Robust Stochastic Programming with Uncertain Probabilities

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Abstract

Stochastic programming has traditionally assumed the exact knowledge of the underlying scenario probabilities. In practice however, such probabilities are difficult to estimate accurately and the optimal variables may be very sensitive to the assumed distributions. This motivates the use of minimax stochastic models, where the decision-maker minimizes the maximum expected cost over the set of possible probability distributions. We use ideas from the field of robust optimization over polyhedral uncertainty sets to reformulate the minimax stochastic programming problem as a single convex problem, and show that it can be solved efficiently using the traditional techniques developed to address sequential decision-making under uncertainty. In the two-stage setting, we describe how the Benders decomposition algorithm can be modified to solve the robust formulation. In the case of multiple stages, we build upon the recursive equations of dynamic programming to formulate an approach as tractable as the multi-stage stochastic problem where the probabilities are known exactly. Key contributions of this work are: (i) we show that the minimax approach is equivalent to the nominal stochastic programming problem with a penalty term, which measures the cost volatility due to the ambiguity on the probability estimates, and (ii) we provide deeper insights into the connection between the value of the recourse function in a given scenario and the worst-case probability associated with that outcome. The robust approach also allows the decision-maker to adjust the parameters defining the uncertainty set to better capture his own trade-off between ambiguity and performance.

1 Introduction

Stochastic programming is concerned with sequential decision-making under uncertainty. In this framework, information is revealed over time and the manager must take action before knowing the actual value of the random parameters. To evaluate the impact of his decisions on the overall costs, he considers an expected value criterion, motivated for instance by the law of large numbers when

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the problem is solved repeatedly. This framework was first proposed by Dantzig [12] in 1955 in the context of linear optimization with uncertain parameters, and has subsequently received much attention in the optimization community, as a modeling tool for a wide array of applications such as financial planning (Mulvey and Vladimirou [22], Ziemba and Vickson [36]) and capacity expansion (Louveaux and Smeers [21]), and as a problem with a special structure which allows for efficient solution techniques, e.g., decomposition methods (Benders [1], Ruszczyński [26]), sampling approximations (Shapiro and Homem-de-Mello [30], Linderoth et. al. [13]). For a thorough treatment of stochastic linear programming, the reader is referred to the numerous monographs available on the topic, for instance Birge and Louveaux [9], Kall and Wallace [19], Prékopa [24], or more recently Ruszczyński and Shapiro [27] and Kall and Mayer [20].

Despite the breadth of the stochastic programming literature, the vast majority of the research efforts mentioned above share one crucial assumption, specifically, that the underlying probability distributions are exactly known. In practice, distributions are computed using historical data, which results in estimation errors, and the optimal decision variables might vary widely depending on the assumed distribution. Efforts to address this issue have led researchers to focus on worst-case probabilities, yielding minimax stochastic models where the decision-maker minimizes the maximum expected cost over a family of probability distributions. This line of work was pioneered in the mid-1960s by Zackova-Dupacova [35, 14, 15, 16]. Other early attempts at modeling the decision-maker's limited knowledge of the distributions include Jagannathan [18], who studies stochastic linear programming with simple recourse when the first two moments of the distributions are known, and Birge and Wets [10], who focus on bounding and approximating stochastic problems. The main drawback of the stochastic minimax approach implemented by these researchers is that the numerical methods they propose (stochastic gradient in Ermoliev et. al. [17], bundle methods in Breton and El Hachem [11], a cutting plane algorithm in Riis and Andersen [25], to name a few), all require finding explicitly the worst-case probability for the current candidate solution at each step of the algorithm, and hence suffer from dimensionality problems. A notable exception is due to Shapiro and his co-authors (Shapiro and Ahmed [29], Shapiro and Kleywegt [31]) who use duality results in semi-infinite linear programming to obtain tractable models when the moments of the underlying distributions are known.

In this paper, we propose an approach based on robust optimization with polyhedral uncertainty sets to address the minimax stochastic programming problem. Robust optimization is concerned with minimizing the objective over the worst-case value of the parameters, and as such is the framework of choice to handle the decision-maker's aversion to ambiguity. (While risk refers to the volatility of a random variable, ambiguity denotes the uncertainty of a deterministic parameter. The reader is referred to Thiele [32] for a robust optimization approach to stochastic programming in the presence of risk aversion.) In this setting, the probability for each scenario is modeled by a confidence interval, centered at the nominal probability estimate. The scaled deviations of the probabilities from their nominal values might be subject to additional (linear) constraints to prevent overconservatism of the solution. A typical constraint we consider is the budget of uncertainty constraint, which was first proposed by Bertsimas and Sim [3] and limits the maximum number of parameters differing from their nominal values. Our choice of polyhedral uncertainty sets is motivated by the body of work documenting the tractability of the robust optimization approach and the theoretical insights available in this framework (Bertsimas and Sim [3, 5], Bertsimas and Thiele [6]). While other researchers have previously considered transition matrices of Markov decision processes lying in polytopes and implemented a min-max approach (Satia and Lave [28], White and Eldeib [34]), they have all proceeded by alternating between minimizing the cost and maximizing over the probability space which, as in the case of stochastic programming, quickly becomes intractable. This has led the research community to believe polyhedral uncertainty sets were not a suitable choice to address the robustness problem. The goal of this paper is to present a methodology that not only yields tractable formulations, but also extends far beyond the obvious (and conservative) choice of box uncertainty sets.

From a methodological standpoint, the use of strong duality in linear programming problems, which is central to robust optimization, connects this paper to the works by Nilim and El-Ghaoui [23] and Shapiro and Ahmed [29], and we recover the same results when the uncertainty set is the collection of the confidence intervals for each scenario, but the approach we develop here differs from [23] and [29] in several important ways:

(i) the choice of polyhedral uncertainty sets allows for deeper insights into the impact of ambiguity on the optimal decision variables,

(ii) we provide a simple characterization of the worst-case probabilities, and emphasize under which conditions the probability of a given outcome is increased or decreased from its nominal estimate in the robust model,

(iii) we extend the approach to multi-stage stochastic problems under finite horizon using dynamic programming arguments, which paves the way for a comprehensive and tractable theory of sequential decision-making under uncertainty and ambiguity using polyhedral sets.

The remainder of the paper is organized as follows. Section 2 develops the robust optimization approach for two-stage stochastic programming. Section 3 provides an extension to multi-stage models. Numerical experiments are presented in Section 4, and Section 5 contains concluding remarks.

2 Problem Setup

2.1 Traditional stochastic programming

The focus of this section is on two-stage stochastic linear optimization problems with fixed recourse and uncertainty-independent cost coefficients. The decision-maker must take a first set of decisions before uncertainty is realized ("here and now"), but can also adjust his strategy after additional information is revealed ("wait and see"). Possible outcomes of the random parameters between the two stages are represented as a finite number of scenarios, for which the traditional approach assumes that occurrence probabilities are known exactly. This problem can be formulated as a large-scale linear programming problem:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \sum_{\omega \in \Omega} \overline{\pi}_\omega \, \mathbf{q}'\mathbf{y}_\omega \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b}, \\ & \mathbf{T}_\omega \, \mathbf{x} + \mathbf{W}\,\mathbf{y}_\omega = \mathbf{h}_\omega, \; \forall \omega \in \Omega, \\ & \mathbf{x}, \; \mathbf{y}_\omega \geq \mathbf{0}, \; \forall \omega \in \Omega. \end{aligned} \tag{1}$$

with the following notations (Birge and Louveaux [9]):

$\mathbf{x}$:	the first-stage decision vector,
$\mathbf{y}$:	the second-stage decision vector,
$\mathbf{c}$:	the first-stage cost vector,
$\mathbf{q}$:	the second-stage cost vector,
$\mathbf{A}, \mathbf{b}$:	the parameters for the first-stage constraints,
$\mathbf{T}_\omega$, $\mathbf{W}$, $\mathbf{h}_\omega$:	the parameters for the coupled constraints,
Ω:	the set of possible scenarios,
$\overline{\pi}_\omega$:	the occurrence probability of scenario ω.

Problem (1) is equivalent to:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \sum_{\omega \in \Omega} \overline{\pi}_\omega \, Q(\mathbf{x}, \omega) \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{2}$$

where the recourse function Q is defined by:

$$\begin{aligned} Q(\mathbf{x}, \omega) = \quad \min \quad & \mathbf{q}'\mathbf{y}_\omega \\ \text{s.t.} \quad & \mathbf{W}\,\mathbf{y}_\omega = \mathbf{h}_\omega - \mathbf{T}_\omega \, \mathbf{x}, \\ & \mathbf{y}_\omega \geq \mathbf{0}, \end{aligned} \tag{3}$$

for any $\omega \in \Omega$ and $\mathbf{x} \in \{\mathbf{x} | \mathbf{A}\,\mathbf{x} = \mathbf{b}, \; \mathbf{x} \geq \mathbf{0}\}$. A powerful solution technique to such problems, first proposed by Benders [1] in the 1960s, is based on delayed constraint generation. We briefly review it below for completeness, as it provides the foundation for the algorithm we propose in Section 2.2 to solve the robust problem. We first need the following lemma. We assume that $\{\mathbf{p} \,|\, \mathbf{W}'\mathbf{p} \leq \mathbf{q}\}$ is nonempty and has at least one extreme point.

Lemma 2.1 (Recourse function)
(a) $Q(\mathbf{x},\omega)$ is finite if and only if $(\mathbf{w}^{\mathbf{j}})'(\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x}) \leq 0$ for any extreme ray $\mathbf{w}^{\mathbf{j}}$, $j = 1,\ldots,J$, of $\{\mathbf{p} \,|\, \mathbf{W}'\mathbf{p} \leq \mathbf{q}\}$.
(b) Whenever $Q(\mathbf{x},\omega)$ is finite, it is equal to:

$$Q(\mathbf{x},\omega) = \max_{k=1,\ldots,K} (\mathbf{p}^{\mathbf{k}})'(\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x}), \tag{4}$$

where the $\mathbf{p}^{\mathbf{k}}$, $k = 1,\ldots,K$, are the extreme points of $\{\mathbf{p} \,|\, \mathbf{W}'\mathbf{p} \leq \mathbf{q}\}$.

