Multiperiod Pricing via Robust Optimization

Aurélie Thiele
Lehigh University

Report No. 05T-006

Multiperiod Pricing via Robust Optimization

Aurélie Thiele*

November 2005

Abstract

We present a robust optimization approach to pricing multiple products over a finite time horizon in the presence of constrained resources and uncertainty on the demand. We consider a broad class of nominal demand functions and the cases of additive as well as multiplicative uncertainty. The robust optimization approach does not require the knowledge of the underlying probability distributions, which are often difficult to obtain in practice, and instead models the random variables as uncertain parameters belonging to a polyhedral uncertainty set. A novelty of the proposed model is that, instead of imposing an upper bound on the number of uncertain parameters that can reach their worst-case value, which has been the polyhedral uncertainty set of choice in the robust optimization literature and is known as budget-of-uncertainty constraints, we are motivated by the specific problem structure at hand to introduce a budget on the amount of the resource that can be used by the uncertain component of the aggregate demand rather than its nominal value. This allows us to derive key insights on the structure of the optimal solution. We establish the existence of a single reference price for each product over the time horizon and show that this new parameter plays a crucial role in understanding the impact of uncertainty on the optimal prices. In particular, it is not always optimal to decrease prices when demand is uncertain. Whether it is optimal or not will instead depend on whether the product price at that time period is above or below the reference price, and whether the maximal amount of uncertainty at that time period exceeds a threshold.

1 Introduction

Uncertain demand in pricing problems has traditionally been addressed by specifying probability distributions and maximizing the resulting expected revenue. Within that framework, researchers in price-based revenue management have investigated a wide array of problems, from the classical single-product problem without replenishment to more complex issues of product markdown and auctions. The reader is referred to Talluri and van Ryzin [12] for a thorough review of these models. However, probabilities are difficult to estimate in practice, which makes the approach vulnerable

*Department of Industrial and Systems Engineering, Lehigh University, Harold S. Mohler Laboratory Room 329, Bethlehem, PA 18015, aurelie.thiele@lehigh.edu.

to estimation errors. Another drawback is the implicit assumption that the decision-maker is risk-neutral. While this second point can be addressed by introducing an expected utility function, such a Talluri and van Ryzin present in [12] a comprehensive treatment of deterministic and stochastic models for dynamic pricing. It has traditionally been modeled as a random variable with a specific distribution; however, distributions are difficult to estimate accurately. Furthermore, the objective of maximizing the expected revenue does not capture the manager's risk preferences, and while this can be addressed by maximizing expected utilities, few managers are able to articulate their own utility function. In contrast, the robust optimization paradigm developed by addresses those issues in a tractable manner by modeling random parameters as uncertain parameters in an uncertainty set.

Specifically, we make the following contributions to the problem of multiperiod pricing under uncertainty:

1. We provide a tractable, convex framework based on robust optimization to incorporate uncertainty without incurring a substantial increase in the size of the problems considered. Specifically, the robust problem can be formulated as a convex problem with no new constraints and $2m$ additional decision variables if m is the number of resources and $m \geq 2$. In the single-resource case, only one new decision variable is required.

2. We establish the existence of a *reference price*, independent of time, for each product considered. The reference price plays a crucial role in understanding the impact of uncertainty on the optimal solution and the marginal value of the resources, as explained below. For each product, the reference price is equal to one of the optimal prices.

3. The optimal price of product j at time t converges further towards the reference price of that product when uncertainty in demand (for j at t) increases. The marginal value of the resources increases with the demand uncertainty for product j at time t if and only if the optimal price of j at t exceeds the reference price of that product.

4. In the single-product case, an increase in demand uncertainty at time t increases the optimal prices at *all* other time periods if and only if the optimal price at t exceeds the reference price. For multiple products, whether an increase in the uncertainty for product j at time t increases the price of other products at other time periods is determined by the position (above or below the reference price) of the optimal price of j at t and by the sign of a coefficient that depends on the products considered but not on the time period.

The structure of the paper is as follows. In Section 2, we develop the robust pricing model for one product in the presence of additive uncertainty. We analyze its counterpart in the case of multiplicative uncertainty in Section 3, and extend our results, both for additive and multiplicative uncertainty, to several products using multiple capacitated resources in Section 4. Section ?? provides computational results. Finally, we conclude the paper in Section 5.

2 Single-Product Pricing With Additive Uncertainty

2.1 The Deterministic Model

We consider here the problem of pricing a single product over a finite horizon of length T, given an initial inventory of C items, without replenishment. Throughout the paper, we assume that the average demand $\overline{d}_t$ at time t, $t = 0, \ldots, T-1$, as a function of prices is fully known and only depends on the prices at that time period. We further assume that the average demand, resp. the revenue, at time t is a convex, resp. concave, function of the prices. This is formulated as:

$$\overline{d}_t(p_t) = f_t(p_t), \; t = 0, \ldots, T-1, \tag{1}$$

where f_t is convex, strictly decreasing, and g_t, defined by $g_t(p_t) = p_t\, f_t(p_t)$, is concave. A widely used choice for f_t in practical implementations is: $f_t(p_t) = a_t - b_t\, p_t$, with a_t, $b_t > 0$, which corresponds to an average demand linear in the price at each time period. This framework also applies to more complex demand functions such as $f_t(p_t) = a_t - b_t\, p_t + c_t/p_t$, with a_t, b_t, $c_t > 0$.

We first review the properties of the nominal problem, which will be useful in the analysis of its robust counterpart. When the demand is deterministic, the problem of finding the optimal prices to maximize the revenue is formulated as a convex problem:

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) \\ \text{s.t.} \quad & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C, \\ & p_t^{\min} \leq p_t \leq p_t^{\max}, \quad \forall t, \end{aligned} \tag{2}$$

where $p_t^{\min}$ and p_t^{max} are lower and upper bounds on the prices. In particular, these bounds enforce that p_t, $\overline{d}_t(p_t) \geq 0$ at each time period. The optimal solution can be characterized as follows.

Theorem 2.1 (Optimal Solution) *Let $\overline{\lambda}^* \geq 0$ be the optimal Lagrange multiplier for the capacity constraint and $\overline{\mu}_t^{*\min}$, $\overline{\mu}_t^{*\max} \geq 0$ the optimal Lagrange multipliers for the bound constraints at time t, $t = 0, \ldots, T-1$. Then the optimal price p_t^* at t satisfies:*

$$\overline{d}_t(p_t^*) + (p_t^* - \overline{\lambda}^*)\,\frac{\partial \overline{d}_t(p_t^*)}{\partial p_t} + \overline{\mu}_t^{*\min} - \overline{\mu}_t^{*\max} = 0. \tag{3}$$

In particular, if $p_t^{\min} < p_t^* < p_t^{\max}$, we have:

$$\overline{d}_t(p_t^*) + (p_t^* - \overline{\lambda}^*)\,\frac{\partial \overline{d}_t(p_t^*)}{\partial p_t} = 0. \tag{4}$$

Proof: Follows by applying the Karush-Kuhn-Tucker conditions to Problem (2) (see Bertsekas [4] for an introduction to nonlinear programming.) By complementarity slackness, $\overline{\mu}_t^{\min} = 0$ if $p_t > p_t^{\min}$ and $\overline{\mu}_t^{\max} = 0$ if $p_t < p_t^{\max}$. □

2.2 Description of the Uncertainty

In practice, the future demand is uncertain, and the manager has only limited knowledge on the structure of the randomness. Following the approach developed by Bertsimas and Sim [5] for uncertain data coefficients and Bertsimas and Thiele [13] for random variables with unknown distributions, we model the random demands d_t as uncertain parameters of known mean and support. In this section, we assume that the uncertainty is *additive*, i.e., the random demand d_t is modeled as:

$$d_t = \overline{d}_t + \delta_t, \tag{5}$$

where $\overline{d}_t$ verifies Equation (1), and δ_t is a zero-mean random variable independent of the prices. For notational convenience, we present the approach when the support of δ_t is symmetric, i.e., $\delta_t \in [-\widehat{\delta}_t, \widehat{\delta}_t]$ for some $\widehat{\delta}_t$, although the results can be extended to the asymmetric case without any difficulty. This yields a box-type description of the uncertainty:

$$d_t(p_t) = \overline{d}_t(p_t) + \widehat{\delta}_t\, z_t, \;\; |z_t| \leq 1, \tag{6}$$

where $\overline{d}_t$ verifies Equation (1). The scalar z_t is called the *scaled deviation* of the demand from its nominal value at time t. To avoid overprotecting the system, we impose an additional constraint on the scaled deviations, specifically:

$$\left| \sum_{t=0}^{T-1} \widehat{\delta}_t\, z_t \right| \leq \Delta, \tag{7}$$

which limits the *total consumption of the resource by the uncertainty*. The parameter Δ is chosen in $\left[0, \sum_{t=0}^{T-1} \widehat{\delta}_t\right]$ using the historical data available to the decision-maker, and is called the *maximum allowable impact of the uncertainty* on the resource, or *budget of uncertainty impact*. This represents a departure from the polyhedral uncertainty sets presented in the literature (Bertsimas and Sim [5, 6], Bertsimas and Thiele [13]), which bound the number of uncertain parameters that can deviate from their nominal values through constraints of the type: $\sum_{t=0}^{T-1} |z_t| \leq \Gamma$ for some Γ. The choice of Constraint (7) is motivated in multiperiod pricing by the specific impact of the uncertainty on the formulation.

2.3 The Robust Optimization Approach

We start by defining the robust problem, and in a second step propose an equivalent convex formulation, which can therefore be solved efficiently. The definition of the robust problem is not straightforward; indeed, if we replace the deterministic demands in Problem (2) by the uncertain parameters in Equation (6), we obtain:

1. an objective value of $\sum_{t=0}^{T-1} p_t \left(\overline{d}_t(p_t) + \widehat{\delta}_t\, z_t\right)$, which is negatively affected by the uncertainty

when there is *less* demand than expected,

2. a constraint on the available capacity of $\sum_{t=0}^{T-1}(\overline{d}_t(p_t) + \widehat{\delta}_t\, z_t) \leq C$, which is negatively affected by the uncertainty when there is *more* demand than expected.

Hence, an approach considering simultaneously these two worst cases, i.e., maximizing the smallest revenue while guaranteeing feasibility for all possible realizations of the demands, will overprotect the system, as no realization of the random demand yields the worst objective *and* resource utilization. To address this issue and connect the robust optimization approach more tightly to the worst-case value of the uncertainty, we consider instead the problem of maximizing the worst-case profit over the set of scaled deviations *that are feasible for the capacity constraint*, i.e.:

$$
\begin{array}{ll}
\max & \left[\sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) + \begin{array}{ll} \min & \sum_{t=0}^{T-1} p_t\, \widehat{\delta}_t\, z_t \\ \text{s.t.} & \sum_{t=0}^{T-1} \widehat{\delta}_t z_t \leq C - \sum_{t=0}^{T-1} \overline{d}_t(p_t), \\ & -\Delta \leq \sum_{t=0}^{T-1} \widehat{\delta}_t z_t \leq \Delta, \\ & |z_t| \leq 1\ \forall t, \end{array} \right] \\
\text{s.t.} & p_t^{\min} \leq p_t \leq p_t^{\max},\ \forall t.
\end{array}
\tag{8}
$$

The following theorem provides a tractable equivalent formulation to the max-min problem given in Equation (8).

Theorem 2.2 (Robust Formulation) *The robust problem can be formulated as a convex programming problem with only one new decision variable, called the* reference price, *and no new constraint:*

$$
\begin{array}{ll}
\max & \sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} \widehat{\delta}_t |p_t - x|\right] \\
s.t. & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta, \\
& p_t^{\min} \leq p_t \leq p_t^{\max},\ \forall t.
\end{array}
\tag{9}
$$

Hence, it can be solved as efficiently as its deterministic counterpart.

Proof: The inner minimization problem in Problem (8) is feasible if and only if the constraint:

$$\sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta \tag{10}$$

holds. Furthermore, the feasible set is obviously bounded and the worst-case scaled deviations correspond to having $\sum_{t=0}^{T-1} \widehat{\delta}_t\, z_t$ as low as possible, so that none of the upper bound constraints on $\sum_{t=0}^{T-1} \widehat{\delta}_t\, z_t$ will be tight at optimality. Hence, we can discard these constraints without affecting the optimal solution. By strong duality, Problem (8) can then be reformulated as:

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} (y_t^+ + y_t^-) \right] \\
\text{s.t.} \quad & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \le C + \Delta, \\
& \widehat{\delta}_t\, x - y_t^+ + y_t^- = p_t\, \widehat{\delta}_t, \qquad \forall t, \\
& p_t^{\min} \le p_t \le p_t^{\max},\ y_t^+, y_t^- \ge 0, \qquad \forall t.
\end{aligned} \tag{11}
$$

Problem (9) follows from interpreting y_t^+, resp. y_t^- as the positive, resp. negative, component of $\widehat{\delta}_t(x - p_t)$, and therefore $y_t^+ + y_t^-$ as its absolute value. □

Analysis: The objective in the robust problem (9) has two components: (i) the nominal revenue, and (ii) a penalty term, which penalizes the deviations (both upside and downside) of the decision variables from a **reference price** x, common to all time periods. The unit penalty is equal to the maximum amount $\widehat{\delta}_t$ of demand uncertainty faced in the time period considered. The constraints are the same as in the deterministic problem where the capacity of the resource has become $C + \Delta$. Section 2.4 provides further theoretical insights.

2.4 Theoretical Insights

Throughout this section, we will assume that no bound constraint is binding, i.e., $p_t^{\min} < p_t^* < p_t^{\max}$ for all t, and that the capacity constraint is tight at optimality $\sum_{t=0}^{T-1} \overline{d}_t(p_t^*) = C + \Delta$.

2.4.1 Optimal reference price

Let $p_{(t)}^*$, $t = 0, \ldots, T-1$, be the optimal prices in Problem (9) ranked in increasing order ($p_{(0)}^* \le \cdots \le p_{(T-1)}^*$). The optimal reference price x^* is characterized as follows.

Theorem 2.3 (Optimal reference price) *At optimality, $x^* = p_{(s)}^*$, where s is the smallest integer such that:*

$$
\sum_{t | p_t^* \le p_{(s)}^*} \widehat{\delta}_t > \frac{1}{2} \left(\sum_{t=0}^{T-1} \widehat{\delta}_t - \Delta \right). \tag{12}
$$

Proof: Let $p_{(s)}^* \le x \le p_{(s+1)}^*$ for some s. ($p_{(-1)}^* = -\infty$ and $p_{(T)}^* = \infty$ by convention.) Then the slope in x of the objective function in Problem (9) is: $-\Delta + \sum_{t | p_t^* \ge p_{(s+1)}^*} \widehat{\delta}_t - \sum_{t | p_t^* \le p_{(s)}^*} \widehat{\delta}_t$, i.e., $-\Delta + \sum_{t=0}^{T-1} \widehat{\delta}_t - 2 \sum_{t | p_t^* \le p_{(s)}^*} \widehat{\delta}_t$, which decreases as x increases. Hence, the maximum over all real numbers is reached at $p_{(s)}$, with s such that the slope is nonnegative on $[p_{(s-1)}, p_{(s)}]$ and negative on $[p_{(s)}, p_{(s+1)}]$. Equation (12) follows immediately. □

Remarks:

1. If the decision-maker is very risk-averse and plans for the maximal amount of uncertainty $\Delta = \sum_{t=0}^{T-1} \widehat{\delta}_t$, the optimal reference x^* is equal to the smallest price $p_{(0)}^*$.

2. The optimal reference price never exceeds the τ-th smallest price ($x^* \leq p^*_{(\tau)}$ for any Δ), where τ is the smallest integer s verifying $\sum_{t|p^*_t \leq p^*_{(s)}} \widehat{\delta}_t > \frac{1}{2}\sum_{t=0}^{T-1} \widehat{\delta}_t$. (This is because $\Delta \geq 0$.)

3. If $\widehat{\delta}_t = \widehat{\delta}$ for all t, the optimal reference price for the item is equal to $p^*_{(s)}$ with $s = \left\lfloor \frac{1}{2}\left(T - \frac{\Delta}{\widehat{\delta}}\right)\right\rfloor$.

