Adjusting Production Decisions for Short Life Cycle Products

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1 Introduction

The purpose of this research is to address strategic as well as operational issues faced by contract manufacturers, who are capable of developing their own market presence. A contract manufacturer (CM) often faces the strategic options of devoting its manufacturing capacity entirely to a brand-carrying customer (BC), or leveraging a portion of its manufacturing capabilities to develop its own market presence. In our research, we aim to analyze these conflicting incentives of a CM, specifically focusing on the following issues; Already serving a BC, whether the CM should develop her market presence and when. If the CM builds a market presence in a local market, how she should allocate the fixed capacity between the BC and the local market demands.

The research is expected to have long-term impact to industries, where contract manufacturing is prevalent, including electronics and computers, semiconductors, communications, automotive, and medical products. These industries and their subsidiaries in the Asia-Pacific region are going through a profound and rapid transformation. Major global corporations are moving aggressively away from vertical integration; instead of owning and operating the entire process of product realization, they are focusing on those aspects with the strongest value proposition, e.g., product development, marketing and sales, and supply chain management, while functions such as manufacturing and assembly are outsourced to regions with skilled and lower cost labors. By consolidating demands from different brand carrying customers and developing highly flexible processes, the contract manufacturers are able to realize a much higher utilization on their equipment, thereby reducing unit costs. Thus, the contract manufacturers can offer their customers a greater variety of products at a significantly lower cost.

Contract manufacturing has grown from a few billion dollar industry in the early 1990's to over $180 billion in 2001. In the U.S., this rate is expected to accelerate rapidly in the next few years with the share of manufacturing done on a contract basis is expected to be well over 50% (Gartner, 2003 [14]). Contract manufacturers, also known as the Original Equipment Manufacturers (OEMs) in some industries, have significant presence in the Asia-Pacific region. They represent a dominating force and a significant economic driver for regions such as Mainland China, Taiwan, Korea, and Malaysia. As a major source of investment capital the Hong Kong economy is strongly influenced by the issues related to contract manufacturing.

Despite its enormous development potentials and promises, contract manufacturing and the issues facing the contract manufacturers are not well studied. There is a significant gap in the literature between the theory and the practice. The goal of this research is to close this gap; not only by answering intellectually challenging research questions but also by producing insights to the Hong Kong industry.

1.1 Problem Setting

We focus on contract manufacturers, who have reached technological and business maturity, and who have the potential to develop their own markets. In a discrete-time finite horizon, suppose a contract manufacturer (CM) produces products for a brand-carrying customer (BC). Meanwhile, she may enter the local market with her own brand, where the products for both the BC and the local market share the same technology and facility.

We assume that the demand from the BC follows a life-cycle growth model with an i.i.d. forecast errors for each period. If the CM decides to enter her own market, the demand from the local market will follow the BC demand with a known integer lag parameter and a scale parameter between $(0, 1)$. The pricing of the products depends on the quantity demanded from both the local market and the BC. Hence, if the CM decides to enter her own market, there will be no priority schemes to deliver the order between the local market or the BC demand. As it will be cleared out in Model Development section, we assume linear unit price for the local market, and decreasing marginal unit price for the BC. As a result, the priority of the orders will be determined by the quantity of the units demanded from both the local market and the BC.

The CM has fixed known production capacity for each review period of the planning horizon. The unfilled demand from both the local market and the BC demand will be lost incurring a penalty cost per unit per time. Additionally we have periodic unit holding cost and unit production cost.

The CM needs to decide: (1) whether the CM should develop her own brand and when, and (2) how much to produce, and how to allocate the available inventory between the BC demand and the CM's own market had she entered the market.

In the next chapter, we will explore the existing literature related to our problem, then we will formalize the notation and formulate the problem. Afterwards, discussion of the preliminary results and special cases of the problem will be made. We conclude our report with future research agenda.

2 Literature Review

Our model reflects various aspects of the high-tech business and manufacturing environment, as a result, the proposed model addresses multiple fields in the literature. First, we will explore the literature on life-cycle growth models, then we will investigate production and inventory management literature for the capacitated systems, and last we will browse the literature for outsourcing incentives.