Proof: The dual of Problem (3) is:

$$\begin{array}{ll} \max & (\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x})'\mathbf{p} \\ \text{s.t.} & \mathbf{W}'\mathbf{p} \leq \mathbf{q}. \end{array} \tag{5}$$

Since $\mathbf{p} = \mathbf{0}$ is feasible, Problem (5) either has a finite optimal value, in which case the optimal solution is reached at an extreme point of the feasible set $\{\mathbf{p} \,|\, \mathbf{W}'\mathbf{p} \leq \mathbf{q}\}$, or Problem (5) has an infinite optimal value, in which case there exists an extreme ray $\mathbf{w}$ yielding a direction of (strict) increase in the objective. □

Using Lemma 2.1, Problem (2) can immediately be reformulated as the master problem:

$$\begin{array}{lll} \min & \mathbf{c}'\mathbf{x} + \sum_{\omega\in\Omega} \overline{\pi}_\omega \, \theta_\omega & \\ \text{s.t.} & \mathbf{A}\,\mathbf{x} = \mathbf{b}, & \\ & \theta_\omega \geq (\mathbf{p}^{\mathbf{k}})'(\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x}), & \forall k = 1,\ldots,K,\ \forall \omega \in \Omega, \\ & (\mathbf{w}^{\mathbf{j}})'(\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x}) \leq 0, & \forall j = 1,\ldots,J,\ \forall \omega \in \Omega, \\ & \mathbf{x} \geq \mathbf{0}, & \end{array} \tag{6}$$

The fact that only a few of the constraints in Problem (6) will be tight at optimality motivates the use of delayed constraint generation techniques, which are summarized in Algorithm 2.2.

Algorithm 2.2 (Benders decomposition) *Each iteration I of the Benders decomposition algorithm consists in the following steps:*

(i) *Solve the relaxed master problem, which includes the constraints $\theta_\omega \geq (\mathbf{p}^{\mathbf{k}})'(\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x})$ and $(\mathbf{w}^{\mathbf{j}})'(\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x}) \leq 0$ for only a small number of extreme points and extreme rays. Let $\tilde{\mathbf{x}}$ and $\tilde{\theta}_\omega$, $\omega = 1,\ldots,\Omega$ be the optimal solution to the relaxed master problem.*

(ii) *Compute* $Q(\tilde{\mathbf{x}}, \omega)$ *for each* $\omega \in \Omega$.

(iii) *If* $\tilde{\theta}_\omega \geq Q(\tilde{\mathbf{x}}, \omega)$ *for all* $\omega \in \Omega$, *stop: we have found the optimal solution to Problem (2). Otherwise:*

- *If* $Q(\tilde{\mathbf{x}}, \omega)$ *is finite but* $\tilde{\theta}_\omega < Q(\tilde{\mathbf{x}}, \omega)$, *there exists an extreme point* $\mathbf{p}^{\mathbf{k}(\mathbf{I})}$ *such that* $\tilde{\theta}_\omega < (\mathbf{p}^{\mathbf{k}(\mathbf{I})})'(\mathbf{h}_\omega - \mathbf{T}_\omega \tilde{\mathbf{x}})$. *Add the constraint:*

$$\theta_\omega \geq (\mathbf{p}^{\mathbf{k}(\mathbf{I})})'(\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x}) \tag{7}$$

  to the relaxed master problem and reiterate.

- *If* $Q(\tilde{\mathbf{x}}, \omega)$ *is infinite, i.e., the problem has an unbounded optimal objective (which corresponds to an infeasible primal problem), there exists an extreme ray* $\mathbf{w}^{\mathbf{j}(\mathbf{I})}$ *such that:* $(\mathbf{w}^{\mathbf{j}(\mathbf{I})})'(\mathbf{h}_\omega - \mathbf{T}_\omega \tilde{\mathbf{x}}) > 0$. *Add the constraint:*

$$(\mathbf{w}^{\mathbf{j}(\mathbf{I})})'(\mathbf{h}_\omega - \mathbf{T}_\omega \mathbf{x}) \leq 0 \tag{8}$$

  to the relaxed master problem and reiterate.

The reader is referred to Birge and Louveaux [9] for more details on this algorithm.

2.2 The robust optimization approach

In practice, scenario probabilities are very difficult to estimate accurately, and the optimal solution in the traditional stochastic programming framework might vary widely for small changes in these estimates. This calls for an approach that will protect system performance despite the ambiguity on the actual probabilities. The approach that we propose here is based on robust optimization techniques and solves the worst-case stochastic programming problem over a set of possible probability values. We model the probabilities as uncertain parameters belonging to confidence intervals centered at the mean, that is:

$$\pi_\omega = \overline{\pi}_\omega + \widehat{\pi}_\omega z_\omega, \ -1 \leq z_\omega \leq 1, \ \forall \omega \in \Omega, \tag{9}$$

where $\overline{\pi}_\omega$ is the nominal estimate of the probability of scenario ω, $\widehat{\pi}_\omega$ is the half-length of the interval and therefore measures the uncertainty on the estimate, and z_ω is called the scaled deviation. We assume that $\widehat{\pi}_\omega \leq \overline{\pi}_\omega$ for all $\omega \in \Omega$. Since all the elements in the uncertainty set are probabilities, as well as the nominal estimates, the scaled deviations must satisfy:

$$\sum_{\omega \in \Omega} \widehat{\pi}_\omega z_\omega = 0. \tag{10}$$

Finally, the decision-maker might wish to enforce additional constraints on the probabilities (for instance to bound the mean), which are represented as:

$$\mathbf{F}\,\mathbf{z} \le \mathbf{g}. \tag{11}$$

This model also allows for equality constraints. Let $\mathcal{Z}$ be the resulting polyhedral uncertainty set:

$$\mathcal{Z} = \left\{ \widehat{\boldsymbol{\pi}}'\mathbf{z} = 0,\ -\mathbf{e} \le \mathbf{z} \le \mathbf{e},\ \mathbf{F}\,\mathbf{z} \le \mathbf{g} \right\}. \tag{12}$$

The nominal probabilities $\overline{\pi}_\omega$, $\omega \in \Omega$, must belong to the uncertainty set, therefore $\mathbf{g} \ge \mathbf{0}$. The robust counterpart to Problem (2) can then be formulated as:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \sum_{\omega\in\Omega} \overline{\pi}_\omega\, Q(\mathbf{x},\omega) + \max_{\mathbf{z}\in\mathcal{Z}} \sum_{\omega\in\Omega} \widehat{\pi}_\omega\, Q(\mathbf{x},\omega)\, z_\omega \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \ge \mathbf{0}. \end{aligned} \tag{13}$$

Hence, the robust optimization approach introduces a penalty term, which measures the upside variability of the objective function when the probabilities differ from their estimates.

Let t be the number of rows in matrix $\mathbf{F}$.

Theorem 2.3 (Robust problem)
(a) The robust problem can be formulated as a convex programming problem with $t+1$ new variables: a scalar r and a vector $\mathbf{s}$ of size t, and no new constraint besides the nonnegativity of the vector $\mathbf{s}$:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \sum_{\omega\in\Omega} \overline{\pi}_\omega\, Q(\mathbf{x},\omega) + \sum_{\omega\in\Omega} \left| \widehat{\pi}_\omega \left[Q(\mathbf{x},\omega) - r \right] - (\mathbf{F}'\mathbf{s})_\omega \right| + \mathbf{g}'\mathbf{s} \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x},\, \mathbf{s} \ge \mathbf{0}. \end{aligned} \tag{14}$$

(b) The robust problem can also be formulated as a large-scale linear programming problem:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \mathbf{g}'\mathbf{s} + \sum_{\omega\in\Omega} \theta_\omega \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \theta_\omega \ge (\overline{\pi}_\omega + \widehat{\pi}_\omega) \left[(\mathbf{p^k})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\mathbf{x}) \right] - \widehat{\pi}_\omega\, r - (\mathbf{F}'\mathbf{s})_\omega, && \forall k,\ \forall \omega \in \Omega, \\ & \theta_\omega \ge (\overline{\pi}_\omega - \widehat{\pi}_\omega) \left[(\mathbf{p^k})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\mathbf{x}) \right] + \widehat{\pi}_\omega\, r + (\mathbf{F}'\mathbf{s})_\omega, && \forall k,\ \forall \omega \in \Omega, \\ & (\mathbf{w^j})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\mathbf{x}) \le 0, && \forall j,\ \forall \omega \in \Omega, \\ & \mathbf{x},\, \mathbf{s} \ge \mathbf{0}, \end{aligned} \tag{15}$$

where the $\mathbf{p^k}$ and $\mathbf{w^j}$ are, respectively, the extreme points and extreme rays of $\{\mathbf{p} \mid \mathbf{W}'\mathbf{p} \le \mathbf{q}\}$.

Proof: The auxiliary problem:

$$\begin{aligned} \max \quad & \sum_{\omega\in\Omega} \widehat{\pi}_\omega \, Q(\mathbf{x},\omega)\, z_\omega \\ \text{s.t.} \quad & \sum_{\omega\in\Omega} \widehat{\pi}_\omega \, z_\omega = 0, \\ & -1 \le z_\omega \le 1, \\ & \mathbf{F}\,\mathbf{z} \le \mathbf{g} \end{aligned} \tag{16}$$

is feasible (since $\mathbf{z} = \mathbf{0}$ is in the set) and bounded. Hence, strong duality allows us to rewrite Problem (16) as:

$$\begin{aligned} \min \quad & \sum_{\omega\in\Omega} (r_\omega^+ + r_\omega^-) + \mathbf{g}'\mathbf{s} \\ \text{s.t.} \quad & \widehat{\pi}_\omega \, r + r_\omega^+ - r_\omega^- + (\mathbf{F}'\mathbf{s})_\omega = \widehat{\pi}_\omega \, Q(\mathbf{x},\omega), \quad \forall \omega \in \Omega, \\ & r_\omega^+, \, r_\omega^-, \, \mathbf{s} \ge \mathbf{0}, \qquad \forall \omega \in \Omega. \end{aligned} \tag{17}$$

At r and $\mathbf{s}$ given, Problem (17) is equivalent to minimizing $r_\omega^+ + r_\omega^-$ for each $\omega \in \Omega$, subject to $r_\omega^+ - r_\omega^- = \widehat{\pi}_\omega \, [Q(\mathbf{x},\omega) - r] - (\mathbf{F}'\mathbf{s})_\omega$ and $r_\omega^+, \, r_\omega^- \ge 0$. Therefore, at optimality $r_\omega^+ + r_\omega^-$ is equal to $|\widehat{\pi}_\omega \, [Q(\mathbf{x},\omega) - r] - (\mathbf{F}'\mathbf{s})_\omega]|$. Reinjecting into Problem (13) yields Problem (14) immediately. The feasible set is convex and the objective is convex as the maximum of convex functions. Hence, Problem (14) is convex. It can be rewritten as:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \mathbf{g}'\mathbf{s} + \sum_{\omega\in\Omega} \theta_\omega \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \theta_\omega \ge (\overline{\pi}_\omega + \widehat{\pi}_\omega)\, Q(\mathbf{x},\omega) - \widehat{\pi}_\omega \, r - (\mathbf{F}'\mathbf{s})_\omega, \ \forall \omega \in \Omega, \\ & \theta_\omega \ge (\overline{\pi}_\omega - \widehat{\pi}_\omega)\, Q(\mathbf{x},\omega) + \widehat{\pi}_\omega \, r - (\mathbf{F}'\mathbf{s})_\omega, \ \forall \omega \in \Omega, \\ & \mathbf{x}, \, \mathbf{s} \ge \mathbf{0}. \end{aligned} \tag{18}$$

Problem (15) follows from rewriting the recourse function using Lemma 2.1. □

Hence, the robust min-max problem can be reformulated as a *single* large-scale linear programming problem, which removes the need to proceed iteratively, alternating between minimizing Problem (13) over $\mathbf{x}$ and maximizing over $\mathbf{z}$, and drastically simplifies the solution technique.