2.4.2 Preliminary results

Let $\mathcal{T}$ be the set of time periods t for which $p^*_t = x^*$ at optimality (from Theorem 2.3, we know that $\mathcal{T}$ is nonempty), and let λ^* be the optimal Lagrange multiplier associated with the capacity constraint in Problem (9). ($\lambda^* \geq 0$.) We only consider changes that do not affect the set $\mathcal{T}$. We first need the following lemma.

Lemma 2.4
(a) For all $t \notin \mathcal{T}$, p^*_t satisfies:

$$(p^*_t - \lambda^*)\, \overline{d}'_t(p^*_t) + \overline{d}_t(p^*_t) = \widehat{\delta}_t\, sgn(p^*_t - x^*). \tag{13}$$

Furthermore, x^* satisfies:

$$(x^* - \lambda^*) \sum_{t\in\mathcal{T}} \overline{d}'_t(x^*) + \sum_{t\in\mathcal{T}} d_t(x^*) = \Delta, \tag{14}$$

as well as:

$$-\widehat{\delta}_t \leq (x^* - \lambda^*)\, \overline{d}'_t(x^*) + \overline{d}_t(x^*) \leq \widehat{\delta}_t,\ \forall t \in \mathcal{T}. \tag{15}$$

(b) All prices exceed the marginal value of the resource λ^* at optimality.
(c) Let $\phi_t(p_t, \lambda) = (p_t - \lambda)\, \overline{d}'_t(p_t) + \overline{d}_t(p_t)$ and $\psi(x, \lambda) = (x - \lambda)\sum_{t\in\mathcal{T}} \overline{d}'_t(x) + \sum_{t\in\mathcal{T}} d_t(x)$. Then:
(c-i) $\phi_t(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ decrease at $\lambda \geq 0$ given.
(c-ii) $\phi_t(p_t, \cdot)$, resp. $\psi(x, \cdot)$, increases at p_t, resp. x, given.

Proof: (a) follows from applying the Karush-Kuhn-Tucker optimality conditions to Problem (9). Equation (13) is obtained by maximizing in p_t the unconstrained objective: $(p_t - \lambda^*)\, \overline{d}_t(p_t) - \widehat{\delta}_t\, sgn(p^*_t - x^*)\,(p_t - x^*)$. Equation (14) is obtained by maximizing in x the unconstrained objective: $(x - \lambda^*)\sum_{t\in\mathcal{T}} \overline{d}_t(x) - \Delta\, x$. Equation (15) is obtained by writing the conditions for x^* to be the global maximum of the unconstrained function nondifferentiable at x^*, with $t \in \mathcal{T}$: $(p_t - \lambda^*)\, \overline{d}_t(p_t) - \widehat{\delta}_t|p_t - x^*|$.
(b) Demand is always nonnegative (including in the worst case). Therefore, $(\lambda^* - p^*_t)\, \overline{d}'_t(p^*_t)$ (for $t \notin \mathcal{T}$) and $(\lambda^* - x^*)\, \overline{d}'_t(x^*)$ (for $t \in \mathcal{T}$) are nonnegative. The fact that demand decreases in prices allows us to conclude.
(c) At λ given, $\phi_t(\cdot, \lambda)$ is the derivative of $(p_t - \lambda)\, \overline{d}_t(p_t)$, which is a concave function (revenue is concave, demand is convex and λ is nonnegative.) This yields (i). Moreover, $\phi_t(p_t, \cdot)$ is linear in λ, with slope $-\overline{d}'_t(p_t)$, which is nonnegative since demand decreases in price. This yields (ii). The proof for ψ is similar. □

Remark: Once the set of $\mathcal{T}$ has been determined, the specific amount of uncertainty $\widehat{\delta}_t$ in the time periods in $\mathcal{T}$ does not affect x^*.

We now characterize the optimal prices. These results will be particularly useful when we investigate the dependence of the optimal solution on the uncertainty in Section 2.4.3.

Lemma 2.5

(a) The optimal reference and product prices are a function of the uncertainty as follows:

(i) For each $t \notin \mathcal{T}$, there exists a function F_{1t} such that:

$$p_t^* = F_{1t}(\widehat{\delta}_t\, sgn(p_t^* - x^*), \lambda^*). \tag{16}$$

Moreover, there exists a function F_1 such that:

$$x^* = F_1(\Delta, \lambda^*). \tag{17}$$

(ii) There exists a function F_2 such that:

$$\lambda^* = F_2((\widehat{\delta}_t\, sgn(p_s^* - x^*))_{s=0,\ldots,T-1}, \Delta). \tag{18}$$

(iii) For each t, there exists a function F_{3t} such that:

$$p_t^* = F_{3t}((\widehat{\delta}_s\, sgn(p_s^* - x^*))_{s=0,\ldots,T-1}, \Delta). \tag{19}$$

(b) The functions F_{1t} (for any $t \notin \mathcal{T}$) and F_1 are monotonic in both arguments. Specifically,

$$\frac{\partial F_{1t}}{\partial u_1}(u_1, u_2) = \left(2\,\overline{d}_t'[F_{1t}(u_1,u_2)] + (F_{1t}(u_1,u_2) - u_2)\,\overline{d}_t''[F_{1t}(u_1,u_2)]\right)^{-1},\ t \notin \mathcal{T}, \tag{20}$$

$$\frac{\partial F_1}{\partial u_1}(u_1, u_2) = \left(2\sum_{t\in\mathcal{T}}\overline{d}_t'[F_1(u_1,u_2)] + (F_1(u_1,u_2) - u_2)\sum_{t\in\mathcal{T}}\overline{d}_t''[F_1(u_1,u_2)]\right)^{-1}, \tag{21}$$

which are both nonpositive, and:

$$\frac{\partial F_{1t}}{\partial u_2}(u_1, u_2) = \left(2\,\overline{d}_t'[F_{1t}(u_1,u_2)] + (F_{1t}(u_1,u_2) - u_2)\,\overline{d}_t''[F_{1t}(u_1,u_2)]\right)^{-1}\overline{d}_t'[F_{1t}(u_1,u_2)],\ t \notin \mathcal{T}, \tag{22}$$

$$\frac{\partial F_1}{\partial u_2}(u_1, u_2) = \left(2\sum_{t\in\mathcal{T}}\overline{d}_t'[F_1(u_1,u_2)] + (F_1(u_1,u_2) - u_2)\sum_{t\in\mathcal{T}}\overline{d}_t''[F_1(u_1,u_2)]\right)^{-1}\sum_{t\in\mathcal{T}}\overline{d}_t'[F_1(u_1,u_2)], \tag{23}$$

which are both nonnegative.

Proof: (a-i) and (a-ii) We know by concavity of the objective function that Equations (13) and (14) have a unique solution, so that F_{1t} (for all $t \notin \mathcal{T}$) and F_1 are well defined. A similar argument applies to F_2 using that $\sum_{t=0}^{T-1}\overline{d}_t(p_t^*) = C + \Delta$ and the fact that the demand is a convex function. (a-iii) combines Equations (16), (17) and (18).
(b) follows from differentiating Equations (13), resp. (14), with respect to $u_1 = \widehat{\delta}_t\, sgn(p_t^* - x^*)$, resp.

$u_1 = \Delta$, and $u_2 = \lambda^*$. The functions $(p_t - \lambda^*)\, d_t(p_t)$, for $t \notin \mathcal{T}$, and $(x - \lambda^*) \sum_{t \in \mathcal{T}} d_t(x)$ are concave at $\lambda^* \geq 0$ given (as sum of concave functions), and the demand is nonincreasing in the price, which yields the sign of the partial derivatives. $\square$

When they can be expressed in closed form, the functions F_{1t}, F_1, F_2 and F_{3t} provide valuable insights into the impact of the problem parameters on the optimal solution. A key advantage of the robust optimization approach is that this is much more frequently the case than in the traditional stochastic framework, hence offering the decision-maker a deeper understanding of the problem at hand. Section 2.5 illustrates this point when the nominal demand is linear in the prices.

2.4.3 Impact of uncertainty

In this section, we investigate the impact of the uncertainty (measured either in terms of total deviation Δ or variability $\widehat{\delta}_t$ at a specific time period t) on the optimal prices and the marginal value of the resource. We will make the following mild assumptions to simplify the analysis:

Assumption 2.6 (Impact of the uncertainty) *Increasing the uncertainty decreases the value of the resource, in the following sense:*

(i) The marginal value λ^* of the resource in the robust framework is always less than or equal to the marginal value $\overline{\lambda}^*$ of the resource in the nominal model.

(ii) The marginal value λ^* of the resource in the robust framework is nonincreasing as the budget Δ of resource consumption by the uncertainty increases.

Remark: We will show below that the condition $\frac{\partial \lambda^*}{\partial \Delta} \leq 0$ is equivalent to:

$$\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \leq 0. \tag{24}$$

By concavity of the revenue function, we already know that:

$$\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \leq -\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*),. \tag{25}$$

where the right-hand side is positive since demand decreases with price. This motivates the claim that Assumptions 2.6 impose only mild restrictions on the demand function. In particular, Equation (24) is trivially satisfied when the nominal demand is linear in the prices, and it is easy to check whether Assumptions 2.6 hold in any application by injecting the specific demand function into the equations defining λ^*.

Furthermore, Assumption 2.6 (i) is the more critical assumption of the two, and is used throughout our analysis. We only use Assumption 2.6 (ii) to study the impact of Δ on the prices.

This allows us to analyze the direction of change in the prices when uncertainty is incorporated into the deterministic model.

Theorem 2.7 (Comparison with nominal prices)
(a) Optimal robust prices are always smaller than their nominal counterparts if they strictly exceed the reference price.
(b) Optimal robust prices that fall strictly below the reference price are smaller than their nominal counterparts if and only if the uncertainty remains below a threshold, specifically:

$$\widehat{\delta}_t \leq -(\overline{\lambda}^* - \lambda^*)\, \overline{d}_t'(p_t^*). \tag{26}$$

Proof: Using the notations of Lemma 2.4, the optimal price at time t, $t \notin \mathcal{T}$, satisfies: $f_t(p_t^*, \lambda^*) = \widehat{\delta}_t\, sgn(p_t^* - x^*)$ with $f_t(p_t^*, \lambda^*) = f_t(p_t^*, \overline{\lambda}^*) + (\overline{\lambda}^* - \lambda^*)\, \overline{d}_t'(p_t^*)$. Therefore, we have: $f_t(p_t^*, \overline{\lambda}^*) = \widehat{\delta}_t\, sgn(p_t^* - x^*) - (\overline{\lambda}^* - \lambda^*)\, \overline{d}_t'(p_t^*)$. Since $f_t(\overline{p}_t^*, \overline{\lambda}^*) = 0$ and $f_t(\cdot, \lambda)$ decreases (from Lemma 2.4 (c)), $p_t^* \leq \overline{p}_t^*$ if and only if:

$$\widehat{\delta}_t\, sgn(p_t^* - x^*) - (\overline{\lambda}^* - \lambda^*)\, \overline{d}_t'(p_t^*) \geq 0. \tag{27}$$

(a) and (b) follow by distinguishing between $p_t^* < x^*$ and $p_t^* > x^*$, using that nominal demand decreases in price and invoking Assumption 2.6 (i). □

Remark: Low-priced items with high uncertainty see a price increase from their nominal values. Intuitively, the decision-maker reduces the sales at that time so that capacity can be reallocated to more profitable time periods.

Theorem 2.8 analyzes the dependence of the optimal prices and the marginal value of the resource on the parameter Δ, as the $\widehat{\delta}_t$ are kept constant. In this context, increasing Δ can be interpreted as increasing the risk aversion of the decision-maker. The key insight of Theorem 2.8 is that, provided that $sgn(p_t^* - x^*)$ remains constant for all t, i.e., provided that we consider changes small enough, the optimal prices, the reference price and the marginal value of the resource all decrease as the decision-maker's degree of conservatism increases.

Theorem 2.8 (Impact of the budget of resource consumption by the uncertainty)
(a) The marginal value of the resource satisfies:

$$\frac{\partial \lambda^*}{\partial \Delta} = \left[\frac{\partial F_1}{\partial u_2}(\Delta, \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + \sum_{t \notin \mathcal{T}} \frac{\partial F_{1t}}{\partial u_2}(\widehat{\delta}_t\, sgn(p_t^* - x^*), \lambda^*)\, \overline{d}_t'(p_t^*) \right]^{-1} \left(1 - \frac{\partial F_1}{\partial u_1}(\Delta, \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) \right), \tag{28}$$

where $\frac{\partial F_1}{\partial u_1}$, $\frac{\partial F_{1t}}{\partial u_2}$ and $\frac{\partial F_1}{\partial u_2}$ are defined in Equations (21)-(23). From Assumption 2.6, the marginal value of the resource decreases in the budget of resource consumption by the uncertainty.
(b) The optimal price at time t (both for $t \in \mathcal{T}$ and $t \notin \mathcal{T}$) satisfies:

$$\frac{\partial p_t^*}{\partial \Delta} \leq 0\ \forall t. \tag{29}$$

Specifically,

$$\frac{\partial p_t^*}{\partial \Delta} = \frac{\partial F_{1t}}{\partial u_2}(\widehat{\delta}_t\, sgn(p_t^* - x^*), \lambda^*)\, \frac{\partial \lambda^*}{\partial \Delta},\ t \notin \mathcal{T}, \tag{30}$$

and:
$$\frac{\partial x^*}{\partial \Delta} = \frac{\partial F_1}{\partial u_1}(\Delta, \lambda^*) + \frac{\partial F_1}{\partial u_2}(\Delta, \lambda^*)\, \frac{\partial \lambda^*}{\partial \Delta}, \tag{31}$$

where $\frac{\partial F_1}{\partial u_1}$, $\frac{\partial F_{1t}}{\partial u_2}$ *and* $\frac{\partial F_1}{\partial u_2}$ *are defined in Equations (21)-(23) and* $\frac{\partial \lambda^*}{\partial \Delta}$ *is defined in Equation (28). A higher degree of risk aversion will decrease all optimal prices.*

Proof: We will prove (a) and (b) simultaneously. Differentiating Equations (13) and (14) with respect to Δ yields Equations (30) and (31). The sign of $\frac{\partial p_t^*}{\partial \Delta}$ follows from injecting $\frac{\partial \lambda^*}{\partial \Delta} \leq 0$ from Assumption 2.6 and the sign of the partial derivatives from Lemma 2.5 into Equations (30) and (31). Furthermore, differentiating $\sum_{t=0}^{T-1} \overline{d}_t(p_t^*) = C + \Delta$ with respect to Δ yields:

$$\frac{\partial x^*}{\partial \Delta} \sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + \sum_{t \notin \mathcal{T}} \frac{\partial p_t^*}{\partial \Delta}\, \overline{d}_t'(p_t^*) = 1. \tag{32}$$

We obtain Equation (28) by reinjecting Equations (30) and (31) into Equation (32). □

Remark: The robust problem with $\Delta = 0$ is not equivalent to the nominal problem since the choice of $\Delta = 0$ does not set the $\widehat{\delta}_t$ to 0. Moreover, the marginal value of the resource and the optimal prices will exhibit discontinuities when the optimal prices move above or below the reference price, i.e., when the set $\mathcal{T}$ changes. This explains why the optimal prices in the robust framework are not always smaller than the optimal prices in the nominal model, as formalized in Theorem 2.7.

Finally, Theorem 2.9 establishes that increasing the uncertainty at time t does not necessarily decrease the optimal price p_t^*. Instead, the optimal price converges further towards the reference price. We keep Δ constant, which is justified for instance when this parameter is selected empirically using historical data, and an increase in $\widehat{\delta}_t$ is not immediately reflected in its value.