The high-tech industry is characterized by its short life-cycled products. Also in our model, the products have short life-cycles, and the demand of a product follows a life-cycle growth model, which exhibits a demand life-cycle of growth, maturity, and decline. Thus, both the market entry and the supply decisions are made under *life-cycle growth model.* There is a rich literature on modeling and describing demand for high-tech products. Meade and Islam (1998) [23] document 29 different growth curves found in the literature. One of the earliest model of the growth models is the famous Bass model. Bass (1969) [2] describes the demand for a new product by the theory of adoption and diffusion. According to him, demand for an innovative product is driven by two sources: Impact of mass-media influences and word of mouth effect of previous buyers. The first effect is modeled as the coefficient of innovation, and the latter as the coefficient of imitation. There are three parameters to be estimated in the model: Coefficient of innovation (p), coefficient of imitation (q), and market potential (m). The parameter estimation is made with regression analysis using past sales data. The model is powerful in estimating the magnitude and timing of the peak sales when the parameters are appropriately estimated. One major complication of the method is when estimation to be made for a new technology or a product, there is no available sales history data. One effective method to estimate the parameters of a new product or technology is to use the sales history of similar or analogous products. Thomas (1985) [31] proposes an evaluation procedure for identifying similar products, and using sales history of these products for the demand estimation of the new product. According to him, four data sources are used to make such an estimation; test markets, market studies, expert judgement, and products with similar characteristics.

After the introduction of Bass model, a large body of literature revisiting the structural and conceptual assumptions together with the research on estimation issues has been formed. Mahajan et al.(1990) [22] provide an excellent survey, and categorize these developments over the years

1969-1990 into five subareas; basic diffusion models, parameter estimation considerations, flexible diffusion models, refinement and extensions, and use of diffusion models. After its introduction, Bass model has been traditionally used for sales forecasting, however one of the other useful applications of the Bass model is to select optimal marketing mix strategies to maximize profitability over planning horizon considering the life-cycle dynamics. Mahajan et al. (1990) [22] summarize the use of the Bass model at the end of their paper. More recently, Kumar and Swaminathan (2003) [21] consider the production and sales decisions of a single item in a capacitated system under life-cycle dynamics. They assume that the demand of the item follows a Bass type model however with one difference; the word of mouth effect is proportional to cumulative sales, not cumulative demand. Hence, the demand at an instantaneous time depends on the cumulative sales up to that time. They conclude that, myopic sales plan is not necessarily optimal, and inventory build up heuristic is a robust approximation of the optimal sales plan. They compare the performance measures of myopic and build up policies over a wide range of parameters. In a special setting, they prove the optimality of build up policy when there is no initial inventory in the system. In a similar setting, Ho et al. (2002) [18] analyze capacity, time to market and demand fulfillment decisions jointly, and they provide closed form solutions for the optimal decisions. Their demand model is same as Kumar and Swaminathan (2003) [21]. They show that delaying the product launch can act as a substitute for capacity. One interesting result of their study is that for fixed values of capacity and product launch time, when faced between the choice of selling an available unit immediately versus delaying the sale in order to reduce future shortages, the firm should favor the first choice. This is due to time benefit of the immediate cash flow outweighs the negative effect of customer loss due to demand acceleration. Our problem setting is similar to Kumar and Swaminathan (2003) [21] and, Ho et al. (2002) [18]. However we significantly depart from their work by considering the strategic decision of entering a new market and allocation of fixed capacity between two demands. If it is optimal for the CM to enter the local market, the CM needs to decide the timing of the entry and the allocation of fixed capacity over the planning horizon, which is not considered in the previous two works. After solving our problem with the Bass model, we plan to gain insights for the CM's decisions under different types of demand models, utilizing from Meade and Islam (1998) [23]. In the future, we also plan to extend our problem by allowing multiple generations of the current technology. When the next generation products make the current ones obsolete, this new demand pattern might suggest different strategic decisions for the CM. Norton and Bass (1987) [26] and Bass and Bass (2001) [3] extend the classical Bass demand model to describe the demand process of multiple generations of a technology. We will use the similar demand model to theirs when we incorporate the future generations of a technology into our model.