Corollary 2.4 (Worst-case probabilities)

(a) If at optimality the cost of the second-stage recourse in scenario ω exceeds a threshold, specifically:

$$Q(\mathbf{x}^*,\omega) > r^* + \frac{(\mathbf{F}'\mathbf{s}^*)_\omega}{\widehat{\pi}_\omega}, \tag{19}$$

the probability of scenario ω in the robust problem is equal to its highest possible value, $\overline{\pi}_\omega + \widehat{\pi}_\omega$. (Equivalently, $z_\omega^* = 1$.)

(b) If at optimality the cost of the second-stage recourse in scenario ω falls below that same threshold:

$$Q(\mathbf{x}^*, \omega) < r^* + \frac{(\mathbf{F}'\mathbf{s}^*)_\omega}{\widehat{\pi}_\omega}, \tag{20}$$

the probability of scenario ω in the robust problem is equal to its lowest possible value, $\overline{\pi}_\omega - \widehat{\pi}_\omega$. (Equivalently, $z^*_\omega = -1$.)

(c) Otherwise, i.e., if:

$$Q(\mathbf{x}^*, \omega) = r^* + \frac{(\mathbf{F}'\mathbf{s}^*)_\omega}{\widehat{\pi}_\omega}, \tag{21}$$

the probability of scenario ω in the robust problem is found by complementarity slackness using the optimal value of r^* and $\mathbf{s}^*$.

Proof: is an immediate consequence of Theorem 2.3(a), where we study the sign of $|\widehat{\pi}_\omega\,[Q(\mathbf{x}, \omega) - r] - (\mathbf{F}'\mathbf{s})_\omega|$. □

Remark: The threshold is not identical for all scenarios in Ω. Instead, it consists in a component, r^*, common to all scenarios, plus a component $\frac{(\mathbf{F}'\mathbf{s}^*)_\omega}{\widehat{\pi}_\omega}$, that is scenario-specific. The optimal r^* and $\mathbf{s}^*$ vary as a function of the $\widehat{\pi}_\omega$, $\omega \in \Omega$, except in the case where $\widehat{\pi}_\omega = \widehat{\pi}$ for all $\omega \in \Omega$. In that case, the threshold decreases as $1/\widehat{\pi}$ when the ambiguity on the probabilities increases.

We now provide a modification of the traditional Benders decomposition algorithm in order to solve the robust problem (15). Since this is an immediate extension of Algorithm 2.2, we give it without further justification.

Algorithm 2.5 (Modified Benders decomposition) *Each iteration I of the modified Benders decomposition algorithm consists in the following steps:*

(i) *Solve the relaxed master problem, which includes the constraints $\theta_\omega \geq (\overline{\pi}_\omega + \widehat{\pi}_\omega)\,[(\mathbf{p^k})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\mathbf{x})] - \widehat{\pi}_\omega\, r - (\mathbf{F}'\mathbf{s})_\omega$, $\theta_\omega \geq (\overline{\pi}_\omega - \widehat{\pi}_\omega)\,[(\mathbf{p^k})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\mathbf{x})] + \widehat{\pi}_\omega\, r + (\mathbf{F}'\mathbf{s})_\omega$ and $(\mathbf{w^j})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\mathbf{x}) \leq 0$ for only a small number of extreme points and extreme rays. Let $(\tilde{\mathbf{x}}, \tilde{r}, \tilde{\mathbf{s}}, (\tilde{\theta}_\omega)_{\omega\in\Omega})$ be the optimal solution to the relaxed master problem.*

(ii) *Compute $Q(\tilde{\mathbf{x}}, \omega)$ for each $\omega \in \Omega$.*

(iii) *If $\tilde{\theta}_\omega \geq \overline{\pi}_\omega\, Q(\tilde{\mathbf{x}}, \omega) + |\widehat{\pi}_\omega\,[Q(\tilde{\mathbf{x}}, \omega) - r] - (\mathbf{F}'\tilde{\mathbf{s}})_\omega|$ for all $\omega \in \Omega$, stop: we have found the optimal solution to Problem (15). Otherwise:*

- *If $Q(\tilde{\mathbf{x}}, \omega)$ is finite but $\tilde{\theta}_\omega < \overline{\pi}_\omega\, Q(\tilde{\mathbf{x}}, \omega) + |\widehat{\pi}_\omega\,[Q(\tilde{\mathbf{x}}, \omega) - r] - (\mathbf{F}'\tilde{\mathbf{s}})_\omega|$, there exists an extreme point $\mathbf{p^{k(I)}}$ such that:*

  Either $\tilde{\theta}_\omega < (\overline{\pi}_\omega + \widehat{\pi}_\omega)\,[(\mathbf{p^{k(I)}})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\tilde{\mathbf{x}})] - \widehat{\pi}_\omega\,\tilde{r} - (\mathbf{F}'\tilde{\mathbf{s}})_\omega$. *Add the constraint:*

$$\tilde{\theta}_\omega \geq (\overline{\pi}_\omega + \widehat{\pi}_\omega)\,[(\mathbf{p^{k(I)}})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\tilde{\mathbf{x}})] - \widehat{\pi}_\omega\,\tilde{r} - (\mathbf{F}'\tilde{\mathbf{s}})_\omega \tag{22}$$

to the relaxed master problem and reiterate.

Or $\tilde{\theta}_\omega < (\overline{\pi}_\omega - \widehat{\pi}_\omega)\,[(\mathbf{p}^{\mathbf{k(I)}})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\tilde{\mathbf{x}})] + \widehat{\pi}_\omega\,\tilde{r} + (\mathbf{F}'\tilde{\mathbf{s}})_\omega$. *Add the constraint:*

$$\tilde{\theta}_\omega \geq (\overline{\pi}_\omega - \widehat{\pi}_\omega)\,[(\mathbf{p}^{\mathbf{k(I)}})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\tilde{\mathbf{x}})] + \widehat{\pi}_\omega\,\tilde{r} + (\mathbf{F}'\tilde{\mathbf{s}})_\omega \tag{23}$$

to the relaxed master problem and reiterate.

- *If* $Q(\tilde{\mathbf{x}},\omega)$ *is infinite, i.e., the problem has an unbounded optimal objective (which corresponds to an infeasible primal problem), there exists an extreme ray* $\mathbf{w}^{\mathbf{j(I)}}$ *such that:* $(\mathbf{w}^{\mathbf{j(I)}})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\tilde{\mathbf{x}}) > 0$. *Add the constraint:*

$$(\mathbf{w}^{\mathbf{j(I)}})'(\mathbf{h}_\omega - \mathbf{T}_\omega\,\mathbf{x}) \leq 0 \tag{24}$$

to the relaxed master problem and reiterate.

2.3 Suboptimality of the nominal solution

In this section, we quantify the amount of ambiguity required to make the nominal solution $\overline{\mathbf{x}}$, i.e., the solution that is optimal when the probabilities are equal to their nominal values, suboptimal in the robust formulation. The approach relies on the following theorem.

Theorem 2.6 (Nominal solution) $\overline{\mathbf{x}}$ *is the optimal solution to Problem (13) if and only if, for any probability vector* $\boldsymbol{\pi}$ *in the uncertainty set, there exists* $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ *such that* $\boldsymbol{\mu} \geq \mathbf{0}$, $\boldsymbol{\mu}'\overline{\boldsymbol{x}} = \mathbf{0}$ *and:*

$$-\mathbf{c} + \mathbf{A}'\boldsymbol{\lambda} + \boldsymbol{\mu} \in \partial Q(\overline{\mathbf{x}}), \tag{25}$$

where $Q(\overline{\mathbf{x}}) = \sum_{\omega\in\Omega} \pi_\omega Q(\overline{\mathbf{x}},\omega)$.

Proof: See Birge and Louveaux [9]. □

Note that the subdifferentials $Q(\overline{\mathbf{x}},\omega)$ are known at the nominal solution $\overline{\mathbf{x}}$, so that the only uncertainty is on the coefficients of the linear combination $\sum_{\omega\in\Omega} \pi_\omega Q(\overline{\mathbf{x}},\omega)$.

To take full advantage of Theorem 2.6, it is necessary to characterize the subgradient of the expected recourse function, $\partial Q(\overline{\mathbf{x}})$. Corollary 2.7 provides an application to the case of simple recourse, i.e., recourse which is exerted componentwise on the second-stage vector. Problem (1) can then be reformulated as:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \sum_{\omega\in\Omega} \overline{\pi}_\omega\,[(\mathbf{q}^+)'\mathbf{y}_\omega^+ + (\mathbf{q}^-)'\mathbf{y}_\omega^-] \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b}, \\ & \mathbf{T}_\omega\,\mathbf{x} + \mathbf{y}_\omega^+ - \mathbf{y}_\omega^- = \mathbf{h}_\omega,\ \forall\omega\in\Omega, \\ & \mathbf{x},\ \mathbf{y}_\omega^+,\ \mathbf{y}_\omega^- \geq \mathbf{0},\ \forall\omega\in\Omega. \end{aligned} \tag{26}$$

We assume that the half-length $\widehat{\pi}_\omega$ of the uncertainty interval in scenario ω can be decomposed as $\alpha \cdot \widehat{\pi}^0_\omega$, where the parameter α is common to all scenarios. This is motivated by the construction of β-confidence intervals (with $0 < \beta < 1$) for large data samples, since it is well-known that in this case, $\widehat{\pi}^0_\omega = s/\sqrt{n}$ with s the observed sample standard deviation and n the number of data points, and α is such that: $P(-\alpha \le Z \le \alpha) = \beta$ with Z the standard normal random variable. (For instance, $\alpha = 2.576$ for $\beta = 99\%$.) It is also possible to set $\widehat{\pi}_\omega$ at the same fraction α of the nominal probabilities $\overline{\pi}_\omega$, in which case $0 < \alpha < 1$ and $\widehat{\pi}^0_\omega = \overline{\pi}_\omega$ for all ω in Ω.