Theorem 2.9 (Impact of the uncertainty at each time period)
(a) The marginal value of the resource increases, resp. decreases, when the demand uncertainty increases at a time where the price is strictly above, resp. strictly below, the reference price.
(b) For $t \notin \mathcal{T}$, *the optimal price* p_t^* *converges towards the reference price* x^* *as* $\widehat{\delta}_t$ *increases.*

Proof: (a) Differentiating $\sum_{s \notin \mathcal{T}} \overline{d}_s[F_{1s}(\widehat{\delta}_s\, sgn(p_s^* - x^*), \lambda^*)] + \sum_{s \in \mathcal{T}} \overline{d}_s[F_1(\Delta, \lambda^*)] = C + \Delta$ with respect to $\widehat{\delta}_t$ for some $t \notin \mathcal{T}$ yields (after dropping the arguments of the partial derivatives for notational convenience):

$$sgn(p_t^* - x^*)\, \frac{\partial F_{1t}}{\partial u_1} \cdot \overline{d}_t'(p_t^*) + \frac{\partial \lambda^*}{\partial \widehat{\delta}_t} \cdot \left[\sum_{s \notin \mathcal{T}} \frac{\partial F_{1s}}{\partial u_2} \cdot \overline{d}_s'(p_s^*) + \frac{\partial F_1}{\partial u_2} \sum_{s \in \mathcal{T}} \overline{d}_s'(x^*) \right] = 0, \tag{33}$$

or equivalently:
$$\frac{\partial \lambda^*}{\partial \widehat{\delta}_t} = -sgn(p_t^* - x^*)\, \frac{\frac{\partial F_{1t}}{\partial u_1} \cdot \overline{d}_t'(p_t^*)}{\sum_{s \notin \mathcal{T}} \frac{\partial F_{1s}}{\partial u_2} \cdot \overline{d}_s'(p_s^*) + \frac{\partial F_1}{\partial u_2} \sum_{s \in \mathcal{T}} \overline{d}_s'(x^*)}. \tag{34}$$

Let a_t be such that $\dfrac{\partial \lambda^*}{\partial \widehat{\delta}_t} = -sgn(p_t^* - x^*)\, a_t$ in Equation (34). From Lemma 2.5 (b) and the fact that the demand decreases in the prices, we have: $a_t \leq 0$. (a) follows immediately.
(b) Differentiating Equation (16) with respect to $\widehat{\delta}_t$ and injecting Equation (34) yields:

$$\frac{\partial p_t^*}{\partial \widehat{\delta}_t} = sgn(p_t^* - x^*)\, \frac{\partial F_{1t}}{\partial u_1} \left(1 - \frac{\frac{\partial F_{1t}}{\partial u_2} \cdot \overline{d}_t'(p_t^*)}{\sum_{s \notin \mathcal{T}} \frac{\partial F_{1s}}{\partial u_2} \cdot \overline{d}_s'(p_s^*) + \frac{\partial F_1}{\partial u_2} \sum_{s \in \mathcal{T}} \overline{d}_s'(x^*)} \right). \tag{35}$$

From Lemma 2.5 (b), the coefficient b_t defined such that $\dfrac{\partial p_t^*}{\partial \widehat{\delta}_t} = sgn(p_t^* - x^*)\, b_t$ is nonpositive. □

2.5 Example

In this section, we illustrate the results above on the example of additive nominal demand, i.e., nominal demand that is linear in the prices:

$$\overline{d}_t(p_t) = a_t - b_t\, p_t, \text{ with } a_t,\, b_t > 0,\ \forall t. \tag{36}$$

Because the mathematical expressions follow directly from injecting Equation (36) into the framework developed in Sections 2.3 and 2.4, we state them without proof. As before, all the results (with the exception of the robust formulation) are derived assuming that the capacity constraint is tight at optimality and that the prices do not reach their bounds. This situation is commonly encountered in practice when demand is linear in the prices.

2.5.1 Robust formulation

The robust problem when the average demand is linear in the prices and uncertainty is additive can be formulated as:

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} p_t\,(a_t - b_t\, p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} \widehat{\delta}_t |p_t - x| \right] \\ \text{s.t.} \quad & \sum_{t=0}^{T-1} b_t\, p_t \geq \sum_{t=0}^{T-1} a_t - (C + \Delta), \\ & p_t^{\min} \leq p_t \leq p_t^{\max},\ \forall t. \end{aligned} \tag{37}$$

2.5.2 Optimal prices

Let $\mathcal{T} = \{t | p_t^* = x^*\}$. The optimal price at time t satisfies:

$$p_t^* = \begin{cases} \dfrac{a_t - \widehat{\delta}_t\, sgn(p_t^* - x^*)}{2\, b_t} + \dfrac{\lambda^*}{2}, & \text{if } t \notin \mathcal{T}, \\ \dfrac{\sum_{t \in \mathcal{T}} a_t - \Delta}{2 \sum_{t \in \mathcal{T}} b_t} + \dfrac{\lambda^*}{2}, & \text{if } t \in \mathcal{T}. \end{cases} \tag{38}$$

This defines the functions F_{1t} and F_1 in Equations (16) and (17).

The impact of uncertainty on the optimal prices at each time period has two components:

- a term that is specific to that time period and depends on the maximal deviation allowed $\widehat{\delta}_t$,
- a term that depends on the change in the marginal value of the capacity and is common to all time periods.

While the fact that uncertainty exists always make the resource as a whole less valuable ($\lambda^* \leq \overline{\lambda}^*$) and drives prices down, the specific uncertainty at each time period might drive prices up or down. In particular, $p_t^* < \overline{p}_t^*$ if and only if (i) $p_t^* > x^*$ or (ii) $p_t^* < x^*$ and $\widehat{\delta}_t < b_t(\overline{\lambda}^* - \lambda^*)$. In other words, it is optimal to *increase* prices when prices are low and uncertainty is high. This allows the decision-maker to re-allocate some of the resource to a more profitable time period.

2.5.3 Optimal Lagrange multiplier

The marginal value of the resource as a function of the uncertainty is given by:

$$\lambda^* = \frac{\sum_{t=0}^{T-1}[a_t + \widehat{\delta}_t\, sgn(p_t^* - x^*)] - (2\,C + \Delta)}{\sum_{t=0}^{T-1} b_t}. \tag{39}$$

This defines the function F_2 in Equation (18). In particular:

$$\lambda^* = \overline{\lambda}^* + \frac{\sum_{t=0}^{T-1} \widehat{\delta}_t\, sgn(p_t^* - x^*) - \Delta}{\sum_{t=0}^{T-1} b_t}. \tag{40}$$

From Theorem 2.3, we know that $\sum_{t=0}^{T-1} \widehat{\delta}_t\, sgn(p_t^* - x^*) \leq \Delta$ at optimality, so that $\lambda^* \leq \overline{\lambda}^*$, which satisfies Assumption 2.6 (i). We also note that:

$$\frac{\partial \lambda^*}{\partial \Delta} = -\frac{1}{\sum_{t=0}^{T-1} b_t}, \tag{41}$$

which is negative and thus satisfies Assumption 2.6 (ii), and:

$$\frac{\partial \lambda^*}{\partial \widehat{\delta}_t} = \frac{1}{\sum_{s=0}^{T-1} b_s} \cdot sgn(p_t^* - x^*), \tag{42}$$

so that λ^* indeed increases, resp. decreases, in the maximum uncertainty $\widehat{\delta}_t$ at time t when the optimal price at time t strictly exceeds, resp. falls strictly below, the reference price x^*.

2.5.4 Optimal prices as a function of uncertainty

The optimal prices are piecewise linear in the uncertainty, measured by $\widehat{\delta}_t$ for all t. Specifically:

$$p_t^* = \frac{1}{2}\,\frac{\sum_{\tau=0}^{T-1}[a_\tau + \widehat{\delta}_\tau\, sgn(p_\tau^* - x^*)] - (2\,C + \Delta)}{\sum_{\tau=0}^{T-1} b_\tau} + \begin{cases} \dfrac{a_t - \widehat{\delta}_t\, sgn(p_t^* - x^*)}{2\,b_t}, & \text{if } t \notin \mathcal{T}, \\[2ex] \dfrac{\sum_{t\in\mathcal{T}} a_t - \Delta}{2\sum_{t\in\mathcal{T}} b_t}, & \text{if } t \in \mathcal{T}. \end{cases} \tag{43}$$

It follows that:

$$\frac{\partial p_t^*}{\partial \Delta} = \begin{cases} -\dfrac{1}{2\sum_{\tau=0}^{T-1} b_\tau}, & \text{if } t \notin \mathcal{T}, \\ -\dfrac{1}{\sum_{\tau=0}^{T-1} b_\tau}, & \text{if } t \in \mathcal{T}. \end{cases} \tag{44}$$

and:

$$\frac{\partial p_t^*}{\partial \widehat{\delta}_s} = \begin{cases} \dfrac{1}{2} \cdot \left[-\dfrac{1}{b_t} + \dfrac{1}{\sum_{\tau=0}^{T-1} b_\tau} \right] \cdot sgn(p_t^* - x^*), & \text{if } t \notin \mathcal{T} \text{ and } s = t, \\ \dfrac{1}{2} \cdot \dfrac{1}{\sum_{\tau=0}^{T-1} b_\tau} \cdot sgn(p_s^* - x^*), & \text{otherwise.} \end{cases} \tag{45}$$

The optimal price at time t decreases when the decision-maker's degree of conservatism increases, and converges further towards the reference price x^* when the uncertainty at that time period increases. Moreover, the optimal price at time t increases, resp. decreases, when the uncertainty at a time period s for which $p_s^* > x^*$, resp. $p_s^* < x^*$, increases.

Therefore, the reference price plays a crucial role in understanding how uncertainty affects the optimal prices in the single-product problem with additive uncertainty.

3 Single-Product Pricing with Multiplicative Uncertainty

3.1 Description of Uncertainty

We now present the robust optimization framework in the case of *multiplicative* uncertainty. As before, we assume the average demand, resp. revenue, at time t is convex, resp. concave in the prices. The random demand is modeled by:

$$d_t = \overline{d}_t \cdot (1 + \delta_t), \tag{46}$$

where δ_t is again a random variable with zero mean and symmetric support $[-\widehat{\delta}_t, \widehat{\delta}_t]$ independent of the price p_t ($\widehat{\delta}_t < 1$). The half-length of the uncertainty interval in the robust formulation will hence be equal to $\overline{d}_t \cdot \widehat{\delta}_t$ for all t, yielding the following model for the demand:

$$d_t = \overline{d}_t + \widehat{\delta}_t \, \overline{d}_t \, z_t, \ |z_t| \leq 1, \tag{47}$$

where $\overline{d}_t$ verifies Equation (1).

As in Section 2.2, we consider a box uncertainty set with a budget limiting the deviation in the utilization of the resource. Here, the uncertainty set depends on the prices:

$$\mathcal{Z}(\mathbf{p}) = \left\{ \left| \sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p_t) \, z_t \right| \leq \Delta, \ |z_t| \leq 1, \forall t \right\}. \tag{48}$$

(Since $p_t^{\min} \leq p_t \leq p_t^{\max}$ for all t, we choose $\Delta \leq \sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p_t^{\min})$.)

3.2 The Robust Optimization Approach

Motivated by the approach developed in Section 2.3, we seek to maximize the worst-case revenue, where the worst case is taken over the set of allowable scaled deviations, and define the **robust problem** in the presence of multiplicative uncertainty as:

$$
\begin{aligned}
\max \quad & \left[\sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) + \min \quad \sum_{t=0}^{T-1} p_t\, \widehat{\delta}_t\, \overline{d}_t(p_t)\, z_t\right] \\
& \qquad\qquad \text{s.t.} \quad -\Delta \leq \sum_{t=0}^{T-1} \widehat{\delta}_t\, \overline{d}_t(p_t)\, z_t \leq \min\left(\Delta, C - \sum_{t=0}^{T-1} \overline{d}_t(p_t)\right), \\
& \qquad\qquad\qquad\quad |z_t| \leq 1,\ \forall t, \\
\text{s.t.} \quad & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta, \\
& p_t^{\min} \leq p_t \leq p_t^{\max},\ \forall t.
\end{aligned}
\tag{49}
$$

Theorem 3.1 provides a simpler nonlinear formulation of the robust problem and describes how it can be solved efficiently.

Theorem 3.1 (Robust Counterpart and Algorithm)

(a) The robust problem in the case of multiplicative uncertainty can be formulated as a nonlinear problem:

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} p_t\, \overline{d}_t(p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} \widehat{\delta}_t\, \overline{d}_t(p_t)\, |p_t - x|\right] \\
s.t. \quad & \sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta, \\
& p_t^{\min} \leq p_t \leq p_t^{\max},\ \forall t.
\end{aligned}
\tag{50}
$$

(b) If the uncertainty at each time period satisfies:

$$
\widehat{\delta}_t \leq 1 + \min_{p_t^{\min} \leq p_t \leq p_t^{\max}} \frac{p_t\, \overline{d}_t^{''}(p_t)}{2\, \overline{d}_t^{'}(p_t)},\ \forall t,
\tag{51}
$$

Problem (50) for any given x *is convex, and hence can be solved efficiently as a function of the reference price.*

(c) Let the function F be such that, for all x, $F(x)$ is equal to the optimal objective of Problem (50) solved at x given. There exists x^* such that F is nondecreasing on $(-\infty, x^*]$ and nonincreasing on $[x^*, \infty)$. Hence, the optimal reference price in Problem (50) is equal to x^* and can be found efficiently using gradient-ascent methods.

Proof: (a) is a direct extension of Theorem 2.2 to the case with multiplicative uncertainty.
(b) At x given, the part of the objective function that depends on p_t is equal to $p_t\, \overline{d}_t(p_t) - \widehat{\delta}_t\, \overline{d}_t(p_t)\, |p_t - x|$, for each t. The second derivative of this function is equal to: $(1-\widehat{\delta}_t)\,(p_t\, \overline{d}_t^{''}(p_t) + 2\, \overline{d}_t^{'}(p_t)) + \widehat{\delta}_t\, \overline{d}_t^{''}(p_t)\, x$ when $p_t > x$, which is smaller than or equal to $p_t\, \overline{d}_t^{''}(p_t) + 2\, \overline{d}_t^{'}(p_t) - 2\, \widehat{\delta}_t\, \overline{d}_t^{'}(p_t)$ (because $p_t > x$ and $\overline{d}_t$ is convex), and $(1+\widehat{\delta}_t)\,(p_t\, \overline{d}_t^{''}(p_t) + 2\, \overline{d}_t^{'}(p_t)) - \widehat{\delta}_t\, \overline{d}_t^{''}(p_t)\, x$ when $p_t \leq x$, which is smaller than or equal

to $p_t\,\overline{d}_t''(p_t) + 2\,\overline{d}_t'(p_t) + 2\,\widehat{\delta}_t\,\overline{d}_t'(p_t)$ (because $p_t \le x$ and $\overline{d}_t$ is convex). Moreover, the average demand decreases in the prices, and straightforward calculations show that the slope at the breakpoint x decreases. Therefore, for the objective function at x given to be concave, it suffices that, for all t and for all p_t such that $p_t^{\min} \le p_t \le p_t^{\max}$, the condition:

$$p_t\,\overline{d}_t''(p_t) + 2\,\overline{d}_t'(p_t) - 2\,\widehat{\delta}_t\,\overline{d}_t'(p_t) \le 0 \tag{52}$$

holds. This yields Condition (51) immediately.

