu Q^2}{2Q^*(Q\mu + \beta D)} \\
&= \left(\frac{KD\mu + D^2p\beta}{hQ^*Q\mu} + \frac{\mu Q^2}{2Q^*Q\mu}\right) \frac{Q\mu}{Q\mu + \beta D} \\
&= \left(\frac{\frac{h\mu}{2}(Q^*)^2 + \beta DhQ^*}{hQQ^*\mu} + \frac{Q}{2Q^*}\right) \frac{Q\mu}{Q\mu + \beta D} \text{ (using Lemma 15)} \\
&= \left(\frac{Q^*}{2Q} + \frac{\beta D}{Q\mu} + \frac{Q}{2Q^*}\right) \frac{Q\mu}{Q\mu + \beta D} \\
&= \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right) \frac{Q\mu}{Q\mu + \beta D} + \frac{\beta D}{Q\mu + \beta D} \\
&= \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right) + \left[1 - \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right)\right] \frac{\beta D}{Q\mu + \beta D} \\
&= \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right) - \frac{(Q - Q^*)^2}{2QQ^*} \cdot \frac{\beta D}{Q\mu + \beta D}
\end{aligned}
$$

as desired. □

Proof of Lemma 13. We first prove the equality, then the inequality.

$$
\begin{aligned}
f\left(\frac{3}{2}\hat{T}\right) - f\left(\frac{3}{4}\hat{T}\right) &= \frac{\frac{h\mu D}{2}\left(\frac{3}{2}\hat{T}\right)^2 + K\mu + Dp\beta}{\frac{3}{2}\hat{T}\mu + \beta} - \frac{\frac{h\mu D}{2}\left(\frac{3}{4}\hat{T}\right)^2 + K\mu + Dp\beta}{\frac{3}{4}\hat{T}\mu + \beta} \\
&= \frac{3\hat{T}\mu[9h\mu D\hat{T}^2 + 18h\beta D\hat{T} - 16(K\mu + Dp\beta)]}{8(3\hat{T}\mu + 2\beta)(3\hat{T}\mu + 4\beta)} \qquad (19)
\end{aligned}
$$

Since

$$
\begin{aligned}
\hat{T}^2 &= \frac{2(\beta h)^2 + \frac{16}{9}h\mu\left(\frac{K\mu}{D} + p\beta\right) - 2\beta h\sqrt{(\beta h)^2 + \frac{16}{9}h\mu\left(\frac{K\mu}{D} + p\beta\right)}}{(h\mu)^2} \\
&= \frac{2}{h\mu}\left[\frac{8}{9}\left(\frac{K\mu}{D} + p\beta\right) - \beta h\hat{T}\right],
\end{aligned}
$$

the numerator of (19) equals

$$
3\hat{T}\mu\left[9h\mu D \cdot \frac{2}{h\mu}\left[\frac{8}{9}\left(\frac{K\mu}{D} + p\beta\right) - \beta h\hat{T}\right] + 18h\beta D\hat{T} - 16(K\mu + Dp\beta)\right] = 0.
$$

This proves that $f\left(\frac{3}{4}\hat{T}\right) = f\left(\frac{3}{2}\hat{T}\right)$. We next prove that $f\left(\frac{3}{2}\hat{T}\right) \leq \frac{3\sqrt{2}}{4} f(T^*)$. By Theorem 11,

$$\begin{aligned}
\frac{f\left(\frac{3}{2}\hat{T}\right)}{f(T^*)} = \frac{g\left(\frac{3}{2}D\hat{T}\right)}{g(Q^*)} &\leq \frac{1}{2}\left(\frac{Q^*}{\frac{3}{2}D\hat{T}} + \frac{\frac{3}{2}D\hat{T}}{Q^*}\right) \\
&= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{\sqrt{1+2\alpha}-1}{\sqrt{1+\frac{16}{9}\alpha}-1} + \frac{3}{2}\cdot\frac{\sqrt{1+\frac{16}{9}\alpha}-1}{\sqrt{1+2\alpha}-1}\right) \\
&= \frac{1}{2}\left(\frac{2\sqrt{1+2\alpha}-2}{\sqrt{9+16\alpha}-3} + \frac{\sqrt{9+16\alpha}-3}{2\sqrt{1+2\alpha}-2}\right) \equiv \psi(\alpha),
\end{aligned}$$

where $\alpha = \frac{h\mu(KD\mu+D^2p\beta)}{(\beta Dh)^2} \geq 0$. We will show that $\psi(\alpha)$ is increasing in α, thus it attains its maximum value in the limit as $\alpha \to \infty$.

$$\frac{d\psi}{d\alpha} = \frac{\left[1-3\sqrt{9+16\alpha}+8\sqrt{1+2\alpha}\right]\left[4\left(\sqrt{1+2\alpha}-1\right)^2 - \left(\sqrt{9+16\alpha}-3\right)^2\right]}{4\sqrt{1+2\alpha}\sqrt{9+16\alpha}\left(\sqrt{1+2\alpha}-1\right)^2\left(\sqrt{9+16\alpha}-3\right)^2}.$$

The denominator is clearly positive, and both terms in the numerator are negative:

$$\begin{aligned}
&1-3\sqrt{9+16\alpha}+8\sqrt{1+2\alpha} < 0 \\
\iff\ &1+16\sqrt{1+2\alpha}+64+128\alpha < 81+144\alpha \\
\iff\ &\sqrt{1+2\alpha} < 1+\alpha,
\end{aligned}$$

which holds since $1+\alpha = \sqrt{1+2\alpha+\alpha^2}$. Similarly,

$$\begin{aligned}
&4\left(\sqrt{1+2\alpha}-1\right)^2 - \left(\sqrt{9+16\alpha}-3\right)^2 < 0 \\
\iff\ &\sqrt{4+8\alpha}+1 < \sqrt{9+16\alpha} \\
\iff\ &4+8\alpha+2\sqrt{4+8\alpha}+1 < 9+16\alpha \\
\iff\ &2\sqrt{4+8\alpha} < 4+8\alpha,
\end{aligned}$$

which holds since $2 < \sqrt{4+8\alpha}$. Thus, $\frac{d\psi}{d\alpha} > 0$, so

$$\begin{aligned}
\max_{\alpha}\psi(\alpha) &= \lim_{\alpha\to\infty}\psi(\alpha) \\
&= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{\sqrt{2}}{\sqrt{\frac{16}{9}}} + \frac{3}{2}\cdot\frac{\sqrt{\frac{16}{9}}}{\sqrt{2}}\right) \\
&= \frac{3\sqrt{2}}{4} \approx 1.06,
\end{aligned}$$

proving the lemma. □