Let m be the size of $\mathbf{h}_\omega$ for all $\omega \in \Omega$. The expected recourse function $Q(\mathbf{x})$ is equal to $\sum_{i=1}^m Q_i(\mathbf{x})$, with:

$$Q_i(\mathbf{x}) = \sum_{\omega\in\Omega} \pi_\omega \max\left\{q_i^+[(\mathbf{h}_\omega)_i - (\mathbf{T}_\omega\,\mathbf{x})_i], q_i^-[(\mathbf{T}_\omega\,\mathbf{x})_i - (\mathbf{h}_\omega)_i]\right\}. \tag{27}$$

Therefore, the subdifferential of the expected recourse function at $\overline{\mathbf{x}}$ for any probability vector is of the type: $\left\{\sum_{i=1}^m \boldsymbol{\nu}_i,\ \boldsymbol{\nu}_i^- \le \boldsymbol{\nu}_i \le \boldsymbol{\nu}_i^+,\ \forall i\right\}$, where $\{\boldsymbol{\nu}_i | \boldsymbol{\nu}_i^- \le \boldsymbol{\nu}_i \le \boldsymbol{\nu}_i^+\}$ is the subdifferential of Q_i at $\overline{\mathbf{x}}$. We will use the following notations:

$$\boldsymbol{\nu}_i^- = \overline{\boldsymbol{\nu}}_i^- + \alpha \sum_{\omega\in\Omega} \widehat{\pi}^0_\omega\, z_\omega\, \widehat{\boldsymbol{\nu}}_{i\omega}^-, \tag{28}$$

$$\boldsymbol{\nu}_i^+ = \overline{\boldsymbol{\nu}}_i^+ + \alpha \sum_{\omega\in\Omega} \widehat{\pi}^0_\omega\, z_\omega\, \widehat{\boldsymbol{\nu}}_{i\omega}^+. \tag{29}$$

In particular, $\{\boldsymbol{\nu}_i | \overline{\boldsymbol{\nu}}_i^- \le \boldsymbol{\nu}_i \le \overline{\boldsymbol{\nu}}_i^+\}$ is the subdifferential of Q_i at $\overline{\mathbf{x}}$ when the probabilities are equal to their nominal values. The exact expressions of $\boldsymbol{\nu}_i^-$ and $\boldsymbol{\nu}_i^+$ are not particularly important here and we omit them for the sake of clarity.

Corollary 2.7 (Simple recourse) *Let* $(\mathbf{u}^j, \mathbf{v}^{+j}, \mathbf{v}^{-j}, w^j)$, $j = 1, \dots, J$, *be the extreme rays of the set defined by:*

$$\mathbf{A}\,\mathbf{u} = \mathbf{0}, \tag{30}$$

$$\mathbf{u} + w\,\overline{\mathbf{x}} \le \mathbf{0}, \tag{31}$$

$$\mathbf{u} + \mathbf{v}_i^+ - \mathbf{v}_i^- = \mathbf{0},\ \forall i = 1, \dots, m, \tag{32}$$

$$\mathbf{v}_i^+, \mathbf{v}_i^- \ge \mathbf{0},\ \forall i = 1, \dots, m. \tag{33}$$

$\overline{\mathbf{x}}$ *is* not *the optimal solution to the robust counterpart of Problem (2) if and only if:*

$$\alpha > \min_{j=1,\dots,J} \frac{|\overline{m}_j|}{\widehat{m}_j}, \tag{34}$$

with:

$$\overline{m}^j = \mathbf{c}'\mathbf{u}^j + \sum_{i=1}^{m} \left[-(\overline{\boldsymbol{\nu}}_i^+)'\mathbf{v}_i^+ + (\overline{\boldsymbol{\nu}}_i^-)'\mathbf{v}_i^- \right], \tag{35}$$

$$\widehat{m}^j = \max_{\mathbf{z}\in\mathcal{Z}} \sum_{\omega\in\Omega} \widehat{\pi}_\omega^0 \, z_\omega \sum_{i=1}^{m} \left[-\left(\widehat{\boldsymbol{\nu}}_{i\omega}^+\right)' \mathbf{v}_i^+ + \left(\widehat{\boldsymbol{\nu}}_{i\omega}^-\right)' \mathbf{v}_i^- \right]. \tag{36}$$

Proof: From Theorem 2.6, $\overline{\mathbf{x}}$ is the optimal solution to the robust counterpart of Problem (2) if and only if, for any probability vector $\boldsymbol{\pi}$ in the uncertainty set, the problem:

$$\begin{aligned} \min \quad & 0 \\ \text{s.t.} \quad & -\mathbf{c} + \mathbf{A}'\boldsymbol{\lambda} + \boldsymbol{\mu} = \sum_{i=1}^{m} \boldsymbol{\nu}_i, \\ & \boldsymbol{\nu}_i^- \le \boldsymbol{\nu}_i \le \boldsymbol{\nu}_i^+, \ \forall i, \\ & \boldsymbol{\mu}'\overline{\boldsymbol{x}} = \mathbf{0}, \\ & \boldsymbol{\mu} \ge \mathbf{0}. \end{aligned} \tag{37}$$

is feasible. From Farkas' lemma, this is equivalent to all extreme rays of the dual of Problem (37) having a nonpositive objective value, for all probability vectors $\boldsymbol{\pi}$ in the uncertainty set. (Note that the set has at least one extreme ray.) Therefore, $\overline{\mathbf{x}}$ is not optimal for the robust problem if and only if there exists at least one extreme ray having a positive objective value for at least one probability vector in the uncertainty set. This can be reformulated as:

$$\exists j,\ \mathbf{z}\in\mathcal{Z} \text{ s.t. } \mathbf{c}'\mathbf{u}^j + \sum_{i=1}^{m} \left[-\left(\overline{\boldsymbol{\nu}}_i^+ + \alpha \sum_{\omega\in\Omega} \widehat{\pi}_\omega^0 \mathbf{z}_\omega \widehat{\boldsymbol{\nu}}_{i\omega}^+ \right)' \mathbf{v}_i^+ + \left(\overline{\boldsymbol{\nu}}_i^- + \alpha \sum_{\omega\in\Omega} \widehat{\pi}_\omega^0 \mathbf{z}_\omega \widehat{\boldsymbol{\nu}}_{i\omega}^- \right)' \mathbf{v}_i^- \right] > 0, \tag{38}$$

which is equivalent to:

$$\exists j \text{ s.t. } \mathbf{c}'\mathbf{u}^j + \sum_{i=1}^{m} \left[-(\overline{\boldsymbol{\nu}}_i^+)'\mathbf{v}_i^+ + (\overline{\boldsymbol{\nu}}_i^-)'\mathbf{v}_i^- \right] + \alpha \max_{\mathbf{z}\in\mathcal{Z}} \sum_{\omega\in\Omega} \widehat{\pi}_\omega^0 \, z_\omega \sum_{i=1}^{m} \left[-\left(\widehat{\boldsymbol{\nu}}_{i\omega}^+\right)' \mathbf{v}_i^+ + \left(\widehat{\boldsymbol{\nu}}_{i\omega}^-\right)' \mathbf{v}_i^- \right] > 0. \tag{39}$$

The term $\mathbf{c}'\mathbf{u}^j + \sum_{i=1}^{m} \left[-(\overline{\boldsymbol{\nu}}_i^+)'\mathbf{v}_i^+ + (\overline{\boldsymbol{\nu}}_i^-)'\mathbf{v}_i^- \right]$ is nonpositive from the optimality of $\overline{\mathbf{x}}$ for the stochastic problem solved for $\overline{\boldsymbol{\pi}}$, and the term $\max_{\mathbf{z}\in\mathcal{Z}} \sum_{\omega\in\Omega} \widehat{\pi}_\omega^0 \, z_\omega \sum_{i=1}^{m} \left[-\left(\widehat{\boldsymbol{\nu}}_{i\omega}^+\right)' \mathbf{v}_i^+ + \left(\widehat{\boldsymbol{\nu}}_{i\omega}^-\right)' \mathbf{v}_i^- \right]$ is positive from the definition of the uncertainty set. □

2.4 Examples of uncertainty sets

In this section, we focus on two specific polyhedral uncertainty sets: the well-known box model of uncertainty where each probability belongs to a confidence interval (Section 2.4.1), and a modified box model limiting the total number of probabilities that can equal their worst-case values, which is modeled by a budget of uncertainty constraint (Section 2.4.2).

2.4.1 The box model of uncertainty

Here, the decision-maker only knows confidence intervals for each probability:

$$\mathcal{Z} = \left\{ \sum_{\omega \in \Omega} \widehat{\pi}_\omega \, z_\omega = 0, \; -1 \leq z_\omega \leq 1, \; \forall \omega \in \Omega \right\}. \tag{40}$$

Theorem 2.8 (Robust problem with box uncertainty set) *The robust problem for the box model of uncertainty can be formulated as a convex programming problem with* 1 *new variable and no new constraint:*

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \sum_{\omega \in \Omega} \overline{\pi}_\omega \, Q(\mathbf{x}, \omega) + \sum_{\omega \in \Omega} \widehat{\pi}_\omega \, |Q(\mathbf{x}, \omega) - r| \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{41}$$

Proof: Immediate application of Theorem 2.3 to the uncertainty set defined by Equation (40). □

Analysis: The robust optimization approach penalizes the dispersion of the second-stage costs $Q(\mathbf{x}, \omega)$, $\omega \in \Omega$, away from a target value r. This dispersion is measured by a weighted combination of the $|Q(\mathbf{x}, \omega) - r|$, where the weights are the half-lengths $\widehat{\pi}_\omega$ of the confidence intervals. Hence, the robust stochastic programming problem seeks to decrease this variability with only a moderate increase in the expected objective. If the second-stage recourse cost in scenario ω, $Q(\mathbf{x}, \omega)$, is strictly greater than r ("high-cost scenario"), the probability of the scenario is increased from $\overline{\pi}_\omega$ to $\overline{\pi}_\omega + \widehat{\pi}_\omega$. Similarly, if the second-stage recourse cost in scenario ω, $Q(\mathbf{x}, \omega)$, is strictly smaller than r ("low-cost scenario"), the probability of the scenario is decreased from $\overline{\pi}_\omega$ to $\overline{\pi}_\omega - \widehat{\pi}_\omega$.

A drawback of this uncertainty model is that it will needlessly yield very conservative solutions, since it is unlikely in practice that every single scenario probability will be equal to its worst-case value. This justifies modifying the uncertainty set to exclude such particularly adverse outcomes, which is the purpose of Section 2.4.2.