(c) Let $\widetilde{F}$ be the objective function of Problem (50) and let F be its maximum value, function of the reference price, over the set of feasible prices. Let also x and x' be any two numbers such that $x < x'$, and let the vectors $\mathbf{p}$, resp. $\mathbf{p}'$ be the optimal prices at x, resp. x' given, that is:

$$F(x) = \widetilde{F}(\boldsymbol{p}, x) = \sum_{t=0}^{T-1} p_t\,\overline{d}_t(p_t) - \left[\Delta\,x + \sum_{t=0}^{T-1} \widehat{\delta}_t\,\overline{d}_t(p_t)\,|p_t - x|\right], \tag{53}$$

and

$$F(x') = \widetilde{F}(\boldsymbol{p}', x') = \sum_{t=0}^{T-1} p_t'\,\overline{d}_t(p_t') - \left[\Delta\,x' + \sum_{t=0}^{T-1} \widehat{\delta}_t\,\overline{d}_t(p_t')\,|p_t' - x'|\right]. \tag{54}$$

It is straightforward to check that:

$$|p_t' - x'| = \begin{cases} x' - x + |x - p_t'|, & \text{if } p_t' < x, \\ x' - x - |p_t' - x|, & \text{if } x \le p_t' \le x', \\ |p_t' - x| + x - x', & \text{otherwise.} \end{cases} \tag{55}$$

This yields:

$$F(x') = \widetilde{F}(\boldsymbol{p}', x) + (x' - x)\,\left[-\Delta + \sum_{t|p_t' > x'} \widehat{\delta}_t\,\overline{d}_t(p_t') - \sum_{t|p_t' \le x'} \widehat{\delta}_t\,\overline{d}_t(p_t')\right] + 2 \sum_{t|x \le p_t' \le x'} \widehat{\delta}_t\,\overline{d}_t(p_t')|p_t' - x|. \tag{56}$$

Since $\widetilde{F}(\boldsymbol{p}', x) \le F(x)$ by definition of F and $\sum_{t|x \le p_t' \le x'} \widehat{\delta}_t\,\overline{d}_t(p_t')|p_t' - x| \le (x' - x)\sum_{t|x \le p_t' \le x'} \widehat{\delta}_t\,\overline{d}_t(p_t')$, we obtain after re-arranging the terms:

$$F(x') \le F(x) + (x' - x)\,\left[-\Delta + \sum_{t|p_t' \ge x} \widehat{\delta}_t\,\overline{d}_t(p_t') - \sum_{t|p_t' < x} \widehat{\delta}_t\,\overline{d}_t(p_t')\right]. \tag{57}$$

Similarly, we derive a lower bound on $F(x')$ by expressing $|p_t - x'|$ as a function of $|p_t - x|$ and using that $F(x') \ge \widetilde{F}(\boldsymbol{p}, x')$ and $\sum_{t|x \le p_t' \le x'} \widehat{\delta}_t\,\overline{d}_t(p_t)|p_t - x| \ge 0$. Combining these results leads to:

$$-\Delta + \sum_{t|p_t > x'} \widehat{\delta}_t\,\overline{d}_t(p_t) - \sum_{t|p_t \le x'} \widehat{\delta}_t\,\overline{d}_t(p_t) \le \frac{F(x') - F(x)}{x' - x} \le -\Delta + \sum_{t|p_t' \ge x} \widehat{\delta}_t\,\overline{d}_t(p_t') - \sum_{t|p_t' < x} \widehat{\delta}_t\,\overline{d}_t(p_t'). \tag{58}$$

From the right-hand side of Equation (58), we conclude that $F(x') < F(x)$ for all $x < x'$ such that $-\Delta + \sum_{t|p_t' \ge x} \widehat{\delta}_t\,\overline{d}_t(p_t') - \sum_{t|p_t' < x} \widehat{\delta}_t\,\overline{d}_t(p_t') < 0$. Since the vector $\mathbf{p}'$ depends on x' but not on x,

$-\Delta + \sum_{t|p'_t \geq x} \widehat{\delta}_t \, \overline{d}_t(p'_t) - \sum_{t|p'_t < x} \widehat{\delta}_t \, \overline{d}_t(p'_t)$ is piecewise constant, nonincreasing in x. Therefore, there exists a threshold value which depends on x' and is noted $x^+(x')$ such that, for all x', F decreases on $[x^+(x'), x']$. $x^+(x')$ is defined as:

$$x^+(x') = \min \left\{ y | -\Delta + \sum_{t|p'_t \geq y} \widehat{\delta}_t \, \overline{d}_t(p'_t) - \sum_{t|p'_t < y} \widehat{\delta}_t \, \overline{d}_t(p'_t) < 0 \right\}. \tag{59}$$

It follows that F decreases on $[\min_{x'} x^+(x'), \infty)$.

Similarly, F is nondecreasing on $(-\infty, \max_x x^-(x)]$, where $x^-(x)$ is defined for all x as:

$$x^-(x) = \max \left\{ y | -\Delta + \sum_{t|p'_t \geq y} \widehat{\delta}_t \, \overline{d}_t(p'_t) - \sum_{t|p'_t < y} \widehat{\delta}_t \, \overline{d}_t(p'_t) \geq 0 \right\}. \tag{60}$$

Therefore, the global maximum of F, noted x^*, belongs to $[\max_x x^-(x), \min_{x'} x^+(x')]$. Let $\mathbf{p}^*$ be such that $F(x^*) = \widetilde{F}(\mathbf{p}^*, x^*)$, that is, $\mathbf{p}^* = \arg\max \widetilde{F}(\cdot, x^*)$. Because x^* is the global maximum of F, we also have: $\mathbf{x}^* = \arg\max \widetilde{F}(\mathbf{p}^*, \cdot)$. Hence, x^* is the point where the slope of $\widetilde{F}(\mathbf{p}^*, \cdot)$ changes sign. It follows that $x^+(x^*) = x^-(x^*) = x^*$, and F is nondecreasing on $(-\infty, x^*]$ and decreasing on $[x^*, \infty)$. □

Remark: As in the case with additive uncertainty, the objective in the robust problem (50) has two components: (i) the nominal revenue, and (ii) a penalty term, which penalizes the deviations (both upside and downside) of the decision variables from a **reference price** x for the item, common to all time periods. The unit penalty is equal to the maximum amount of demand uncertainty faced in that time period, measured by the half-length of the uncertainty interval $\widehat{\delta}_t \, \overline{d}_t(p_t)$. The constraints are the same as in the deterministic problem where the quantity of resource available is $C + \Delta$.

3.3 Theoretical Insights

3.3.1 Optimal reference price

Let $p^*_{(t)}$, $t = 0, \ldots, T-1$, be the optimal prices ranked in increasing order $(p^*_{(0)} \leq \cdots \leq p^*_{(T-1)})$.

Theorem 3.2 (Optimal reference price) *The optimal price satisfies: $x^* = p^*_{(s)}$, where s is the smallest integer such that:*

$$\sum_{t|p^*_t \leq p^*_{(s)}} \widehat{\delta}_t \, \overline{d}_t(p^*_t) > \frac{1}{2} \left[\sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p^*_t) - \Delta \right]. \tag{61}$$

Proof: When the prices p_t are set to their optimal value p^*_t, $t = 0, \ldots, T-1$, the objective in Problem (50) is piecewise linear in the reference price x, with slope $\sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p^*_t) \, sgn(p^*_t - x^*) - \Delta$ or equivalently: $\sum_{t=0}^{T-1} \widehat{\delta}_t \, \overline{d}_t(p^*_t) - \Delta - 2 \sum_{t|x > p^*_t} \widehat{\delta}_t \, \overline{d}_t(p^*_t)$ (for $x^* \neq p^*_t$, $t = 0, \ldots, T-1$.) The value of the slope decreases as x^* increases, and optimality is reached at x^* for which the slope changes sign, which yields Equation (61). $\square$

3.3.2 Preliminary results

As in Section 2.4.2, let $\mathcal{T}$ be the set of time periods for which $p_t^* = x^*$ (from Theorem 3.2, the set $\mathcal{T}$ is nonempty.) Let λ^* be the optimal Lagrange multiplier associated with the capacity constraint. We only consider changes which do not affect the set $\mathcal{T}$ and assume $p_t^{\min} < p_t^* < p_t^{\max}$ for all t at optimality.

Lemma 3.3
(a) For $t \notin \mathcal{T}$, p_t^* satisfies:

$$(p_t^* - \lambda^*)\,\overline{d}_t'(p_t^*) + \overline{d}_t(p_t^*) = \widehat{\delta}_t\, sgn(p_t^* - x^*)\,\left[(p_t^* - x^*)\,\overline{d}_t'(p_t^*) + \overline{d}_t(p_t^*)\right], \tag{62}$$

or equivalently:

$$p_t^* + \frac{\overline{d}_t(p_t^*)}{\overline{d}_t'(p_t^*)} = \frac{\lambda^* - x^*\,\widehat{\delta}_t\, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}. \tag{63}$$

Furthermore, x^* satisfies:

$$(x^* - \lambda^*)\sum_{t\in\mathcal{T}} \overline{d}_t'(x^*) + \sum_{t\in\mathcal{T}} \overline{d}_t(x^*) = \Delta, \tag{64}$$

and for any $t \in \mathcal{T}$:

$$-\widehat{\delta}_t\,\overline{d}_t(x^*) \leq (x^* - \lambda^*)\,\overline{d}_t'(x^*) + \overline{d}_t(x^*) \leq \widehat{\delta}_t\,\overline{d}_t(x^*). \tag{65}$$

(b) At optimality, all prices exceed the marginal value of the resource λ^*.
(c) Let $\phi_t(p_t, \lambda) = (p_t - \lambda)\,\overline{d}_t'(p_t) + \overline{d}_t(p_t)$ and $\psi(x, \lambda) = (x - \lambda)\sum_{t\in\mathcal{T}} \overline{d}_t'(x) + \sum_{t\in\mathcal{T}} d_t(x)$. Then:
(c-i) $\phi_t(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ decrease at $\lambda \geq 0$ given.
(c-ii) $\phi_t(p_t, \cdot)$, resp. $\psi(x, \cdot)$, increases at p_t, resp. x, given.

Proof: The proof of (a) and (c) is similar to Lemma 2.4 and we omit it here. For (b), we use Condition (65) and note that worst-case demand is nonnegative, so that $(x^* - \lambda^*)\,\overline{d}_t'(p_t^*) \leq -(1 - \widehat{\delta}_t)\,\overline{d}_t(p_t^*)$ requires $x^* \geq \lambda^*$. It follows that $p_t^* \geq \lambda^*$ for all p_t^* greater than x^*. For $p_t^* < x^*$, Condition (63) yields: $p_t^* + \overline{d}_t(p_t^*)/\overline{d}_t'(p_t^*) = (\lambda^* + x^*\,\widehat{\delta}_t)/(1 + \widehat{\delta}_t)$ or equivalently:

$$p_t^* - \lambda^* = \frac{(x^* - \lambda^*)\,\widehat{\delta}_t}{1 + \widehat{\delta}_t} - \frac{\overline{d}_t(p_t^*)}{\overline{d}_t'(p_t^*)}. \tag{66}$$

Since $x^* \geq \lambda^*$, the right-hand side is nonnegative and $p_t^* \geq \lambda^*$. Therefore, all prices are greater than or equal to the marginal value of the resource. If at least one price was equal to λ^*, then in particular $p_t^* = \lambda^*$ for the time period t corresponding to the smallest product price. If $p_t^* < x^*$ for that t, then from Equation (66) we have $x^* = \lambda^*$ (and $\overline{d}_t(p_t^*) = 0$), and if x^* is indeed the smallest price then the condition $x^* = \lambda^*$ is trivial. Reinjecting into Condition (65) yields $\overline{d}_t(x^*) = 0$ for all $t \in \mathcal{T}$ (since $\widehat{\delta}_t < 1$ for all t), which would violate Equation (64). Hence, $p_t^* > \lambda^*$ for all t. $\square$

We now provide a general characterization of the optimal prices.

Lemma 3.4

(a) The optimal reference and product prices are a function of the uncertainty as follows:

(i) For each $t \notin \mathcal{T}$, there exists a function G_{1t} such that:

$$p_t^* = G_{1t}(\widehat{\delta}_t \, sgn(p_t^* - x^*), \lambda^*, x^*). \tag{67}$$

Moreover, there exists a function G_1 such that:

$$x^* = G_1(\Delta, \lambda^*). \tag{68}$$

(ii) There exists a function G_2 such that:

$$\lambda^* = G_2((\widehat{\delta}_t \, sgn(p_s^* - x^*))_{s=0,\dots,T-1}, \Delta). \tag{69}$$

(iii) For each t, there exists a function G_{3t} such that:

$$p_t^* = G_{3t}((\widehat{\delta}_s \, sgn(p_s^* - x^*))_{s=0,\dots,T-1}, \Delta). \tag{70}$$

(b) The function G_1 is nonincreasing in its first argument and nondecreasing in its second. Specifically:

$$\frac{\partial G_1}{\partial u_1}(u_1, u_2) = \left(2 \sum_{t \in \mathcal{T}} \overline{d}_t'[G_1(u_1, u_2)] + (G_1(u_1, u_2) - u_2) \sum_{t \in \mathcal{T}} \overline{d}_t''[G_1(u_1, u_2)]\right)^{-1}, \tag{71}$$

$$\frac{\partial G_1}{\partial u_2}(u_1, u_2) = \left(2 \sum_{t \in \mathcal{T}} \overline{d}_t'[G_1(u_1, u_2)] + (G_1(u_1, u_2) - u_2) \sum_{t \in \mathcal{T}} \overline{d}_t''[G_1(u_1, u_2)]\right)^{-1} \sum_{t \in \mathcal{T}} \overline{d}_t'[G_1(u_1, u_2)]. \tag{72}$$

Proof: Is a straightforward extension of Lemma 2.5. □

While it is possible to obtain partial derivatives of p_t^*, $t \notin \mathcal{T}$, in this broad framework, we will focus on a special case that allows for more powerful results. We make the following assumption:

Assumption 3.5 *The function $p_t \mapsto p_t + \dfrac{\overline{d}_t(p_t)}{\overline{d}_t'(p_t)}$ increases in p_t, for all t.*

This assumption enforces that the ratio of the marginal revenue over the absolute value of the marginal demand decreases in the price (since $\overline{d}_t'(p_t) < 0$). This is equivalent to limiting the curvature of the nominal demand at each time period:

$$\overline{d}_t''(p_t) \leq 2 \frac{[\overline{d}_t'(p_t)]^2}{\overline{d}_t(p_t)}, \ \forall t, \, p_t. \tag{73}$$

Note that the curvature is already limited by the concavity of the revenue. Nominal demands that are linear in the prices obviously satisfy Condition (73).

Lemma 3.6 *Under Assumption 3.5, there exists a nondecreasing function $\widetilde{G}_{1t}$, $t \notin \mathcal{T}$, such that:*

$$p_t^* = \widetilde{G}_{1t}\left(\frac{\lambda^* - x^* \, \widehat{\delta}_t \, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t \, sgn(p_t^* - x^*)}\right). \tag{74}$$

Proof: Is a direct consequence of Equation (63) combined with Assumption 3.5. $\square$

Remark: Since $\widetilde{G}_{1t}$ only depends on the average demand, the impact of the uncertainty on the prices p_t^* (with $p_t^* \neq x^*$) is captured *in its entirety* by the parameter $\dfrac{\lambda^* - x^* \, \widehat{\delta}_t \, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t \, sgn(p_t^* - x^*))}$. This depends on the time period only through $\widehat{\delta}_t \, sgn(p_t^* - x^*)$. If $p_t^* < x^*$, this parameter is a convex combination of λ^* and x^*, with x^* receiving more weight if there is more uncertainty. If $p_t^* > x^*$, this parameter remains strictly smaller than λ^*, since $x^* > \lambda^*$ from Theorem 3.3, and the coefficient of x^* becomes more negative as the uncertainty increases.

Section 3.4 illustrates the insights that the decision-maker can derive from the robust optimization approach in the case of average demand linear in the prices.

3.3.3 Impact of uncertainty

The results in this section require that Assumptions 2.6 and 3.5 hold. We start by comparing the optimal prices with those obtained in the deterministic model.

Theorem 3.7 (Comparison with the nominal model)
(a) If $x^* \leq \overline{\lambda}^*$, then all the optimal prices in the robust model have decreased from their nominal values.
(b) If $x^* > \overline{\lambda}^*$, then the robust prices have always decreased from their nominal values when they exceed the reference price in the robust model, and have decreased from their nominal values when they fall below the reference price if and only if the uncertainty is below a threshold:

$$\widehat{\delta}_t < \frac{\overline{\lambda}^* - \lambda^*}{x^* - \overline{\lambda}^*}. \tag{75}$$

Proof: We prove (a) and (b) simultaneously. We know that $p_t \mapsto p_t + \overline{d}_t(p_t)/\overline{d}_t'(p_t)$ increases in p_t for all t. Hence, from Equation (63) with $t \notin \mathcal{T}$, $p_t^* < \overline{p}_t^*$ if and only if the condition:

$$\frac{\lambda^* - x^* \, \widehat{\delta}_t \, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t \, sgn(p_t^* - x^*)} < \overline{\lambda}^* \tag{76}$$

holds. This is equivalent to:

$$\widehat{\delta}_t \, sgn(p_t^* - x^*) \, (x^* - \overline{\lambda}^*) > \lambda^* - \overline{\lambda}^*. \tag{77}$$

This condition is always satisfied if $x^* = \overline{\lambda}^*$. If $x^* < \overline{\lambda}^*$, Equation (77) becomes:

$$\widehat{\delta}_t \, sgn(p_t^* - x^*) < \frac{\overline{\lambda}^* - \lambda^*}{\overline{\lambda}^* - x^*}, \tag{78}$$

which is always satisfied because the right-hand side is greater than 1 (since $\lambda^* < x^* < \overline{\lambda}^*$) and $\widehat{\delta}_t < 1$ for all t. If $x^* > \overline{\lambda}^*$, Equation (77) becomes:

$$\widehat{\delta}_t \, sgn(p_t^* - x^*) > -\frac{\overline{\lambda}^* - \lambda^*}{\overline{\lambda}^* - x^*}. \tag{79}$$

We conclude by distinguishing between $p_t^* < x^*$ and $p_t^* > x^*$. $\square$

Remark: In contrast with the case of additive uncertainty, the threshold in Condition (75) does not depend on the time period considered.