2.4.2 The box model of uncertainty with budget constraint

In this model, the decision-maker knows confidence intervals for each probability and limits the number of scenarios in which the probabilities are equal to their worst-case value:

$$\mathcal{Z} = \left\{ \sum_{\omega \in \Omega} \widehat{\pi}_\omega \, z_\omega = 0, \; \sum_{\omega \in \Omega} |z_\omega| = \Gamma, \; -1 \leq z_\omega \leq 1, \; \forall \omega \in \Omega \right\}. \tag{42}$$

The integer parameter Γ is called the budget of uncertainty. This uncertainty set can be formulated as the following polyhedron:

$$\mathcal{Z} = \left\{ \sum_{\omega\in\Omega} \widehat{\pi}_\omega \left(z_\omega^+ - z_\omega^-\right) = 0,\ \sum_{\omega\in\Omega} (z_\omega^+ + z_\omega^-) = \Gamma,\ 0 \le z_\omega^- \le 1,\ 0 \le z_\omega^+ \le 1,\ \forall\omega\in\Omega \right\}. \tag{43}$$

Theorem 2.9 (Robust problem with budget constraint) *The robust problem for the box model of uncertainty with budget constraint can be formulated as a convex programming problem with* 2 *new variables and no new constraint (beside nonnegativity):*

$$\begin{aligned} \min\quad & \mathbf{c}'\mathbf{x} + \sum_{\omega\in\Omega} \overline{\pi}_\omega\, Q(\mathbf{x},\omega) + \sum_{\omega\in\Omega} \max(s, \widehat{\pi}_\omega\, |Q(\mathbf{x},\omega) - r|) + (\Gamma - |\Omega|)\, s \\ s.t.\quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x},\ s \ge \mathbf{0}. \end{aligned} \tag{44}$$

Proof: Applying Theorem 2.3 to the uncertainty set defined by Equation (43) yields the following robust problem:

$$\begin{aligned} \min\quad & \mathbf{c}'\mathbf{x} + \sum_{\omega\in\Omega} \overline{\pi}_\omega\, Q(\mathbf{x},\omega) + \sum_{\omega\in\Omega} [\max(0, \widehat{\pi}_\omega\, [Q(\mathbf{x},\omega) - r] - s) + \max(0, -\widehat{\pi}_\omega\, [Q(\mathbf{x},\omega) - r] - s)] + \Gamma\, s \\ \text{s.t.}\quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x},\ s \ge \mathbf{0}, \end{aligned} \tag{45}$$

which can be reformulated as Problem (44) upon noticing that:

$$\max(0, \widehat{\pi}_\omega\, [Q(\mathbf{x},\omega) - r] - s) + \max(0, -\widehat{\pi}_\omega\, [Q(\mathbf{x},\omega) - r] - s) = \max(0, \widehat{\pi}_\omega\, |Q(\mathbf{x},\omega) - r| - s). \tag{46}$$

□

Analysis: For this modified uncertainty set, the optimal r minimizes the total weighted deviation of the *outlier* scenarios only, which are defined such that the distance of $Q(\mathbf{x},\omega)$ from r is greater than or equal to $s/\widehat{\pi}_\omega$. By studying the slope of the objective function in s, it is easy to see that an optimal s will be equal to the Γ-th greatest $\widehat{\pi}_\omega|Q(\mathbf{x},\omega) - r|$. Scenarios for which $Q(\mathbf{x},\omega)$ is strictly greater than $r + s/\widehat{\pi}_\omega$ see their probability increase from $\overline{\pi}_\omega$ to $\overline{\pi}_\omega + \widehat{\pi}_\omega$, and scenarios for which $Q(\mathbf{x},\omega)$ is strictly smaller than $r + s/\widehat{\pi}_\omega$ see their probability decrease from $\overline{\pi}_\omega$ to $\overline{\pi}_\omega - \widehat{\pi}_\omega$. Note that the threshold depends on the scenario through the parameter $\widehat{\pi}_\omega$, so that high-cost scenarios will see their probability increase only if the ambiguity $\widehat{\pi}_\omega$ is large enough.

2.5 Data-driven stochastic programming

Finally, we analyze the robust stochastic programming problem when the decision-maker incorporates the historical data directly into the formulation without estimating scenario probabilities. This is appropriate in particular when there are not enough data points to compute meaningful confidence intervals, and when the decision-maker wants to completely remove the risk of estimation errors, which always affect the nominal probabilities.

The robust optimization approach consists here in removing some of the past outcomes from the data set. Discarding data points is justified in this framework because of the decision-maker's aversion to ambiguity; hence he considers some outcomes to have been "too favorable to ever happen again." Deciding which realizations to remove depends on the first-stage decision variables implemented, and an attractive feature of the robust model proposed below is that the decision-maker does not have to decide which scenarios to delete before solving the robust problem: the data analysis and optimization process are performed simultaneously through a *single* mathematical programming problem. The reader is referred to Bertsimas and Thiele [7] for various applications of this technique.

In the data-driven framework, $|\Omega|$ is the number of past data points and the nominal probabilities are equal to $1/|\Omega|$. (An outcome realized multiple times is treated as multiple indistinguishable outcomes.) Allowing for the deletion of data points corresponds to setting the worst-case (lowest) occurrence probability to 0, i.e., $\widehat{\pi}_\omega = 1/|\Omega|$ for all $\omega \in \Omega$. In a model with symmetric confidence intervals, this implies that the highest occurrence probability is set to to $2/|\Omega|$ for each scenario.

2.5.1 General model

Plugging in the values of $\overline{\pi}_\omega$ and $\widehat{\pi}_\omega$ for all $\omega \in \Omega$, Problem (14) is reformulated as:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \mathbf{g}'\mathbf{s} + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \left(Q(\mathbf{x}, \omega) + \left| Q(\mathbf{x}, \omega) - r - |\Omega| \, (\mathbf{F}'\mathbf{s})_\omega \right| \right) \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b}, \\ & \mathbf{x}, \, \mathbf{s} \geq \mathbf{0}. \end{aligned} \tag{47}$$

This can again be solved using Benders' decomposition. We remark that at optimality, r is the median of $Q(\mathbf{x}, \omega) - |\Omega| \, (\mathbf{F}'\mathbf{s})_\omega$. A scenario ω such that $Q(\mathbf{x}, \omega) - |\Omega| \, (\mathbf{F}'\mathbf{s})_\omega$ is strictly above, resp. below, its median sees its occurrence probability increase from $1/|\Omega|$ to $2/|\Omega|$, resp. decrease from $1/|\Omega|$ to 0. Therefore, scenarios ω for which the second-stage cost $Q(\mathbf{x}, \omega)$ is strictly smaller than the (scenario-dependent) threshold $r + |\Omega| \, (\mathbf{F}'\mathbf{s})_\omega$ are removed from the data set.

2.5.2 Box model

The box model described in Section 2.4.1 allows for a very simple formulation of the robust problem (47) with only one new variable:

$$\begin{aligned}
\min \quad & \mathbf{c}'\mathbf{x} + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \left(Q(\mathbf{x},\omega) + |Q(\mathbf{x},\omega) - r| \right) \\
\text{s.t.} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}.
\end{aligned} \tag{48}$$

The robust formulation only takes into account the scenarios for which the second-stage cost $Q(\mathbf{x},\omega)$ exceeds the optimal r, i.e., the median of all $Q(\mathbf{x},\omega)$, $\omega \in \Omega$. In other words, it doubles the occurrence probability of scenarios corresponding to large second-stage cost, as the realization of those scenarios would affect the total cost negatively, and removes the scenarios which have smaller impact on that cost. Here, this leads to half the scenarios being discarded.

2.5.3 Box model with budget constraint

The box model with budget of uncertainty constraint presented in Section 2.4.2 yields the following robust model in data-driven stochastic programming:

$$\begin{aligned}
\min \quad & \mathbf{c}'\mathbf{x} + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} \left[Q(\mathbf{x},\omega) + \max(|\Omega|\, s, |Q(\mathbf{x},\omega) - r|) \right] + (\Gamma - |\Omega|)\, s \\
\text{s.t.} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b}, \\
& \mathbf{x}, s \geq \mathbf{0}.
\end{aligned} \tag{49}$$

At optimality, r is the median of the $Q(\mathbf{x},\omega)$ corresponding to the Γ greatest $|Q(\mathbf{x},\omega) - r|$ and s is equal to the Γ-th greatest $|Q(\mathbf{x},\omega) - r|$. If the second-stage optimal cost in scenario ω exceeds a threshold, specifically, if $Q(\mathbf{x},\omega) > r + |\Omega|\, s$, the probability of scenario ω is doubled to $2/|\Omega|$. If the second-stage optimal cost in scenario ω falls below the threshold $r + |\Omega|\, s$, scenario ω is discarded from the computation of the expected recourse.

Problem (49) is equivalent to:

$$\begin{aligned}
\min \quad & \mathbf{c}'\mathbf{x} + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} Q(\mathbf{x},\omega) + \frac{1}{|\Omega|} \sum_{\omega \in \Omega(\mathbf{x},\Gamma)} |Q(\mathbf{x},\omega) - r| \\
\text{s.t.} \quad & \mathbf{A}\,\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0},
\end{aligned} \tag{50}$$

where the set $\Omega(\mathbf{x},\Gamma)$ has Γ elements and is defined by:

$$\Omega(\mathbf{x},\Gamma) = \left\{ \omega \in \Omega \text{ s.t. } |Q(\mathbf{x},\omega) - r| \geq |Q(\mathbf{x},\cdot) - r|_{(\Gamma)} \right\}, \tag{51}$$

with the convention $a_{(1)} \geq \ldots \geq a_{(|\Omega|)}$, so that $a_{(\Gamma)}$ is the Γ-th greatest number among $a_1, \ldots, a_{|\Omega|}$. This further illustrates the impact of the budget of uncertainty Γ on the conservatism of the solution.

Specifically, Γ affects the choice of the scenarios that are summed upon in the penalty term:

- If Γ is equal to zero, $\Omega(\mathbf{x}, \Gamma)$ is empty and the formulation is equivalent to the traditional stochastic programming problem with known probabilities.
- If Γ is equal to $|\Omega|$, $\Omega(\mathbf{x}, \Gamma)$ is Ω and the formulation is equivalent to the robust stochastic programming problem for the box model of ambiguity.

Choosing Γ between 0 and Ω allows for a trade-off between the two extremes of performance with no protection against parameter uncertainty ($\Gamma = 0$) and full protection against parameter uncertainty with mediocre performance ($\Gamma = |\Omega|$).

2.6 Special case: single-stage stochastic programming with quadratic objective function

When the recourse function is available in closed form, the underlying stochastic programming problem is considered to have a single stage. We illustrate the impact of the robust optimization approach in this setting when the problem is of the type:

$$\min_{\mathbf{x}} E_{\mathbf{w}}[f(\mathbf{x} - \mathbf{w})], \tag{52}$$

with f quadratic: $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}'\mathbf{A}\mathbf{x} + \mathbf{b}'\mathbf{x}$. The random vector $\mathbf{w}$ can take $|\Omega|$ values, and the probability of value $\mathbf{w}_\omega$ is denoted by π_ω, where the probabilities are subject to ambiguity. The optimal solution $\overline{\mathbf{x}}$ to the stochastic problem where the probabilities are equal to their nominal estimates satisfies:

$$\sum_{\omega \in \Omega} \overline{\pi}_\omega\, f'(\overline{\mathbf{x}} - \mathbf{w}_\omega) = 0, \tag{53}$$

or equivalently:

$$\overline{\mathbf{x}} = \sum_{\omega \in \Omega} \overline{\pi}_\omega\, \mathbf{w}_\omega - \mathbf{A}^{-1}\,\mathbf{b}. \tag{54}$$

Similarly, the optimal solution $\mathbf{x}^*$ to the robust problem satisfies:

$$\mathbf{x}^* = \sum_{\omega \in \Omega} [\overline{\pi}_\omega + \widehat{\pi}_\omega\, z^*_\omega]\, \mathbf{w}_\omega - \mathbf{A}^{-1}\,\mathbf{b}, \tag{55}$$

where the z^*_ω have been found by complementarity slackness upon solving Problem (14). It follows that:

$$\mathbf{x}^* = \overline{\mathbf{x}} + \sum_{\omega \in \Omega} \widehat{\pi}_\omega\, z^*_\omega\, \mathbf{w}_\omega. \tag{56}$$

Therefore, the change in the optimal solution is equal to the difference between the mean of the random parameter in the robust model and its mean in the nominal model. When $\widehat{\pi}_\omega = \widehat{\pi}$ for all

$\omega \in \Omega$ for some $\widehat{\pi}$, the optimal z_ω^* do not depend on the amount of ambiguity $\widehat{\pi}$, and the difference vector $\mathbf{x}^* - \overline{\mathbf{x}}$ is linear in $\widehat{\pi}$.

3 Multi-stage stochastic programming

3.1 Problem setup

Section 2 focused on two-stage stochastic programming, which assumes that information is revealed at once between the two time periods. In that framework, the first-stage decision variables are chosen before observing the value taken by the uncertainty, while the second-stage decision variables are implemented after the random parameters have become known. In practice however, information is revealed sequentially and the manager must decide in stages, using the partial information that he has collected up until that time period to guide his behavior. This motivates the use of multi-stage stochastic programming. With T the total number of stages, the multi-stage stochastic programming problem takes the following form when the probability distributions of the random events are known exactly (see Birge and Louveaux [9]):

$$\begin{aligned}
\min \quad & \mathbf{c}_1' \mathbf{x}_1 + E_{\boldsymbol{\xi}_2}[\; \min \quad \mathbf{c}_2' \mathbf{x}_2(\zeta_2) + \ldots + E_{\boldsymbol{\xi}_T}[\min \mathbf{c}_T' \mathbf{x}_T(\zeta_T)]] \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{s.t } \mathbf{W}_T \mathbf{x}_T(\zeta_T) = \mathbf{h}_T(\zeta_T) - \mathbf{T}_{T-1}(\zeta_T)\mathbf{x}_{T-1}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad \mathbf{x}_T(\zeta_T) \geq \mathbf{0}, \\
& \qquad\qquad \text{s.t.} \quad \mathbf{W}_2 \mathbf{x}_2(\zeta_2) = \mathbf{h}_2(\zeta_2) - \mathbf{T}_1(\zeta_2)\mathbf{x}_1, \\
& \qquad\qquad \qquad \mathbf{x}_2(\zeta_2) \geq \mathbf{0}, \\
\text{s.t.} \quad & \mathbf{W}_1 \mathbf{x}_1 = \mathbf{h}_1, \\
& \mathbf{x}_1 \geq \mathbf{0},
\end{aligned} \tag{57}$$

where $\boldsymbol{\xi}_t$ represents the random events yet to be realized between time periods t and T, and $\boldsymbol{\zeta}_t$ the history up to time t. The other notations are straightforward. Note that the multi-stage stochastic programming framework still assumes finite discrete distributions. Let $\boldsymbol{\omega}_t$ be the random event to be realized at time period t, i.e., $\boldsymbol{\zeta}_{t+1} = (\boldsymbol{\zeta}_t, \boldsymbol{\omega}_{t+1})$. To develop the robust counterpart to multi-stage stochastic programming, we approach Problem (57) from a dynamic programming perspective and reformulate the problem using a series of recursive equations:

$$\begin{aligned}
Q_t(\mathbf{x}_{t-1}, \boldsymbol{\zeta}_t) = \min \quad & \mathbf{c}_t' \mathbf{x}_t + E_{\omega_{t+1}}[Q_{t+1}(\mathbf{x}_t, \boldsymbol{\zeta}_{t+1})] \\
\text{s.t.} \quad & \mathbf{W}_t \mathbf{x}_t = \mathbf{h}_t(\boldsymbol{\zeta}_t) - \mathbf{T}_{t-1}(\boldsymbol{\zeta}_t)\mathbf{x}_{t-1}, \\
& \mathbf{x}_t \geq \mathbf{0},
\end{aligned} \tag{58}$$

for $t < T$, with the terminal condition:

$$Q_T(\mathbf{x_{T-1}}, \zeta_T) = \min \quad \mathbf{c'_T x_T}$$
$$\text{s.t.} \quad \mathbf{W_T x_T = h_T(\zeta_T) - T_{T-1}(\zeta_T) x_{T-1}}, \tag{59}$$
$$\mathbf{x_T \geq 0},$$

The reader is referred to Bertsekas [2] for a thorough treatment of this technique when the underlying distributions are given.

3.2 The robust problem

We formulate here the robust counterpart to Equation (58) (Equation (59) is not subject to probability ambiguity.) In line with Bertsimas and Thiele [6] and El Ghaoui and Nilim [23], we consider an uncertainty set at each time period. (In Bertsimas and Thiele [6], this uncertainty set is defined for the outcomes of the random variables rather than their probabilities.) Specifically, if the history at time period t is $\boldsymbol{\zeta_t}$ and the space of possible outcomes is $\Omega_{t+1}(\boldsymbol{\zeta_t})$, the uncertainty set of the scaled deviations of the probabilities π_ω^{t+1} is given by:

$$\mathcal{Z}_{t+1}(\zeta_t) = \left\{ (\widehat{\boldsymbol{\pi}}_{t+1}(\boldsymbol{\zeta_t}))' z_{t+1} = 0, \ -\mathbf{e} \leq \mathbf{z_{t+1}} \leq \mathbf{e}, \ \mathbf{F\, z_{t+1}} \leq \mathbf{g} \right\}. \tag{60}$$

It is also possible to define one single uncertainty set which encompasses all the remaining time periods. The decision-maker must then determine how much of the total ambiguity (deviation from the nominal probabilities allowed by the uncertainty set) to allocate to the current time period, and how much to keep for later. While this approach (moderately) increases the complexity of the problem, it has excellent practical performance in terms of the trade-off between risk and return. The reader is referred to Thiele [33] for details.

Using the uncertainty set (60), we define the robust counterpart to Equation (58) as:

$$\begin{aligned} Q_t(\mathbf{x_{t-1}}, \zeta_t) = \min \quad & \mathbf{c'_t x_t} + \sum_{\omega_{t+1} \in \Omega_{t+1}(\zeta_t)} \overline{\pi}_{\omega_{t+1}} Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) \\ & + \max_{\mathbf{z_{t+1}} \in \mathcal{Z}_{t+1}(\zeta_t)} \sum_{\omega_{t+1} \in \Omega_{t+1}(\zeta_t)} \widehat{\pi}_{\omega_{t+1}} z_{\omega_{t+1}} Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) \\ \text{s.t.} \quad & \mathbf{W_t x_t = h_t(\zeta_t) - T_{t-1}(\zeta_t) x_{t-1}}, \\ & \mathbf{x_t \geq 0}, \end{aligned} \tag{61}$$

Theorem 3.1 (Robust dynamic programming) *The recursive equation at time t in robust multistage stochastic programming is given by:*

$$
\begin{aligned}
Q_t(\mathbf{x_{t-1}}, \zeta_t) = \min \quad & \mathbf{c_t'}\,\mathbf{x_t} + \sum_{\omega_{t+1}\in\Omega_{t+1}(\zeta_t)} \overline{\pi}_{\omega_{t+1}} Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) + \mathbf{g'}\mathbf{s_t} \\
& \qquad + \sum_{\omega_{t+1}\in\Omega_{t+1}(\zeta_t)} \left|\widehat{\pi}_{\omega_{t+1}}[Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) - r_t] - (\mathbf{F'}\mathbf{s_t})_{\omega_{t+1}}\right| \\
s.t. \quad & \mathbf{W_t}\,\mathbf{x_t} = \mathbf{h_t}(\zeta_\mathbf{t}) - \mathbf{T_{t-1}}(\zeta_\mathbf{t})\,\mathbf{x_{t-1}}, \\
& \mathbf{x_t},\ \mathbf{s_t} \geq \mathbf{0},
\end{aligned}
\tag{62}
$$

The minimization problem remains convex.

Proof: Follows from Theorem 2.3. We prove that the function to be minimized remains convex by induction, since the maximum of convex functions is convex. □

Remark: The fact that convexity is preserved in the robust framework ensures that the optimum of Problem (62) can be found efficiently using standard convex optimization packages. Moreover, the impact of incorporating ambiguity into the formulation mainly affects the objective of the function, through a penalty term $\mathbf{g'}\mathbf{s_t} + \sum_{\omega_{t+1}\in\Omega_{t+1}(\zeta_t)} |\widehat{\pi}_{\omega_{t+1}}[Q_{t+1}(\mathbf{x_t}, \omega_{t+1}) - r_t] - (\mathbf{F'}\mathbf{s_t})_{\omega_{t+1}}|$, rather than through the constraints (besides the nonnegativity of the variables $\mathbf{s_t}$). Therefore, the robust problem has the same structure as its nominal counterpart where the probabilities are known exactly.