The parameter Δ, which represents the maximum allowable use of the resource by the uncertainty and can be interpreted as the decision-maker's degree of risk aversion, affects the optimal solution as follows.

Theorem 3.8 (Impact of the maximum use of the resource by the uncertainty)
(a) The optimal product prices below or equal to the reference price, the reference price and the marginal value of the resource are all piecewise nonincreasing in the budget of uncertainty impact.
(b) Let t be a time period for which the optimal price exceeds the reference price. p_t^* decreases with Δ if and only if:

$$\widehat{\delta}_t \leq \frac{\partial \lambda^*}{\partial \Delta} \left(\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \right), \tag{80}$$

which is equivalent to the following two conditions holding simultaneously:

$$\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \leq 0, \tag{81}$$

and

$$\frac{\partial \lambda^*}{\partial \Delta} \leq \widehat{\delta}_t \left[\sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \right]^{-1}. \tag{82}$$

Proof: (a) $\frac{\partial \lambda^*}{\partial \Delta} \leq 0$ is a direct consequence of Assumption 2.6. Differentiating Equation (64) with respect to Δ yields:

$$\frac{\partial x^*}{\partial \Delta} = \left(1 + \frac{\partial \lambda^*}{\partial \Delta} \sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) \right) \left[2 \sum_{t \in \mathcal{T}} \overline{d}_t'(x^*) + (x^* - \lambda^*) \sum_{t \in \mathcal{T}} \overline{d}_t''(x^*) \right]^{-1}. \tag{83}$$

Since $\frac{\partial \lambda^*}{\partial \Delta} \leq 0$, it follows that $\frac{\partial x^*}{\partial \Delta} \leq 0$ by concavity of the revenue, convexity and monotonicity of the demand. Moreover, differentiating Equation (63) with respect to Δ yields, for $t \notin \mathcal{T}$:

$$\frac{\partial p_t^*}{\partial \Delta} = \frac{1}{1 - \widehat{\delta}_t \, sgn(p_t^* - x^*)} \left[2 - \frac{\overline{d}_t(p_t^*) \, \overline{d}_t''(p_t^*)}{[\overline{d}_t'(p_t^*)]^2} \right]^{-1} \left(\frac{\partial \lambda^*}{\partial \Delta} - \frac{\partial x^*}{\partial \Delta} \widehat{\delta}_t \, sgn(p_t^* - x^*) \right). \tag{84}$$

$\frac{\partial p_t^*}{\partial \Delta} \leq 0$ for $p_t^* < x^*$ (i.e., $sgn(p_t^* - x^*) = -1$) follows immediately from Assumption 3.5.
For $p_t^* > x^*$, $\frac{\partial p_t^*}{\partial \Delta} \leq 0$ is equivalent to:

$$\frac{\partial \lambda^*}{\partial \Delta} \leq \frac{\partial x^*}{\partial \Delta} \widehat{\delta}_t. \tag{85}$$

We obtain Condition (80) by injecting Equation (83) into Equation (85). □

Remark: High prices (those above the reference price) decrease as the decision-maker's risk aversion Δ increases if the uncertainty $\widehat{\delta}_t$ is "small enough", i.e., below a threshold. If the uncertainty is too large, then it is optimal to increase those prices in order to decrease the nominal demand and allocate the resource to more valuable time periods.

We now investigate the impact of the maximum uncertainty at each time period, $\widehat{\delta}_t$, on the optimal prices, for $t \notin \mathcal{T}$.

Theorem 3.9 (Impact of the uncertainty at each time period) *The optimal prices converge further towards the reference price as the uncertainty increases. Specifically:*

$$\frac{\partial p_t^*}{\partial \widehat{\delta}_t} = \frac{(\lambda^* - x^*)\, sgn(p_t^* - x^*)}{\widetilde{G}'_{1t}(p_t^*)\,[1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)]^2}. \tag{86}$$

Proof: Follows by differentiating Equation (63) as a function of $\widehat{\delta}_t$ and invoking Lemma 3.3 (b) and Lemma 3.6. □

3.4 Example

In this section, we apply the robust optimization approach with multiplicative uncertainty to the case where the average demand is linear in the prices, i.e.:

$$\overline{d}_t(p_t) = a_t - b_t\, p_t, \text{with } a_t,\, b_t > 0,\ \forall t. \tag{87}$$

Linear functions satisfy Assumption 3.5. The results follow immediately from injecting Equation (87) into the approach developed in Sections 3.2 and 3.3. Therefore, we state them without proof.

3.4.1 Robust formulation

The robust problem when the average demand is linear in the prices and uncertainty is multiplicative can be formulated as:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} p_t\,(a_t - b_t\, p_t) - \left[\Delta\, x + \sum_{t=0}^{T-1} \widehat{\delta}_t\,(a_t - b_t\, p_t)\,|p_t - x|\right] \\
\text{s.t.} \quad & \sum_{t=0}^{T-1} b_t\, p_t \geq \sum_{t=0}^{T-1} a_t - (C + \Delta), \\
& p_t^{\min} \leq p_t \leq p_t^{\max},\ \forall t.
\end{aligned} \tag{88}$$

3.4.2 Optimal prices

As in Section 2.5.2, let x^*, $\mathbf{p}^*$ be the optimal decision variables of Problem (50), and λ^* the optimal Lagrange multiplier for the capacity constraint $\sum_{t=0}^{T-1} \overline{d}_t(p_t) \leq C + \Delta$. Let also $\overline{\mathbf{p}}^*$, $\overline{\lambda}^*$ be the optimal prices and Lagrange multiplier in the nominal problem. We assume that the capacity constraint is tight both in the nominal and the robust models, that $p_t^{\min} < p_t^* < p_t^{\max}$ for all t, and that $\mathcal{T} = \{t | p_t^* = x^*\}$ does not change with infinitesimal changes in the parameters. As before, we denote by $\mathcal{T}$ the set $\{t | p_t^* = x^*\}$.

The optimal price at time t satisfies:

$$p_t^* = \begin{cases} \dfrac{a_t}{2\,b_t} + \dfrac{\lambda^* - x^*\,\widehat{\delta}_t\, sgn(p_t^* - x^*)}{2\,(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))}, & \text{if } t \notin \mathcal{T} \\ \dfrac{\sum_{t\in\mathcal{T}} a_t - \Delta}{2\sum_{t\in\mathcal{T}} b_t} + \dfrac{\lambda^*}{2}, & \text{if } t \in \mathcal{T}. \end{cases} \tag{89}$$

As expected, $p_t^* < \overline{p}_t^*$ if and only if $(x^* - \overline{\lambda}^*)\,\widehat{\delta}_t\, sgn(p_t^* - x^*) > \lambda^* - \overline{\lambda}^*$. The robust prices p_t^* with $t \notin \mathcal{T}$ differ at optimality from their nominal counterparts as follows:

$$p_t^* = \overline{p}_t^* + \frac{\lambda^* - \overline{\lambda}^*}{2} + \frac{(\lambda^* - x^*)\,\widehat{\delta}_t\, sgn(p_t^* - x^*)}{2\,(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))}. \tag{90}$$

Hence, the impact of uncertainty on the optimal prices at each time period has two components:

- a term that depends on the change in the marginal value of the resource $\lambda^* - \overline{\lambda}^*$ and is common to all time periods,
- a term that depends on the time period through $\widehat{\delta}_t\, sgn(p_t^* - x^*)$ and is proportional to the difference between the reference price x^* and the marginal value of the resource λ^*.

3.4.3 Optimal Lagrange multiplier

The marginal value of the resource as a function of the uncertainty, obtained by injecting Equation (89) into $\sum_{t\notin\mathcal{T}} \overline{d}_t(p_t^*) + \sum_{t\in\mathcal{T}} \overline{d}_t(x^*) = C + \Delta$, is given by:

$$\lambda^* = \frac{\sum_{t=0}^{T-1} a_t + (B/2) - (2\,C + \Delta)}{A/2}. \tag{91}$$

where:

$$\begin{cases} A = \displaystyle\sum_{t=0}^{T-1} b_t + \sum_{t=0}^{T-1} \frac{b_t}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}, \\ B = \left(\displaystyle\sum_{t=0}^{T-1} \frac{\widehat{\delta}_t\, b_t\, sgn(p_t^* - x^*)}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}\right) \cdot \left(\dfrac{\sum_{t\in\mathcal{T}} a_t - \Delta}{\sum_{t\in\mathcal{T}} b_t}\right). \end{cases} \tag{92}$$

Hence, the marginal value of the resource decreases as the decision-maker's risk aversion, i.e., the maximum allowable use of the resource by the uncertainty, increases:

$$\frac{\partial \lambda^*}{\partial \Delta} = -\frac{2}{A}. \tag{93}$$

The marginal value of the resource increases, resp. decreases, in the maximum uncertainty $\widehat{\delta}_t$ when the optimal price at time t strictly exceeds, resp. falls strictly below, the reference price x^*. Specifically:

$$\frac{\partial \lambda^*}{\partial \widehat{\delta}_t} = \frac{2\, b_t\,(x^* - \lambda^*)\, sgn(p_t^* - x^*)}{A\,(1 - \widehat{\delta}_t sgn(p_t^* - x^*))^2}. \tag{94}$$

3.4.4 Optimal prices as a function of uncertainty

The optimal prices are piecewise rational functions in the uncertainty (where the piecewise part comes from the dependence in t):

$$p_t^* = \begin{cases} \dfrac{\sum_{t\in\mathcal{T}} a_t - \Delta}{2\sum_{t\in\mathcal{T}} b_t} + \dfrac{\lambda^*}{2}, & t \in \mathcal{T}, \\ \dfrac{a_t}{2\, b_t} - \dfrac{\widehat{\delta}_t\, sgn(p_t^* - x^*)}{4(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))} \cdot \left(\dfrac{\sum_{s\in\mathcal{T}} a_s - \Delta}{\sum_{s\in\mathcal{T}} b_s}\right) + \lambda^* \cdot \dfrac{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)/2}{2\,(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))}, & t \notin \mathcal{T}. \end{cases} \tag{95}$$

where λ^* is given by Equation (91). It follows that:

$$\frac{\partial p_t^*}{\partial \Delta} = \begin{cases} -\left(\dfrac{1}{2\sum_{s\in\mathcal{T}} b_s} + \dfrac{1}{A}\right), & t \in \mathcal{T}, \\ \dfrac{1}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)} \left[\dfrac{1}{2}\, \widehat{\delta}_t\, sgn(p_t^* - x^*) \left(\dfrac{1}{2\sum_{s\in\mathcal{T}} b_s} + \dfrac{1}{A}\right) - \dfrac{1}{A}\right], & t \notin \mathcal{T}. \end{cases} \tag{96}$$

As expected, we find that p_t^* at t such that $p_t^* < x^*$ and x^* decrease with Δ, and that p_t^* at t such that $p_t^* > x^*$ decreases with Δ if and only if the uncertainty at that time period falls below a threshold, here:

$$\widehat{\delta}_t \leq 2\left[1 + \frac{A}{2\sum_{s\in\mathcal{T}} b_s}\right]^{-1}. \tag{97}$$

Since $2\sum_{s\in\mathcal{T}} b_s < A$ by definition of A, the right-hand side is less than 1. Prices increasing with Δ are therefore prices that are already high (above the reference level) and correspond to time periods with high uncertainty. This allows the capacitated resource to be reallocated to more profitable time periods.

We can also formulate the dependence of the prices on the uncertainty at each time period explicitly:

$$\frac{\partial p_t^*}{\partial \widehat{\delta}_s} = \begin{cases} -\dfrac{(x^* - \lambda^*)\, sgn(p_t^* - x^*)}{2\,(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))^2} \left[1 - \dfrac{b_t}{A}\left(1 + \dfrac{1}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)}\right)\right], & \text{if } t \notin \mathcal{T} \text{ and } s = t, \\ \dfrac{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)/2}{1 - \widehat{\delta}_t\, sgn(p_t^* - x^*)} \cdot \dfrac{b_s\,(x^* - \lambda^*)\, sgn(p_s^* - x^*)}{A\,(1 - \widehat{\delta}_s\, sgn(p_s^* - x^*))^2}, & \text{if } t \notin \mathcal{T} \text{ and } s \neq t, \\ \dfrac{b_s\,(x^* - \lambda^*)\, sgn(p_s^* - x^*)}{A\,(1 - \widehat{\delta}_s\, sgn(p_s^* - x^*))^2}, & \text{if } t \in \mathcal{T}. \end{cases} \tag{98}$$

As established for general demand functions, if $t \notin \mathcal{T}$ and $s = t$, the price converges further the reference price, since $b_t(1 + 1/(1 - \widehat{\delta}_t\, sgn(p_t^* - x^*))) < A$ by definition of A.

Therefore, whether a price is above or below the reference level plays a key role in understanding the impact of an increase in the uncertainty at that time period on all optimal prices.

4 Multi-Product, Multi-Resource Pricing

4.1 The Deterministic Problem

In this section, we extend the frameworks described in Sections 2 and 3 to the case of multiple products and multiple resources. We first present the deterministic model. Let n be the number of products. The (nominal) demand at time t, which is a vector of size n, is given by:

$$\overline{\mathbf{d}}_\mathbf{t} = \mathbf{a_t} - \mathbf{B_t}\, \mathbf{p_t}, \tag{99}$$

where $\mathbf{a_t}$ is a known vector of size n and $\mathbf{B_t}$ is a known positive definite matrix of size $n \times n$, whose diagonal, resp. off-diagonal, elements are positive, resp. nonpositive. This models the fact that an increase in a product's price decreases the nominal demand for that product, and increases or leaves unchanged the nominal demand for other items through a substitution effect.

There are m resources, each available in quantity C_i, $i = 1, \ldots, m$, at the start of the horizon. The resources are never replenished, and can for instance represent the total number of seats available on an aircraft on a specific leg or rooms available in an hotel on a specific night. Let $\mathbf{F}$ be the resource-product allocation matrix of size $m \times n$, whose (i, j) element is equal to 1 if item j uses resource i and 0 otherwise. Let also $\mathbf{p}_\mathbf{t}^{\min}$, $\mathbf{p}_\mathbf{t}^{\max}$ be the lower and upper bounds on the price vector at time t.

When the demand is deterministic, the problem of finding the optimal prices to maximize revenue is formulated as:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} \mathbf{p}_\mathbf{t}' \overline{\mathbf{d}}_\mathbf{t} \\
\text{s.t.} \quad & \sum_{t=0}^{T-1} \mathbf{F}\, \overline{\mathbf{d}}_\mathbf{t} \le \mathbf{C}, \\
& \overline{\mathbf{d}}_\mathbf{t} = \mathbf{a_t} - \mathbf{B_t}\, \mathbf{p_t}, \ \forall t, \\
& \mathbf{p}_\mathbf{t}^{\min} \le \mathbf{p_t} \le \mathbf{p}_\mathbf{t}^{\max}, \ \forall t.
\end{aligned} \tag{100}$$

The following theorem characterizes the deterministic problem and the optimal prices.

Theorem 4.1 (Deterministic Problem)
(a) Problem (100) is equivalent to the quadratic programming problem with linear constraints:

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} \mathbf{p}_t'(\mathbf{a_t} - \mathbf{B_t}\mathbf{p_t}) \\
s.t. \quad & \sum_{t=0}^{T-1} \mathbf{F}\,\mathbf{B_t}\mathbf{p_t} \geq \sum_{t=0}^{T-1} \mathbf{F}\,\mathbf{a_t} - \mathbf{C}, \\
& \mathbf{p}_t^{\min} \leq \mathbf{p_t} \leq \mathbf{p}_t^{\max}, \ \forall t.
\end{aligned}
\tag{101}
$$

(b) Let $\overline{\boldsymbol{\lambda}}^* \geq \mathbf{0}$ be the optimal Lagrange multiplier for the capacity constraints. Then the optimal price vector at time t verifies:

$$
\overline{\mathbf{p}}_t^* = \max\left(\mathbf{p}_t^{\min}, \min\left\{\mathbf{p}_t^{\max}, \frac{1}{2}\left[\mathbf{B}_t^{-1}\,\mathbf{a_t} + \mathbf{F}'\overline{\boldsymbol{\lambda}}^*\right]\right\}\right), \tag{102}
$$

where maximum and minimum are taken componentwise.