As in Section 2.2, outcomes ω_{t+1} for which $Q_{t+1}(\mathbf{x_t^*}, \zeta_{t+1}) > r_t^* + (\mathbf{F'}\mathbf{s_t})_{\omega_{t+1}}/\widehat{\pi}_{\omega_{t+1}}$, resp. $Q_{t+1}(\mathbf{x_t^*}, \zeta_{t+1}) < r_t^* + (\mathbf{F'}\mathbf{s_t})_{\omega_{t+1}}/\widehat{\pi}_{\omega_{t+1}}$, see their occurrence probability increase, resp. decrease, by $\widehat{\pi}_{\omega_{t+1}}$ in the robust model. The expressions for the two examples of uncertainty sets are straightforward extensions of the work presented in Section 2 and we do not repeat them here.

A key feature of the multi-stage model is that, for time horizons large enough, probabilities (evaluated at time 1) of the end scenarios (at time T) can differ significantly from their nominal estimates, even when the value of the parameter $\widehat{\pi}_{\omega_{t+1}}$ is small for each outcome ω_{t+1} for all t. In particular, a sample path $\xi_T = (\omega_1, \dots, \omega_T)$ for which $z^*_{\omega_t} = 1$, for all $t = 1, \dots, T$, see its probability increase by a factor $\prod_{t=1}^{T}\left(1 + \widehat{\pi}_{\omega_{t+1}}/\overline{\pi}_{\omega_{t+1}}\right)$. With $T = 10$ and $\widehat{\pi}_{\omega_{t+1}}/\overline{\pi}_{\omega_{t+1}} = 0.2$ for each t, this corresponds to a six-fold increase. Similarly, a sample path $\xi_T = (\omega_1, \dots, \omega_T)$ for which $z^*_{\omega_t} = -1$, for all $t = 1, \dots, T$, see its probability decrease by a factor $\left[\prod_{t=1}^{T}\left(1 - \widehat{\pi}_{\omega_{t+1}}/\overline{\pi}_{\omega_{t+1}}\right)\right]^{-1}$. With the previous numerical values, this corresponds to a nine-fold decrease.

This underlines the importance to incorporate robustness into multi-stage stochastic programming, as small amounts of ambiguity at each time period will be compounded to yield probabilities of the end scenarios differing widely from their nominal estimates.

Another compounding effect appears in the recursive equation as the sum of two ambiguity-related terms. If we denote by E, resp. $\overline{E}$, the expectation taken with respect to the robust, resp. nominal, probabilities, $\overline{Q}_t$ the optimal cost-to-go function in the nominal problem at time t, and $\mathbf{X_t}(\mathbf{x_{t-1}}, \zeta_t)$

the feasible set for the recourse, Equation (62) becomes:

$$Q_t(\mathbf{x_{t-1}}, \zeta_t) = \min_{\mathbf{x_t} \in \mathbf{X_t}(\mathbf{x_{t-1}}, \zeta_t)} \left[\mathbf{c}_t' \, \mathbf{x_t} + E \, Q_{t+1}(\mathbf{x_t}, \zeta_{t+1})\right] \tag{63}$$

$$= \min_{\mathbf{x_t} \in \mathbf{X_t}(\mathbf{x_{t-1}}, \zeta_t)} \Big[\mathbf{c}_t' \, \mathbf{x_t} + \overline{E} \, \overline{Q}_{t+1}(\mathbf{x_t}, \zeta_{t+1}) \tag{64}$$

$$+ E \, Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) - \overline{E} \, \overline{Q}_{t+1}(\mathbf{x_t}, \zeta_{t+1})\Big]. \tag{65}$$

The penalty term can thus be decomposed as the sum of:

(i) $E \, Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) - \overline{E} \, Q_{t+1}(\mathbf{x_t}, \zeta_{t+1})$, the difference in the expected value of the cost-to-go function at the next time period in the robust and nominal models, and

(ii) $\overline{E} \left[Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) - \overline{Q}_{t+1}(\mathbf{x_t}, \zeta_{t+1})\right]$, the expected value (computed with the nominal probabilities) of the difference in the robust and nominal cost-to-go function at the next time period.

The second term, which did not appear in the two-period model of Section 2, creates a compound effect as follows: if, for a given $\mathbf{x_{t-1}}$ and ζ_t, there exists a scalar a_{t+1} (where we have dropped the reference to the state and the history for the clarity of the exposition) such that:

$$Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) \geq \overline{Q}_{t+1}(\mathbf{x_t}, \zeta_{t+1}) + a_{t+1} \text{ for all } \mathbf{x_t} \text{ and } \omega_t, \tag{66}$$

and if there exists a scalar b_{t+1} such that:

$$E \, Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) \geq \overline{E} \, Q_{t+1}(\mathbf{x_t}, \zeta_{t+1}) + b_{t+1} \text{ for all } \mathbf{x_t} \text{ and } \omega_t, \tag{67}$$

then we have:

$$Q_t(\mathbf{x_{t-1}}, \zeta_t) \geq \overline{Q}_t(\mathbf{x_{t-1}}, \zeta_t) + a_{t+1} + b_{t+1}. \tag{68}$$

Note that the scalars a_{t+1} and b_{t+1} are nonnegative by definition of the robust optimization approach. It follows that the lower bound on the difference between the nominal and robust cost-to-go functions increases (as we proceed *backward* in time) at each step of the recursion. Consequently, this bound will be the largest at time 0.

3.3 Special case: Decision trees

In this section, we apply the robust framework to the special case where the multi-stage problem can be modeled using a decision tree. Since we have already assumed that the set of possible scenarios is finite, this only requires the additional assumption of a finite action space. Decision trees represent a fundamental tool of decision-making under uncertainty taught in many management science courses, in particular in business schools, and one of their most attractive features is that the optimal solution can be computed with only a calculator, without resorting to optimization software. This is particularly appealing for students or practitioners who have limited experience with mathematical programming software. The purpose of this section is to outline how the robust decision model can also be solved without a computer, to foster its practical implementation in the classroom and beyond.

We focus on the two uncertainty sets described in Section 2.4, specifically, the simple box model (where only confidence intervals on the probabilities are known) and the box model with budget of uncertainty (where we further bound the number of parameters that can deviate from their nominal value.) First, we state the algorithm.

3.3.1 High-level algorithm

The procedure to solve the robust problem, where the probabilities at each event node i belong to the uncertainty set $\mathcal{Z}_i$, is the one used for any traditional decision tree problem once the robust probabilities have been found (see Bertsimas and Freund [4]):

1. Start with the end nodes of the decision tree, and evaluate each node as follows:

   Event node Compute the *robust* expected monetary value (EMV) of the node by computing the weighted average of the EMV of each branch, weighted by its probability (in the robust model).

   Decision node Compute the *robust* expected monetary value (EMV) of the node by selecting the branch emanating from this node with the highest robust EMV.

2. The decision tree is solved when all nodes have been evaluated.

The critical issue in the robust framework is to determine the probabilities associated with each outcome, at each event node.

3.3.2 Box uncertainty set

We first consider the case where the probabilities are only known to belong to a confidence interval. Let i be an event node, and let $j = 1, \ldots, J_i$ be the possible nodes following the realization of the event at node i, each occurring with probability π^i_j. These probabilities are unknown, but fall within $[\overline{\pi}^i_j - \widehat{\pi}^i_j, \overline{\pi}^i_j + \widehat{\pi}^i_j]$ for each j. Let v_j be the value of node j, $j = 1, \ldots, J_i$. The value of node i in the traditional decision model is given by:

$$v_i = \sum_{j=1}^{J_i} \overline{\pi}^i_j v_j. \tag{69}$$

In the robust framework, we know from Theorem 2.3 that this value is given by:

$$v_i = \sum_{j=1}^{J_i} \left(\overline{\pi}_j^i \, v_j + \widehat{\pi}_j^i \, |v_j - r| \right), \tag{70}$$

where r minimizes $\sum_{j=1}^{J_i} \widehat{\pi}_j^i \, |v_j - r|$. By studying the breakpoints of this function, it is easy to see r is one of the v_j, $j = 1, \ldots, J_i$. Therefore, Equation (70) becomes:

$$v_i = \sum_{j=1}^{J_i} \overline{\pi}_j^i \, v_j + \min_{k=1,\ldots,J_i} \sum_{j=1}^{J_i} \widehat{\pi}_j^i \, |v_j - v_k|. \tag{71}$$

This requires computing $|J_i|$ expressions at each node i; however, in most decision trees used in practice, the number of children nodes $|J_i|$ is moderate. The method is even simpler in the data-driven approach, since r is then the median of the v_j and is found immediately by sorting these values.

As indicated in Section 2.3, this approach also allows the decision-maker to gain a deeper insight into the connection between the optimal solution and the degree of accuracy of the underlying probabilities, as the amount of ambiguity $\widehat{\pi}_j^i$ can in general be written as $\alpha \, \widehat{\pi}_j^{i0}$, with the nonnegative parameter α being common to all the nodes. In that framework, Equation (71) can be rewritten as:

$$v_i = \sum_{j=1}^{J_i} \overline{\pi}_j^i \, v_j + \alpha \min_{k=1,\ldots,J_i} \sum_{j=1}^{J_i} \widehat{\pi}_j^{i0} \, |v_j - v_k|, \tag{72}$$

or with obvious notations:

$$v_i = \overline{v}_i + \alpha \, \widehat{v}_i. \tag{73}$$

Let S be the set of actions s the manager can take at a decision node 0, each leading to a node also called s. Then the value of the decision node as a function of the ambiguity is given by:

$$v_0(\alpha) = \max_{s \in S} [\overline{v}_s + \alpha \, \widehat{v}_s]. \tag{74}$$

(The decision tree approach usually maximizes profit rather than minimizes cost.) This function is piecewise linear in α, and the action t optimal in the nominal model ($\alpha = 0$) remains optimal in the robust model as long as:

$$\alpha \leq \min_{s \in S | \widehat{v}_s > \widehat{v}_t} \frac{\overline{v}_t - \overline{v}_s}{\widehat{v}_s - \widehat{v}_t}. \tag{75}$$

3.3.3 Box uncertainty set with budget of uncertainty

When the decision-maker further imposes a constraint on the number of scenarios whose probability differs from its nominal estimate, the value of node i becomes, also from Theorem 2.3:

$$v_i = \min_{(r),s\geq 0} \sum_{j=1}^{J_i} \left(\overline{\pi}_j^i \, v_j + \max\left\{ s, \widehat{\pi}_j^i \, |v_j - r| \right\} \right) + (\Gamma - |J_i|)\, s, \tag{76}$$

The optimal r is again the v_k which yields the smallest objective value, and for $r = v_k$, we have seen in Theorem 2.3 that s is the Γ-th greatest $\widehat{\pi}_j^i \, |v_j - v_k|$, which we denote by v_k^Γ. Hence, Equation (76) becomes:

$$v_i = \sum_{j=1}^{J_i} \overline{\pi}_j^i \, v_j + \min_{k=1,\ldots,J_i} \left[\sum_{j=1}^{J_i} \max\left\{ v_k^\Gamma, \widehat{\pi}_j^i \, |v_j - v_k| \right\} + (\Gamma - |J_i|)\, v_k^\Gamma \right]. \tag{77}$$

It is again possible to study for which amounts of ambiguity the solution of the traditional decision tree problem remains optimal.