Proof: Is an immediate extension of Theorem 2.1. □

4.2 The Case of Additive Uncertainty

4.2.1 Generalities

We now incorporate uncertainty to Problem (100). We first assume that uncertainty is additive, i.e., the demand is modeled by:

$$
\mathbf{d_t} = \overline{\mathbf{d}}_t + (\mathbf{diag}\,\widehat{\boldsymbol{\delta}}_t)\,\mathbf{z_t}, \tag{103}
$$

where $\mathbf{z_t}$ is the vector of scaled deviations of the demand at time t. The uncertainty set is a box uncertainty set (enforcing that scaled deviations cannot exceed 1 in absolute value) cut by budgets of resource utilization by the uncertainty for each resource. Let $\boldsymbol{\Delta}$ be the vector of size m of these budgets. The uncertainty set is:

$$
\mathcal{Z} = \left\{ \left| \sum_{t=0}^{T-1} \mathbf{F}(\mathbf{diag}\,\widehat{\boldsymbol{\delta}}_t)\mathbf{z_t} \right| \leq \boldsymbol{\Delta}, \ |\mathbf{z_t}| \leq \mathbf{e}, \forall t \right\}, \tag{104}
$$

where the absolute value is taken componentwise and $\mathbf{e}$ is the vector of all one's.
Following the same reasoning as in Section 2, the counterpart of Problem (??) is formulated as:

$$
\begin{aligned}
\max \quad & \left[\sum_{t=0}^{T-1} \mathbf{p}_t'\,(\mathbf{a_t} - \mathbf{B_t}\,\mathbf{p_t}) + \min \quad \sum_{t=0}^{T-1} \mathbf{p}_t'\,(\mathbf{diag}\,\widehat{\boldsymbol{\delta}}_t)\,\mathbf{z_t} \right] \\
& \qquad \text{s.t.} \quad -\boldsymbol{\Delta} \leq \sum_{t=0}^{T-1} \mathbf{F}(\mathbf{diag}\,\widehat{\boldsymbol{\delta}}_t)\,\mathbf{z_t} \leq \min\left(\boldsymbol{\Delta}, \mathbf{C} - \sum_{t=0}^{T-1} \mathbf{F}(\mathbf{a_t} - \mathbf{B_t}\,\mathbf{p_t})\right), \\
& \qquad\qquad |\mathbf{z_t}| \leq \mathbf{e} \ \forall t, \\
\text{s.t.} \quad & \sum_{t=0}^{T-1} \mathbf{F}\,\mathbf{B_t}\,\mathbf{p_t} \geq \sum_{t=0}^{T-1} \mathbf{F}\,\mathbf{a_t} - (\mathbf{C} + \boldsymbol{\Delta}), \\
& \mathbf{p}_t^{\min} \leq \mathbf{p_t} \leq \mathbf{p}_t^{\max} \ \forall t.
\end{aligned}
\tag{105}
$$

We will distinguish two cases in analyzing Problem (105): (i) multiple products and one resource, (ii) multiple products and two or more resources.

4.2.2 Formulation with one resource

The case with only one resource ($\mathbf{F} = \mathbf{e}'$) is an immediate extension of the framework described in Section 2, as Lemma ?? still applies. Hence, we state the results without proof.

Problem (105) is equivalent to a quadratic programming problem with linear constraints, which involves $2T+1$ new variables and T new constraints (beside nonnegativity):

$$\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} \mathbf{p}_\mathbf{t}' (\mathbf{a}_\mathbf{t} - \mathbf{B}_\mathbf{t}\, \mathbf{p}_\mathbf{t}) - \left[\Delta\, x + \sum_{t=0}^{T-1} \mathbf{e}'(\mathbf{y}_\mathbf{t}^- + \mathbf{y}_\mathbf{t}^+)\right] \\
\text{s.t.} \quad & \mathbf{y}_\mathbf{t}^+ - \mathbf{y}_\mathbf{t}^- = (\mathbf{diag}\, \widehat{\delta}_\mathbf{t})\,(x\,\mathbf{e} - \mathbf{p}_\mathbf{t}), \ \forall t, \\
& \sum_{t=0}^{T-1} \mathbf{e}'\, \mathbf{B}_\mathbf{t}\, \mathbf{p}_\mathbf{t} \geq \sum_{t=0}^{T-1} \mathbf{e}'\mathbf{a}_\mathbf{t} - (C + \Delta), \\
& x \geq 0,\ \mathbf{y}_\mathbf{t}^-, \mathbf{y}_\mathbf{t}^+ \geq \mathbf{0},\ \mathbf{p}_\mathbf{t}^{\min} \leq \mathbf{p}_\mathbf{t} \leq \mathbf{p}_\mathbf{t}^{\max},\ \forall t.
\end{aligned} \tag{106}$$

It can also be formulated as a convex programming problem with linear constraints, which involves only one new variable and no new constraint (beside nonnegativity):

$$\begin{aligned}
\max \quad & \sum_{t=0}^{T-1} \mathbf{p}_\mathbf{t}' (\mathbf{a}_\mathbf{t} - \mathbf{B}_\mathbf{t}\, \mathbf{p}_\mathbf{t}) - \left[\Delta\, x + \sum_{t=0}^{T-1} \widehat{\delta}_t' |\mathbf{p}_\mathbf{t} - x\,\mathbf{e}|\right] \\
\text{s.t.} \quad & \sum_{t=0}^{T-1} \mathbf{e}'\, \mathbf{B}_\mathbf{t}\, \mathbf{p}_\mathbf{t} \geq \sum_{t=0}^{T-1} \mathbf{e}'\mathbf{a}_\mathbf{t} - (C + \Delta), \\
& x \geq 0,\ \mathbf{p}_\mathbf{t}^{\min} \leq \mathbf{p}_\mathbf{t} \leq \mathbf{p}_\mathbf{t}^{\max},\ \forall t.
\end{aligned} \tag{107}$$

Problem (107) can be interpreted as a deterministic problem with capacity $C + \Delta$ and a penalty term in the objective, which penalizes deviations in the prices from a *target level* x. This target price is common to all products and all time periods.

4.2.3 Formulations with two or more resources

We now consider the case with at least two resources. We will need the following lemma.

Lemma 4.2 (Bilinear robust formulation) *Problem (105) is equivalent to the non-convex prob-*

lem:

$$\max \quad \sum_{t=0}^{T-1} \mathbf{p}_t' (\mathbf{a}_t - \mathbf{B}_t \mathbf{p}_t) \;-\; \left[\boldsymbol{\Delta}'(\mathbf{x}^+ + \mathbf{x}^-) + \left(\mathbf{C} - \sum_{t=0}^{T-1} \mathbf{F}(\mathbf{a}_t - \mathbf{B}_t \mathbf{p}_t) \right)' \tilde{\mathbf{x}} + \sum_{t=0}^{T-1} \mathbf{e}'(\mathbf{y}_t^+ + \mathbf{y}_t^-) \right]$$
$$\begin{aligned} s.t. \quad & (\mathbf{diag}\, \widehat{\boldsymbol{\delta}}_t)\, \mathbf{F}'(\mathbf{x}^- - \mathbf{x}^+ - \tilde{\mathbf{x}}) - \mathbf{y}_t^+ + \mathbf{y}_t^- = (\mathbf{diag}\, \widehat{\boldsymbol{\delta}}_t)\, \mathbf{p}_t, \ \forall t, \\ & \sum_{t=0}^{T-1} \mathbf{F}\, \mathbf{B}_t\, \mathbf{p}_t \geq \sum_{t=0}^{T-1} \mathbf{F}\, \mathbf{a}_t - (\mathbf{C} + \boldsymbol{\Delta}), \\ & \mathbf{p}_t^{\min} \leq \mathbf{p}_t \leq \mathbf{p}_t^{\max}, \ \forall t, \\ & \mathbf{x}^-, \mathbf{x}^+, \tilde{\mathbf{x}}, \mathbf{y}_t^-, \mathbf{y}_t^+ \geq \mathbf{0}, \ \forall t. \end{aligned} \tag{108}$$

Proof: Follows from rewriting the constraint $\sum_{t=0}^{T-1} \mathbf{F}(\mathbf{diag}\, \widehat{\boldsymbol{\delta}}_t)\, \mathbf{z}_t \leq \min \left(\boldsymbol{\Delta}, \mathbf{C} - \sum_{t=0}^{T-1} \mathbf{F}(\mathbf{a}_t - \mathbf{B}_t \mathbf{p}_t) \right)$ as:

$$\sum_{t=0}^{T-1} \mathbf{F}(\mathbf{diag}\, \widehat{\boldsymbol{\delta}}_t)\, \mathbf{z}_t \leq \boldsymbol{\Delta} \text{ and } \sum_{t=0}^{T-1} \mathbf{F}(\mathbf{diag}\, \widehat{\boldsymbol{\delta}}_t)\, \mathbf{z}_t \leq \mathbf{C} - \sum_{t=0}^{T-1} \mathbf{F}(\mathbf{a}_t - \mathbf{B}_t \mathbf{p}_t), \tag{109}$$

and invoking strong duality for the inner minimization problem in Problem (105). □

The non-convexity of Problem (108) is due to the bilinear term $\left(\mathbf{C} - \sum_{t=0}^{T-1} \mathbf{F}(\mathbf{a}_t - \mathbf{B}_t \mathbf{p}_t) \right)' \tilde{\mathbf{x}}$ in the objective function. In the discussion below, we motivate the choice of $\tilde{\mathbf{x}} = \mathbf{0}$, which leads to a slight increase in conservatism but yields significant computational gains, as the objective becomes quadratic and the problem convex.

The main challenge in the case of two or more resources is that, contrary to the single-resource case where the smallest revenue is always reached when the resource utilization is lower than average, no such simple relation exists when the demand for a specific product affects several raw materials. Specifically, some resources might be utilized as much as possible to yield the worst-case revenue. This is best explained on an example.

Example: Consider 2 products and 3 resources. Product 1 uses resources 1 and 2, while product 2 uses resources 1 and 3. The budget vector $\boldsymbol{\delta}$ of allowable deviations is equal to $(0, 1, 1)'$. Capacities are infinite and $\boldsymbol{\delta}_t$ is the vector of all one's for all t. Parameters are chosen so that the worst case is to have demand for item 1 as low as possible ($z_1 = -1$). Then we need to select $z_2 = 1$ to satisfy the constraint on the utilization level of resource 1 ($z_1 + z_2 = 0$.) As a result, the utilization level of resource 3 reaches its *upside* limit.

Hence, while we cannot guarantee that $\tilde{\mathbf{x}}$ (or $\mathbf{x}^+$, for that matter) will be zero at optimality, the more important resources will be *under*-utilized, to reflect low demand. The resources i for which $\tilde{\mathbf{x}}_i > 0$, which create the nonconvexity of Problem (108), correspond to less critical resources. As a result, the additional complexity due to the bilinear term in the objective function is not warranted in the problem at hand, and we will force $\tilde{\mathbf{x}} = \mathbf{0}$ in the remainder of this section. This yields the following robust formulation as a quadratic programming problem with linear constraints:

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} \mathbf{p}_t' \left(\mathbf{a}_t - \mathbf{B}_t \mathbf{p}_t\right) - \left[\mathbf{\Delta}'(\mathbf{x}^+ + \mathbf{x}^-) + \sum_{t=0}^{T-1} \mathbf{e}'(\mathbf{y}_t^+ + \mathbf{y}_t^-)\right] \\ \text{s.t.} \quad & (\mathbf{diag}\, \widehat{\delta}_t)\, \mathbf{F}'(\mathbf{x}^- - \mathbf{x}^+) - \mathbf{y}_t^+ + \mathbf{y}_t^- = (\mathbf{diag}\, \widehat{\delta}_t)\, \mathbf{p}_t, \ \forall t, \\ & \sum_{t=0}^{T-1} \mathbf{F}\, \mathbf{B}_t\, \mathbf{p}_t \geq \sum_{t=0}^{T-1} \mathbf{F}\, \mathbf{a}_t - (\mathbf{C} + \mathbf{\Delta}), \\ & \mathbf{p}_t^{\min} \leq \mathbf{p}_t \leq \mathbf{p}_t^{\max}, \ \forall t, \\ & \mathbf{x}^-, \mathbf{x}^+, \mathbf{y}_t^-, \mathbf{y}_t^+ \geq \mathbf{0}, \ \forall t. \end{aligned} \tag{110}$$

Theorem 4.3 (Convex Robust Formulation) *Problem (110) is equivalent to the convex programming problem with linear constraints:*

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} \mathbf{p}_t' \left(\mathbf{a}_t - \mathbf{B}_t \mathbf{p}_t\right) - \left[\mathbf{\Delta}'(\mathbf{x}^+ + \mathbf{x}^-) + \sum_{t=0}^{T-1} \widehat{\delta}_t' \left|\mathbf{p}_t - \mathbf{F}'(\mathbf{x}^- - \mathbf{x}^+)\right|\right] \\ s.t. \quad & \sum_{t=0}^{T-1} \mathbf{F}\, \mathbf{B}_t\, \mathbf{p}_t \geq \sum_{t=0}^{T-1} \mathbf{F}\, \mathbf{a}_t - (\mathbf{C} + \mathbf{\Delta}), \\ & \mathbf{p}_t^{\min} \leq \mathbf{p}_t \leq \mathbf{p}_t^{\max}, \ \forall t, \\ & \mathbf{x}^-, \mathbf{x}^+ \geq \mathbf{0}. \end{aligned} \tag{111}$$

Proof: We have $\mathbf{y}_t^+ - \mathbf{y}_t^- = (\mathbf{diag}\, \widehat{\delta}_t)\, [\mathbf{F}'(\mathbf{x}^- - \mathbf{x}^+) - \mathbf{p}_t]$ and $\mathbf{y}_t^+ + \mathbf{y}_t^-$ is equal to its absolute value, for all t. □

Remark: At $\mathbf{x}^-$, $\mathbf{x}^+$ given, set to their optimal values, the objective in the robust problem is equal to the objective in the nominal problem minus a penalty term. This new term penalizes deviations of the price vector at time t, $\mathbf{p}_t$, from a vector of *reference prices* for each of the items, $\mathbf{F}'(\mathbf{x}^- - \mathbf{x}^+)$.

4.2.4 Theoretical insights

We now characterize the optimal solutions of Problems (110) and (111).

Theorem 4.4 (Optimal prices as a function of the Lagrange multiplier)
(a) Let $\boldsymbol{\lambda}^*$ the optimal Lagrange vector corresponding to the capacity constraints in Problem (111) and let $\mathbf{x}^{-*}$, $\mathbf{x}^{+*}$ be the optimal auxiliary variables. The optimal price vector $\mathbf{p}_t^*$ satisfies:

$$\mathbf{p}_t^* = \max\left(\mathbf{p}_t^{\min},\, \min\left\{\mathbf{p}_t^{\max},\, \frac{\mathbf{1}}{\mathbf{2}}\left[\mathbf{B}_t^{-1}\, \mathbf{a}_t - \mathbf{B}_t^{-1}\, (\mathbf{diag}\, \widehat{\delta}_t)\, \mathbf{sgn}(\mathbf{p}_t^* - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})) + \mathbf{F}'\lambda^*\right]\right\}\right). \tag{112}$$

(b) Assume that $(\mathbf{p}_t^{\min})_j < (\mathbf{p}_t^*)_j < (\mathbf{p}_t^{\max})_j$ and $(\mathbf{p}_t^{\min})_j < (\overline{\mathbf{p}}_t^*)_j < (\mathbf{p}_t^{\max})_j$ for some product j and time t, where $\overline{\mathbf{p}}_t$ is the optimal price vector at time t in the nominal problem. Then, if the optimal robust price $(\mathbf{p}_t^*)_j$ exceeds the target level $[\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j$, it is strictly lower than its nominal counterpart $(\overline{\mathbf{p}}_t^*)_j$ if and only if the amount of uncertainty for that product at that time

period exceeds a threshold, specifically:

$$(\widehat{\delta}_t)_j > \frac{1}{(\mathbf{B}_t^{-1})_{jj}} \left\{ [\mathbf{F}'(\boldsymbol{\lambda}^* - \overline{\boldsymbol{\lambda}}^*)]_j - \sum_{k \neq j} (\mathbf{B}_t^{-1})_{jk} (\widehat{\delta}_t)_k \, sgn((\mathbf{p}_t^*)_k - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})_k) \right\}. \tag{113}$$

If the optimal robust price $(\mathbf{p}_t^*)_j$ falls below the target level $[\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j$, it is strictly greater than its nominal counterpart $(\overline{\mathbf{p}}_t^*)_j$ if and only if the amount of uncertainty for that product at that time period exceeds the opposite value of that threshold.
(c) $\boldsymbol{\lambda}^* \leq \overline{\boldsymbol{\lambda}}^*$ and the optimal target price of each item j, $[\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j$ is equal to the optimal price of that item j at some time period.