4 Applications and numerical experiments

4.1 Production problem

In this section, we implement the robust stochastic programming approach to a multi-period production planning problem, where the state of the demand (base, high or low) is revealed after the first time period. A company must decide how many bands and coils to produce, sell and store to maximize its total profit. This example has been implemented by the AMPL developers at `www.ampl.com` using Benders decomposition when the scenario probabilities are equal to .45, .35 and .20, respectively, and there are 4 time periods. We refer the reader to the AMPL website for the numerical values of the data. Since there are only three possible outcomes, it is not necessary to incorporate a budget of uncertainty to the model, and we only consider a box uncertainty set, where the amount of ambiguity in each scenario is a given fraction of the nominal probability: $\widehat{\pi}_\omega = \alpha \, \overline{\pi}_\omega$ for all $\omega \in \Omega$. The purpose of this example is to provide preliminary results regarding the tractability of Benders decomposition in the robust approach and the impact on the first-stage solution, when α increases from 0 to 1 by increments of 0.1.

The conclusions of this numerical experiment are:

- The Benders decomposition always terminates after 5 iterations. Hence, incorporating parameter ambiguity does not result in any increased computational burden in this case.
- The base-demand scenario (with the highest nominal probability) always sees its probability increase to $0.45\,(1 + \alpha)$. The high-demand scenario (with the lowest nominal probability) always sees its probability decrease to $0.20\,(1 - \alpha)$. The median second-stage cost is reached for the low-demand scenario, whose probability decreases to $0.35 - 0.25\,\alpha$, in order to keep the sum of probabilities equal to 1. Therefore, the probability of both extreme scenarios

decreases, and the robust approach gives more weight to the baseline case. If $\alpha = 1$, the high-demand scenario is discarded altogether.

- The optimal first-stage variables (amounts to be produced, put in storage and sold, respectively) do not change for α between 0 and 0.4, and are equal to:

:	Make1	Inv1	Sell1	:=
bands	2590	600	2000	
coils	3787	787	3000	

  The optimal first-stage variables do not change for α between 0.5 and 1, and are equal to:

:	Make1	Inv1	Sell1	:=
bands	590	600	0	
coils	5187	2187	3000	

- Inventory at time 0 is 10 bands and 0 coil, with a market demand (i.e., maximum sales) in the first time period of 2000 and 3000, respectively. Therefore, the optimal solution for $\alpha \leq 0.4$ is to meet the demand for both products at time 1 and put the rest in storage. If however the level of probability ambiguity exceeds 0.5, then the decision-maker does not take advantage of the market demand for the first product and prefers to put all the bands in inventory, in order to wait and see which scenario is realized. He also produces more coils and hence put more of them in inventory.

4.2 Asset selling

Finally, we implement the robust optimization approach in the case of asset selling. A person is selling an asset (e.g., a car, a piece of land) and is offered an amount of money for that item at each time period. She must then decide whether to accept the offer, in which case she invests the money at an interest rate r for the remainder of the time horizon, or to wait for the next offer. Past offers are not renewed and the last offer, made at time T, must be accepted. We assume that the offers $w_1, \ldots, w_T$ are independent and identically distributed. When the probability distributions of the offers are available, it is well-known (see Bertsekas [2]) that this problem can be solved using dynamic programming and that the optimal solution at time t, for $t = 1, \ldots, T-1$, is to accept the offer x_t if $x_t \geq \alpha_t$ and to reject it if $x_t < \alpha_t$, where the thresholds α_t are determined recursively by:

$$\alpha_{T-1} = \frac{E[w]}{1+r}, \tag{78}$$

$$\alpha_t = \frac{E[\max(w, \alpha_{t+1})]}{1+r}, \ \forall t = 1, \ldots, T-2. \tag{79}$$

The cost-to-go function at time t, which is defined (before the asset is sold) as:

$$J_t(x_t) = \max\left((1+r)^{T-t}\, x_t, E[J_{t+1}(w)]\right) \tag{80}$$

becomes at optimality:

$$J_t(x_t) = (1+r)^{T-t} \max(x_t, \alpha_t). \tag{81}$$

We implement this framework when the offers obey a discrete distribution on 11 points at equal intervals between 100 and 200, i.e., the possible values taken by the random variable are 100, 110, 120, ..., 200. The nominal probabilities correspond to a discretization of a Gaussian random variable with mean 150 and standard deviation 30. The maximum deviation $\widehat{\pi}_\omega$ of the probabilities from their nominal values is set to $\alpha\,\overline{\pi}_\omega$, for all 11 outcomes ω. We consider a time horizon of 21 periods, which yields 20 thresholds.

Worst-case probabilities

Figure 1 shows the impact of α in the interval model of uncertainty. We note that the choice of α does not affect the shape of the curve, but higher values of α will decrease the threshold value further. For $\alpha = 0.3$, the optimal thresholds differ from their nominal values (obtained for $\alpha = 0$)

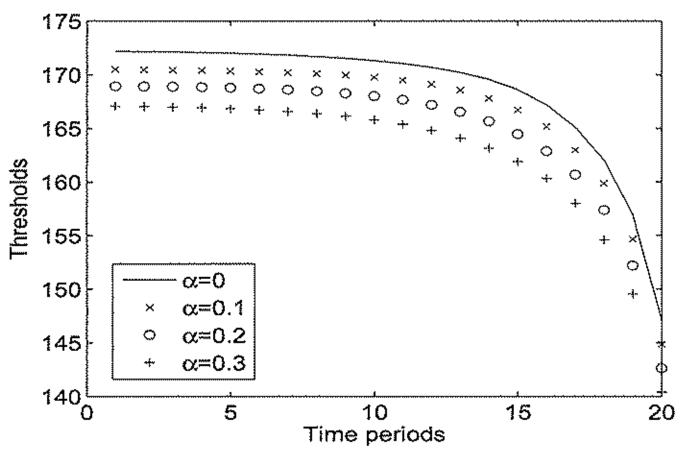

Figure 1: Optimal thresholds as a function of time.

by 3% at time 1 and 4.5% at time 20.

Impact of the budget of uncertainty

Figure 2 shows the impact of Γ on the optimal thresholds, in the box model with budget of uncertainty, when $\widehat{\pi}_\omega = 0.3\,\overline{\pi}_\omega$ for all ω. The thresholds decrease significantly for small values of Γ, which indicates that the thresholds in the nominal model are quite sensitive to the choice of the probabilities. As Γ increases from 0 to 11, the threshold values keep decreasing, but less and less rapidly, i.e., the marginal value of increasing the degree of robustness of the model decreases. For instance, the first threshold decreases by 1.0% as Γ increases from 0 (no robustness) to 1, and by 0.7%, resp. 0.5%, as Γ increases from 1 to 2, resp. from 2 to 3. As a rule of thumb in line with

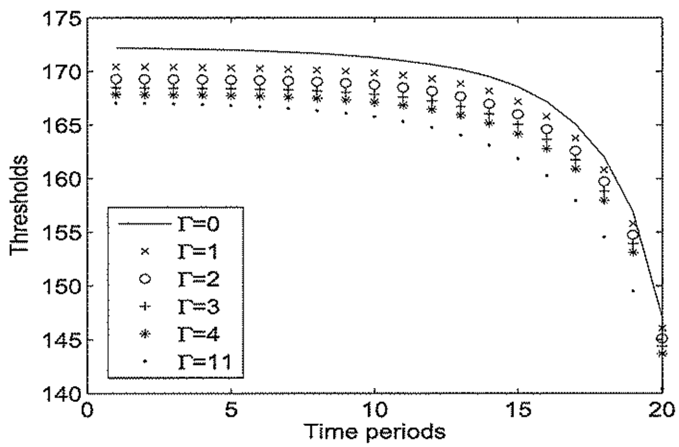

Figure 2: Optimal thresholds for various budgets of uncertainty.

the insights of Bertsimas and Sim [3], Γ should be approximately equal to the square root of the number of ambiguous parameters. Figure 3, which plots the histogram of the terminal wealth in the nominal ($\Gamma = 0$) and the robust ($\Gamma = 3$) frameworks, illustrates how the robust optimization approach reduces the mean but also the standard deviation of the decision-maker's wealth at the end of the time horizon (measured at time 0 by generating 100 distributions, each used for 1000 runs to draw distributions). Specifically, the average wealth in the nominal and robust frameworks is 260.7, resp. 259.7, while the standard deviation is 24.9, resp. 21.1. Therefore, the ratio of standard deviation to expected wealth, which is a measure of the trade-off between risk and return, decreases from 9.6% in the nominal case to 8.1% in the robust model. This represents a decrease of 15.1%.

5 Conclusions

We have proposed an approach to address probability ambiguity in two-stage and multi-stage stochastic programming that models probabilities as uncertain parameters in a polyhedral set, and finds the optimal solution for the worst-case values of the probabilities within that set. We have shown that this framework does not increase the complexity of the problem considered, in the sense that: (i) it introduces a convex penalty term in the objective without any new constraints beside nonnegativity, (ii) in two-stage stochastic programming, the well-known Benders decomposition can easily be modified to incorporate the robust optimization approach directly into the algorithm, and (iii) in multi-stage problems, this model preserves the convexity of the recursive equations used

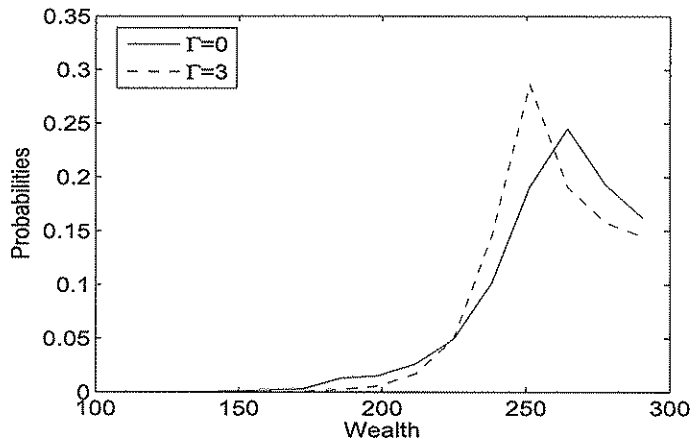

Figure 3: Histogram of terminal wealth in nominal and robust models.

to compute the optimal policy. We have also provided insights into the worst-case probabilities and the impact of the ambiguity on the optimal solution.

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