Proof: (a) Let $\mathbf{x}^{-*}$, $\mathbf{x}^{+*}$, $\mathbf{p}_t^*$ be the optimal variables in Problem (111), and $\boldsymbol{\lambda}^*$ be the optimal Lagrange vector associated with the capacity constraint. Problem (111) is then equivalent to:

$$\begin{aligned} \max \quad & \sum_{t=0}^{T-1} \mathbf{p}_t'(\mathbf{a}_t - \mathbf{B}_t \, \mathbf{p}_t) - \sum_{t=0}^{T-1} \widehat{\delta}_t'(\mathbf{p}_t - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})) \, \mathbf{sgn}(\mathbf{p}_t^* - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})) \\ & \qquad\qquad + (\boldsymbol{\lambda}^*)' \sum_{t=0}^{T-1} \mathbf{F} \, \mathbf{B}_t \, \mathbf{p}_t \\ \text{s.t.} \quad & \mathbf{p}_t^{\min} \leq \mathbf{p}_t \leq \mathbf{p}_t^{\max}, \forall t. \end{aligned} \tag{114}$$

This problem is separable in the $\mathbf{p}_t$ and Equation (112) follows by differentiating the objective function in Problem (114) in $\mathbf{p}_t$ and incorporating the bounds.
(b) If neither the nominal nor the robust price for item j at time t is equal to its (lower or upper) bound, Equation (102), resp. Equation (112), becomes:

$$(\overline{\mathbf{p}}_t^*)_j = \frac{1}{2} \left[\mathbf{B}_t^{-1} \, \mathbf{a}_t + \mathbf{F}' \overline{\boldsymbol{\lambda}}^* \right]_j, \tag{115}$$

$$(\mathbf{p}_t^*)_j = \frac{1}{2} \left[\mathbf{B}_t^{-1} \, \mathbf{a}_t - \mathbf{B}_t^{-1} \, \mathbf{diag}(\widehat{\delta}_t) \, \mathbf{sgn}(\mathbf{p}_t^* - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})) + \mathbf{F}' \boldsymbol{\lambda}^* \right]_j. \tag{116}$$

Hence, $(\mathbf{p}_t^*)_j - (\overline{\mathbf{p}}_t^*)_j$ is of the same sign as $\left[\mathbf{F}'(\boldsymbol{\lambda}^* - \overline{\boldsymbol{\lambda}}^*) - \mathbf{B}_t^{-1} \, \mathbf{diag}(\widehat{\delta}_t) \, \mathbf{sgn}(\mathbf{p}_t^* - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})) \right]_j$. The result follows immediately.
(c) $\boldsymbol{\lambda}^* \leq \overline{\boldsymbol{\lambda}}^*$ is a consequence of the law of diminishing returns. Finally, the objective function of Problem (111) when the prices are set to their optimal values is piecewise linear in $\mathbf{x}^{-*}$, $\mathbf{x}^{+*}$, with breakpoints such that $\mathbf{F}'(\mathbf{x}^- - \mathbf{x}^+)_j$ becomes equal to $(\mathbf{p}_t^*)_j$, for some t and j, which concludes the proof. □

We also investigate the value of the optimal Lagrange multiplier, under the assumption that the optimal price vector is strictly within its bounds, i.e., $\mathbf{p}_{\min}^t < \mathbf{p}_t^* < \mathbf{p}_{\max}^t$ (componentwise) for all t.

Let $\tilde{\boldsymbol{\lambda}}$, resp. $\tilde{\mathbf{F}}$, $\tilde{\mathbf{C}}$ be the optimal Lagrange vector, resp. topology matrix, capacity vector, where the rows corresponding to inactive capacity constraints have been erased.

Theorem 4.5 (Optimal Lagrange multiplier) $\tilde{\boldsymbol{\lambda}}$ *satisfies:*

$$\tilde{\boldsymbol{\lambda}} = \left[\tilde{\mathbf{F}}\left(\sum_{t=0}^{T-1}\mathbf{B_t}\right)\tilde{\mathbf{F}}'\right]^{-1}\left\{\tilde{\mathbf{F}}\sum_{t=0}^{T-1}\left(\mathbf{a_t} + (\mathbf{diag}\,\widehat{\boldsymbol{\delta}}_\mathbf{t})\,\mathbf{sgn}[\mathbf{p_t^*} - \mathbf{F}'(\mathbf{x^{-*}} - \mathbf{x^{+*}})]\right) - 2\tilde{\mathbf{C}}\right\}. \quad (117)$$

Moreover, the Lagrange multiplier corresponding to inactive constraints is equal to 0.

Hence, the marginal value of the resources is piecewise linear in the (maximum) uncertainty.

Proof: The vector of resource utilization by the nominal demand vector is given by:

$$\begin{aligned}\mathbf{F}\left(\sum_{t=0}^{T-1}\overline{\mathbf{d}}_\mathbf{t}\right) &= \mathbf{F}\sum_{t=0}^{T-1}(\mathbf{a_t} - \mathbf{B_t}\,\mathbf{p_t^*}) \\ &= \frac{1}{2}\,\mathbf{F}\sum_{t=0}^{T-1}\left[\mathbf{a_t} + \mathbf{diag}(\widehat{\boldsymbol{\delta}}_\mathbf{t})\,\mathbf{sgn}(\mathbf{p_t^*} - \mathbf{F}'(\mathbf{x^{-*}} - \mathbf{x^{+*}})) - \mathbf{B_t}\,\mathbf{F}'\boldsymbol{\lambda}^*\right].\end{aligned} \quad (118)$$

We define $\tilde{\boldsymbol{\lambda}}$, resp. $\tilde{\mathbf{F}}$, $\tilde{\mathbf{C}}$ as the optimal Lagrange vector, resp. topology matrix, capacity vector, where the rows corresponding to inactive capacity constraints have been erased. Therefore, $\mathbf{F}'\boldsymbol{\lambda}^* = \tilde{\mathbf{F}}'\tilde{\boldsymbol{\lambda}}^*$. Then $\tilde{\mathbf{F}}\sum_{t=0}^{T-1}\mathbf{d_t}(\mathbf{p_t}) = \tilde{\mathbf{C}}$ creates a system of m' equations with m' unknowns, where m' is the number of tight constraints, which is solved in $\tilde{\boldsymbol{\lambda}}$ to yield Equation (117). □

Corollary 4.6 (Sensitivity analysis) *The effect on the Lagrange multiplier $\tilde{\boldsymbol{\lambda}}$ of increasing the maximum uncertainty $\widehat{\boldsymbol{\delta}}_t$ in the demand at time t is given by:*

$$\frac{\partial\tilde{\boldsymbol{\lambda}}}{\partial\widehat{\boldsymbol{\delta}}_t} = \mathbf{diag}[\mathbf{sgn}(\mathbf{p_t^*} - \mathbf{F}'(\mathbf{x^{-*}} - \mathbf{x^{+*}}))]\,\tilde{\mathbf{F}}'\left[\tilde{\mathbf{F}}\left(\sum_{t=0}^{T-1}\mathbf{B_t}\right)\tilde{\mathbf{F}}'\right]^{-1}. \quad (119)$$

In particular, it only depends on t through the sign of $\mathbf{p_t^*} - \mathbf{F}'(\mathbf{x^{-*}} - \mathbf{x^{+*}})$, that is, whether the prices at time t are above or below the reference level for the corresponding products.

Proof: Follows from differentiating Equation (117). □

We can now characterize the structure of the optimal prices as a function of the cost and uncertainty parameters, for a specific set of tight constraints and using our knowledge of which products are priced above or below their reference level throughout the time horizon. We assume that an increase in the uncertainty for any product j at any time period t has a greater impact (in absolute value) on the optimal price for product j at that same time t than on the opportunity cost $\tilde{\mathbf{F}}'\tilde{\boldsymbol{\lambda}}$ for that product. This is a natural assumption to make, as it models the fact that the direct impact of a change in the uncertainty is greater than an indirect one. We also assume that $\mathbf{p_t^{min}} < \mathbf{p_t^*} < \mathbf{p_t^{max}}$ componentwise for all t. To simplify notations, we denote by $\mathbf{sgn_t^*}$ the vector $\mathbf{sgn}(\mathbf{p_t^*} - \mathbf{F}'(\mathbf{x^{-*}} - \mathbf{x^{+*}}))$.

Corollary 4.7 (Optimal prices) *The optimal prices satisfy for all t:*

$$\mathbf{p_t^*} = \frac{1}{2}\left[\mathbf{B_t^{-1}}\left\{\mathbf{a_t} - (\mathbf{diag}\,\widehat{\boldsymbol{\delta}}_\mathbf{t})\,\mathbf{sgn_t^*}\right\} + \tilde{\mathbf{F}}^{-1}\left[\tilde{\mathbf{F}}\left(\sum_{s=0}^{T-1}\mathbf{B_t}\right)\tilde{\mathbf{F}}'\right]^{-1}\left\{\tilde{\mathbf{F}}\sum_{s=0}^{T-1}\left(\mathbf{a_s} + \mathbf{diag}(\widehat{\boldsymbol{\delta}}_\mathbf{s})\,\mathbf{sgn_s^*}\right) - 2\,\tilde{\mathbf{C}}\right\}\right]. \quad (120)$$

It follows that:

$$\frac{\partial \mathbf{p_t^*}}{\partial \widehat{\delta}_s} = \begin{cases} \frac{1}{2}\,\mathbf{diag}(\mathbf{sgn_s^*}) \left(\tilde{\mathbf{F}}^{-1} \left[\tilde{\mathbf{F}} \left(\sum_{s=0}^{T-1} \mathbf{B_s} \right) \tilde{\mathbf{F}}' \right]^{-1} \tilde{\mathbf{F}} \right)', & \textit{if } s \neq t, \\ -\frac{1}{2}\,\mathbf{diag}(\mathbf{sgn_t^*}) \left(\mathbf{B_t^{-1}} - \tilde{\mathbf{F}}^{-1} \left[\tilde{\mathbf{F}} \left(\sum_{s=0}^{T-1} \mathbf{B_s} \right) \tilde{\mathbf{F}}' \right]^{-1} \tilde{\mathbf{F}} \right)', & \textit{if } s = t. \end{cases} \tag{121}$$

In particular, the price of product j at time t, for some j and t, converges further towards the reference price of that product, $\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})$, when the uncertainty in the demand for product j at that same time period increases.

Proof: Follows from Theorem 4.4 (a) and Corollary 4.5. Equation (121) is obtained by differentiating Equation (120). To prove that the optimal price converges towards the reference price of that product when the uncertainty increases, we use the assumption that the impact of a change in the uncertainty in product j at time t yields a greater change (in absolute value) in the price of that product at that time period than on the opportunity cost of the product. In mathematical terms:

$$\frac{1}{2}\left|(\mathbf{B_t}^{-1})_{jj} - \left(\tilde{\mathbf{F}}^{-1} \left[\tilde{\mathbf{F}} \left(\sum_{s=0}^{T-1} \mathbf{B_s} \right) \tilde{\mathbf{F}}' \right]^{-1} \tilde{\mathbf{F}} \right)_{jj}\right| \geq \left| \left(\tilde{\mathbf{F}}^{-1} \left[\tilde{\mathbf{F}} \left(\sum_{s=0}^{T-1} \mathbf{B_s} \right) \tilde{\mathbf{F}}' \right]^{-1} \tilde{\mathbf{F}} \right)_{jj} \right|. \tag{122}$$

We use the fact that $|a+b| \leq |a| + |b|$ and that $(\mathbf{B_t^{-1}})_{jj} > 0$ to conclude that:

$$(\mathbf{B_t}^{-1})_{jj} \geq \left| \left(\tilde{\mathbf{F}}^{-1} \left[\tilde{\mathbf{F}} \left(\sum_{s=0}^{T-1} \mathbf{B_s} \right) \tilde{\mathbf{F}}' \right]^{-1} \tilde{\mathbf{F}} \right)_{jj} \right| \geq \left(\tilde{\mathbf{F}}^{-1} \left[\tilde{\mathbf{F}} \left(\sum_{s=0}^{T-1} \mathbf{B_s} \right) \tilde{\mathbf{F}}' \right]^{-1} \tilde{\mathbf{F}} \right)_{jj}. \tag{123}$$

It follows that $\partial p_{tj}^* / \partial \widehat{\delta}_{tj}$, as given by Equation (121), is nonpositive, resp. nonnegative, if the optimal price is above, resp. below the reference price. □

4.3 The Case of Multiplicative Uncertainty

4.3.1 Generalities

In the presence of multiplicative uncertainty, the demand vector at time t is defined as:

$$\mathbf{d_t} = (\mathbf{Id} + \mathbf{diag}\,\widehat{\delta}_\mathbf{t} \cdot \mathbf{diag}\,\mathbf{z_t})\,\overline{\mathbf{d}}_\mathbf{t}, \tag{124}$$

where Id is the identity matrix, the nominal demand $\overline{\mathbf{d}}_\mathbf{t}$ is given by Equation (99) and $\mathbf{z_t}$ is the vector of scaled deviations, or equivalently:

$$\mathbf{d_t} = \overline{\mathbf{d}}_\mathbf{t} + (\mathbf{diag}\,\widehat{\delta}_\mathbf{t} \cdot \mathbf{diag}\,\overline{\mathbf{d}}_\mathbf{t})\,\mathbf{z_t}. \tag{125}$$

The uncertainty set is defined as:

$$\mathcal{Z}(\mathbf{p}) = \left\{ \left| \sum_{t=0}^{T-1} \mathbf{F}\,\mathbf{diag}(\mathbf{a_t} - \mathbf{B_t}\,\mathbf{p_t})\,\mathbf{diag}\widehat{\delta}_\mathbf{t}\,\mathbf{z_t} \right| \leq \mathbf{\Delta},\ |\mathbf{z_t}| \leq \mathbf{e},\ \forall \mathbf{t} \right\}. \tag{126}$$

Since the results in this section are extensions of the methods presented in Sections 3 and 4.2 and rely on the same insights, we state most of them without proof.

4.3.2 Robust Formulation

Theorem 4.8 (Robust model) *The robust pricing problem in the presence of multiplicative uncertainty is formulated as a convex problem:*

$$\begin{array}{ll} \max & \sum_{t=0}^{T-1} \mathbf{p}_\mathbf{t}'(\mathbf{a_t} - \mathbf{B_t}\,\mathbf{p_t}) - \left[\mathbf{\Delta}'(\mathbf{x}^{-*} + \mathbf{x}^{+*}) + \sum_{t=0}^{T-1} (\mathbf{a_t} - \mathbf{B_t}\,\mathbf{p_t})'\,(\mathbf{diag}\,\widehat{\delta}_\mathbf{t})\,|\mathbf{p_t} - \mathbf{F}'(\mathbf{x}^- - \mathbf{x}^+)| \right] \\ s.t. & \sum_{t=0}^{T-1} \mathbf{F}\,\mathbf{B_t}\,\mathbf{p_t} \geq \sum_{t=0}^{T-1} \mathbf{F}\,\mathbf{a_t} - (\mathbf{C} + \mathbf{\Delta}), \\ & \mathbf{p}_\mathbf{t}^{\min} \leq \mathbf{p_t} \leq \mathbf{p}_\mathbf{t}^{\max},\ \forall t, \\ & \mathbf{x}^-, \mathbf{x}^+ \geq \mathbf{0}. \end{array} \tag{127}$$

Problem (127) is equivalent to the problem of finding the prices maximizing the worst-case revenue when there is only one resource and is slightly more conservative otherwise.

4.3.3 Optimal prices as a function of the Lagrange multipliers

Let $\mathbf{x}^{-*}$, $\mathbf{x}^{+*}$, $\mathbf{p}_\mathbf{t}^*$ be the optimal variables, and let $\tilde{\mathbf{p}}_\mathbf{t}^*$ be the optimal solution to Problem (127) without the box constraints on the prices. To simplify notations, we denote by $\mathbf{sgn}_\mathbf{t}^*$ the vector $\mathbf{diag}[\mathbf{sgn}(\mathbf{p}_\mathbf{t}^* - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*}))]$ and $\widehat{\mathbf{D}}_\mathbf{t}$ the diagonal matrix $\mathbf{diag}\,\widehat{\delta}_\mathbf{t} \cdot \mathbf{diag}\,\mathbf{sgn}_\mathbf{t}^*$. We have:

$$\tilde{\mathbf{p}}_\mathbf{t}^* = \frac{1}{2}\left\{ \mathbf{B}_\mathbf{t}^{-1}\,\mathbf{a_t} + [\mathbf{Id} - \widehat{\mathbf{D}}_\mathbf{t}]^{-1} \left[\mathbf{F}'\boldsymbol{\lambda} - \widehat{\mathbf{D}}_\mathbf{t}\,\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*}) \right] \right\} \tag{128}$$

This yields: $\mathbf{p}_\mathbf{t}^* = \max\left(\mathbf{p}_\mathbf{t}^{\min}, \min\{\mathbf{p}_\mathbf{t}^{\max}, \tilde{\mathbf{p}}_\mathbf{t}^*\}\right)$ and, if neither nominal nor robust prices reach their bounds:

$$\mathbf{p}_\mathbf{t}^* = \overline{\mathbf{p}}_\mathbf{t}^* + \frac{1}{2}\left[(\mathbf{Id} - \widehat{\mathbf{D}}_\mathbf{t})^{-1} \left\{ \mathbf{F}'\boldsymbol{\lambda}^* - \widehat{\mathbf{D}}_\mathbf{t}\,\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*}) \right\} - \mathbf{F}'\overline{\boldsymbol{\lambda}}^* \right],\ \forall t, \tag{129}$$

where $\overline{\boldsymbol{\lambda}}^*$, resp. $\boldsymbol{\lambda}^*$, is the Lagrange multiplier of the capacity constraints in the nominal, resp. robust, model. From the law of diminishing returns, we have: $\boldsymbol{\lambda}^* \leq \overline{\boldsymbol{\lambda}}^*$.

In what follows, we assume that $\mathbf{p}_\mathbf{t}^{\min} < \mathbf{p}_\mathbf{t}^* < \mathbf{p}_\mathbf{t}^{\max}$ componentwise for all t.

Theorem 4.9 (Optimal and nominal prices)
(a) If the reference price $[\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j$ of a product j is below the opportunity cost $[\mathbf{F}'\overline{\boldsymbol{\lambda}}^*]_j$ of that product in the deterministic model, then introducing uncertainty decreases the optimal price for product j at time t, i.e., $(\mathbf{p}_\mathbf{t}^*)_j < (\overline{\mathbf{p}}_\mathbf{t}^*)_j$, if and only if:

<u>either</u> the robust price of product j at time t is below the reference price,

or the robust price is above the reference price and the uncertainty is small enough, specifically:

$$(\widehat{\delta}_{\mathbf{t}})_j < \frac{[\mathbf{F}'(\overline{\boldsymbol{\lambda}}^* - \boldsymbol{\lambda}^*)]_j}{[\mathbf{F}'\overline{\boldsymbol{\lambda}}^* - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j}. \tag{130}$$

(b) If the reference price $[\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j$ of a product j is above the opportunity cost $[\mathbf{F}'\overline{\boldsymbol{\lambda}}^*]_j$ of that product in the deterministic model, then introducing uncertainty decreases the optimal price for product j at time t, i.e., $(\mathbf{p}_\mathbf{t}^*)_j < (\overline{\mathbf{p}}_\mathbf{t}^*)_j$, if and only if:

either the robust price of product j at time t is above the reference price,

or the robust price is below the reference price and the uncertainty is small enough, specifically:

$$(\widehat{\delta}_{\mathbf{t}})_j < \frac{[\mathbf{F}'(\overline{\boldsymbol{\lambda}}^* - \boldsymbol{\lambda}^*)]_j}{[\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*}) - \mathbf{F}'\overline{\boldsymbol{\lambda}}^*]_j}. \tag{131}$$

Proof: From Equation (129), $(\mathbf{p}_\mathbf{t}^*)_j < (\overline{\mathbf{p}}_\mathbf{t}^*)_j$ if and only if:

$$\left[(\mathbf{Id} - \widehat{\mathbf{D}}_\mathbf{t})^{-1}\left\{\mathbf{F}'\boldsymbol{\lambda}^* - \widehat{\mathbf{D}}_\mathbf{t}\,\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})\right\}\right]_j < [\mathbf{F}'\,\overline{\boldsymbol{\lambda}}^*]_j. \tag{132}$$

Since $\mathbf{Id} - \widehat{\mathbf{D}}_\mathbf{t}$ is a diagonal matrix, with the positive elements $1 - (\mathbf{sgn}_\mathbf{t}^*)_j\,(\widehat{\delta}_t)_j$ on the diagonal, Condition (132) is equivalent to:

$$[\mathbf{F}'\boldsymbol{\lambda}^*]_j - (\mathbf{sgn}_\mathbf{t}^*)_j\,(\widehat{\delta}_t)_j\,[\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j < \{1 - (\mathbf{sgn}_\mathbf{t}^*)_j\,(\widehat{\delta}_t)_j\}[\mathbf{F}'\,\overline{\boldsymbol{\lambda}}^*]_j. \tag{133}$$

This yields:

$$\left[\mathbf{F}'(\overline{\boldsymbol{\lambda}}^* - \boldsymbol{\lambda}^*)\right]_j > (\mathbf{sgn}_\mathbf{t}^*)_j\,(\widehat{\delta}_t)_j\left[\mathbf{F}'\overline{\boldsymbol{\lambda}}^* - \mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})\right]_j. \tag{134}$$

Since $\boldsymbol{\lambda}^* \leq \overline{\boldsymbol{\lambda}}^*$, the left-hand side is always nonnegative. The results follows from exploiting the side of the right-hand side and expressing Equation (134) as a bound on $(\widehat{\delta}_t)_j$. □

Hence, understanding the change in the optimal price of a product at a given time period, as uncertainty is introduced in the model, requires the knowledge of the following three quantities: the reference price (in the robust framework) and the opportunity costs of that product, both in the deterministic and robust models.

Note that low uncertainty guarantees that the price will decrease, but the case with high uncertainty is more complex, as the time periods where the prices increase are time periods with high amounts of uncertainty, although not all periods with high uncertainty will see a price increase. High uncertainty is quantified as $(\widehat{\delta}_t)_j$ exceeding a threshold that, in contrast with the additive case, depends neither on the *uncertainty for other products* at that time period, nor on the *time period* considered.

4.3.4 Optimal Lagrange multipliers

Let $\tilde{\boldsymbol{\lambda}}$, resp. $\tilde{\mathbf{F}}$, $\tilde{\mathbf{C}}$ be the optimal Lagrange vector, resp. topology matrix, capacity vector, where the rows corresponding to inactive capacity constraints (and hence zero Lagrange multipliers) have been erased. As before, $\widehat{\mathbf{D}}_{\mathbf{t}}$ is the diagonal matrix $\mathbf{diag}(\mathbf{sgn}_{\mathbf{t}}^{*})\,\mathbf{diag}(\widehat{\boldsymbol{\delta}}_{\mathbf{t}})$.

Theorem 4.10 (Optimal Lagrange multiplier and sensitivity analysis)
(a) $\tilde{\boldsymbol{\lambda}}$ satisfies:

$$\tilde{\boldsymbol{\lambda}} = \left[\tilde{\mathbf{F}}\left(\sum_{t=0}^{T-1}\mathbf{B}_{\mathbf{t}}\,(\mathbf{Id}-\widehat{\mathbf{D}}_{\mathbf{t}})^{-1}\right)\tilde{\mathbf{F}}'\right]^{-1}\left\{\tilde{\mathbf{F}}\sum_{t=0}^{T-1}\mathbf{a}_{\mathbf{t}} - 2\,\tilde{\mathbf{C}} + \tilde{\mathbf{F}}\left(\sum_{t=0}^{T-1}\mathbf{B}_{\mathbf{t}}\,(\mathbf{Id}-\widehat{\mathbf{D}}_{\mathbf{t}})^{-1}\,\widehat{\mathbf{D}}_{\mathbf{t}}\right)\mathbf{F}'(\mathbf{x}^{-*}-\mathbf{x}^{+*})\right\}. \tag{135}$$

Hence, the marginal value of the resources is piecewise rational in the (maximum) uncertainty.
(b) For all time periods t and products j, the vector $\dfrac{\partial\tilde{\boldsymbol{\lambda}}}{\partial(\widehat{\boldsymbol{\delta}}_{\mathbf{t}})_j}$ satisfies:

$$\frac{\partial\tilde{\boldsymbol{\lambda}}}{\partial(\widehat{\boldsymbol{\delta}}_{\mathbf{t}})_j} = \frac{(\mathbf{sgn}_{\mathbf{t}}^{*})_j}{[1-(\widehat{\boldsymbol{\delta}}_{\mathbf{t}})_j\,(\mathbf{sgn}_{\mathbf{t}}^{*})_j]^2}\,(\mathbf{F}'(\mathbf{x}^{-*}-\mathbf{x}^{+*})-\tilde{\mathbf{F}}'\tilde{\boldsymbol{\lambda}})_j\left[\tilde{\mathbf{F}}\left(\sum_{t=0}^{T-1}\mathbf{B}_{\mathbf{t}}\,(\mathbf{Id}-\widehat{\mathbf{D}}_{\mathbf{t}})^{-1}\right)\tilde{\mathbf{F}}'\right]^{-1}(\tilde{\mathbf{F}}\,\mathbf{B}_{\mathbf{t}})_{\cdot j}, \tag{136}$$

where $(\mathbf{A})_{\cdot j}$ is the j-th column of matrix $\mathbf{A}$.

Proof: The proof of (a) closely follows the proof of Theorem 4.5. We establish (b) by differentiating:

$$\left[\tilde{\mathbf{F}}\left(\sum_{t=0}^{T-1}\mathbf{B}_{\mathbf{t}}\,(\mathbf{Id}-\widehat{\mathbf{D}}_{\mathbf{t}})^{-1}\right)\tilde{\mathbf{F}}'\right]\tilde{\boldsymbol{\lambda}} = \tilde{\mathbf{F}}\sum_{t=0}^{T-1}\mathbf{a}_{\mathbf{t}} - 2\,\tilde{\mathbf{C}} + \tilde{\mathbf{F}}\left(\sum_{t=0}^{T-1}\mathbf{B}_{\mathbf{t}}\,(\mathbf{Id}-\widehat{\mathbf{D}}_{\mathbf{t}})^{-1}\,\widehat{\mathbf{D}}_{\mathbf{t}}\right)\mathbf{F}'(\mathbf{x}^{-*}-\mathbf{x}^{+*}), \tag{137}$$

obtained in (a), with respect to $(\widehat{\boldsymbol{\delta}}_{\mathbf{t}})_j$. □

Remark: Whether the marginal value of each resource increases or decreases as demand uncertainty for product j at time t increases depends on (i) whether the price for product j at time t is above or below the reference price of that product, and (ii) whether the reference price is above or below the opportunity cost of that product.

4.3.5 Optimal prices

The following theorem extends many of the results presented in the case with additive uncertainty; hence, we state it without proof.

Theorem 4.11 (Optimal prices and sensitivity analysis)
(a) The optimal auxiliary variables $\mathbf{x}^{-*}$, $\mathbf{x}^{+*}$ are such that, for each product j, there is a time period $s(j)$ which verifies: $(\mathbf{p}_{\mathbf{s(j)}}^{*})_j = [\mathbf{F}'(\mathbf{x}^{-*}-\mathbf{x}^{+*})]_j$.
(b) The impact of an increase in demand uncertainty on the optimal prices is such that:

$$
\frac{\partial (p_t^*)_i}{\partial (\widehat{\delta}_s)_j} = \begin{cases} \frac{1}{2}\left([\mathbf{Id} - \widehat{\mathbf{D}}_\mathbf{t}]^{-1} \tilde{\mathbf{F}}' \frac{\partial \tilde{\boldsymbol{\lambda}}}{\partial (\widehat{\boldsymbol{\delta}}_s)_j} \right)_i, \ \textit{if } s \neq t, \textit{ or } s = t \textit{ and } i \neq j, \\ \frac{1}{2}\left[\frac{(\mathbf{sgn}_\mathbf{t}^*)_j}{[1 - (\widehat{\boldsymbol{\delta}}_t)_j \, (\mathbf{sgn}_\mathbf{t}^*)_j]^2} \left\{ (\tilde{\mathbf{F}}'\tilde{\boldsymbol{\lambda}})_j - (\widehat{\boldsymbol{\delta}}_t)_j \, (\mathbf{sgn}_\mathbf{t}^*)_j \, [\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j \right\} \right] \\ +\frac{1}{2}[1 - (\widehat{\boldsymbol{\delta}}_t)_j \, (\mathbf{sgn}_\mathbf{t}^*)_j]^{-1} \left[\tilde{\mathbf{F}}' \frac{\partial \tilde{\boldsymbol{\lambda}}}{\partial (\widehat{\boldsymbol{\delta}}_t)_j} - (\mathbf{sgn}_\mathbf{t}^*)_j [\mathbf{F}'(\mathbf{x}^{-*} - \mathbf{x}^{+*})]_j \right], \ \textit{if } s = t \textit{ and } i = j. \end{cases} \tag{138}
$$

Remark: By reinjecting Equation (136) into Equation (138), we note that whether prices increase or decrease as uncertainty increases is determined (in part) by whether the prices are above or below their reference price.

5 Conclusions

We have presented an approach to multiperiod pricing of multiple products in the presence of capacitated resources and demand uncertainty that does not require the exact knowledge of the underlying probability distributions, which are difficult to estimate in practice, but instead models the random variables as uncertain parameters belonging to a polyhedral set. We established the existence of a parameter called the reference price for each product and have derived key insights into the impact of the uncertainty on the decision variables. In particular, under mild conditions, the optimal prices converge further towards the reference price as the uncertainty increases. This approach allows the decision-maker to gain a deeper understanding of the structure of the optimal solution.

References

[1] Ben-Tal, A., A. Nemirovski. 1998. Robust convex optimization. *Math. Oper. Res.* **23** 769-805.

[2] Ben-Tal, A., A. Nemirovski. 1999. Robust solutions to uncertain programs. *Oper. Res. Lett.* **25** 1-13.

[3] Ben-Tal, A., A. Nemirovski. 2000. Robust solutions of linear programming problems contaminated with uncertain data. *Math. Progr.* **88** 411-424.

[4] D. Bertsekas. Nonlinear programming, 2nd Ed. Athena Scientific, 1999.

[5] D. Bertsimas, M. Sim. The price of robustness. *Oper. Res.* **52** 35-53, 2004.

[6] D. Bertsimas, M. Sim. Robust discrete optimization and network flows. *Math. Prog.* **98** 49-71, 2003.

[7] D. Bertsimas, A. Thiele. A robust optimization approach to inventory theory. *Oper. Res.*, 2006.

[8] L. Eeckhoudt, C. Gollier, H. Schlesinger. The risk-averse (and prudent) newsboy. *Man. Sci.*, **41** 786-794, 1995.

[9] Y. Levin, J. McGill, M. Nediak. Risk in revenue management and dynamic pricing. Working paper, School of Business, Queen's University, 2005.

[10] Y. Levin, J. McGill, M. Nediak. Price guarantees in dynamic pricing and revenue management. Working paper, School of Business, Queen's University, 2005.

[11] A. Lim, G. Shanthikumar. Relative entropy, exponential utility and robust dynamic pricing. Working paper, Department of Industrial Engineering and Operations Research, University of California at Berkeley, Berkeley, CA.

[12] K. T. Talluri, G. J. van Ryzin. The theory and practice of revenue management. Kluwer Academic Publishers, 2004.

[13] A. Thiele. Robust dynamic optimization: a distribution-free approach. Working paper, Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA, 2005.