As described above, if the CM decides to enter its own market, she has to decide how to allocate the limited capacity to the demands from the BC and the local market. If the CM produces different products to BC and the local market, then the problem can be classified as a *capacitated production-inventory problem*, where the objective is to minimize the total discounted or average costs over a finite or infinite horizon under limited capacity. The earliest formulation and the analysis of this problem seems to be by Evans (1967) [9]. He formulates the problem as a dynamic program under stationary demand assumption. In cases where the capacity is not binding, he was able to characterize the optimal policy in which, each item attains a maximal stock level. However, when the capacity is binding he was unable to characterize the optimal policy. Federgruen and Zipkin (1986a, 1986b) [10] [11] consider capacitated single item, periodic review inventory model under stochastic demand. They prove the optimality of modified base-stock policy for both discrete and continuous demand distribution assuming stationary data and convex one period cost function. The modified base-stock policy is described as; follow a base-stock policy whenever possible, and produce to capacity when the prescribed production would exceed the capacity. Glasserman (1996) [13] addresses the capacitated, multi-item production-inventory system with continuous review. Under a subclass of allocation policies in which, some fraction of the total capacity is permanently dedicated to each of the items throughout the planning horizon, production of an item follows a base-stock policy. He also presents procedures for choosing asymptotically optimal base-stock levels and capacity allocations. DeCroix and Arreola-Risa (1998) [7] generalize Glasserman (1996) [13] for periodic review systems and show the optimality of modified base-stock policy for capacitated, multi-item production-inventory system. When the products are homogenous (i.e identical demand distributions and cost parameters), they show that symmetric resource allocation policy is optimal for both finite-horizon and infinite-horizon problem. Kapuściński and Tayur (1998) [20] study the similar system as Federgruen and Zipkin (1986a, 1986b) [10] [11] studied, with an exception. They consider, stochastic and cyclic demand. They provide the optimal policy for finite-horizon, discounted infinite-horizon, and infinite-horizon average costs. The optimal policy is the modified base-stock policy as described by Federgruen and Zipkin (1986a, 1986b) [10] [11], but with different base-stock levels for each period. For the similar system as Kapuściński and Tayur (1998) [20], with a different method, Aviv and Federgruen (1997) [1] independently proved the optimality of modified base-stock level for the infinite-horizon case.

Above we investigated the literature for the case, in which the supplied products to the BC and the local market are different. If the CM supplies the same product to the BC and the local market, then our problem can also be classified as *rationing problem*. In this problem, the decision maker first decides the production or order quantity of product, later when the demands are realized, the decision maker allocates the products to the different classes of customers according to the *rationing rule*, which is to be determined. One of the earliest formulations of this problem for an uncapacitated discrete time system is made by Veinott (1965) [33]. His focus is to find how much and when to replenish the orders. In a nonstationary environment he proves the optimality of a base-stock policy, but he does not consider any rationing levels. Later, Topkis (1968) [32] extends the results of Veinott (1965) [33] by considering how inventory should be allocated in a single period of a periodic review model. He does the analysis by breaking down the single period into finite number of subperiods, and as the demands are realized, he decides between satisfying the demand now, or reserving the inventory to fill higher-class demands in the subsequent periods. He proves that, for each review subperiod, there exist optimal nonnegative rationing levels for each demand class such that one satisfies demand of a given class only if there is no unsatisfied demand of higher class, and inventory level is above the rationing level for that class. However, he does not allow replenishment of inventory within the subperiods. Most of the rationing problem literature deals with the continuous time environment such as Nahmias and Demmy (1981) [25], Desphande et al. (2003) [8], or the queueing control environment such as Ha (1997a, 1997b, 2000) [15], [16], [17]. Rationing problem is also similar to *assortment problem* and *substitution problem*. In the assortment problem, a firm has the ability to produce n different items, where each item has its own demand, but the firm must satisfy all the demand while producing m ($m < n$) items. In this problem demand for any inferior item can be substituted by a superior or more costly item, and the objective is to minimize all the costs. Pentico (1974) [28] analyzes the problem with stochastic demands and in multi-period environment. By making some assumptions on the pattern of demand, he was able to simplify the problem. In generalized version of the single period newsvendor problem, Parlar and Goyal (1984) [27] investigated two-way substitution problem. Gerchak et al. (1996) [12] study single period production systems with random yield and downward substitutable demand. They prove that expected profit functions are concave and derive the optimality conditions. Bassok et al. (1999) [4] analyze single period, multi-product substitution problem with downward substitution. They prove that greedy allocation policy is optimal.

In our research, the CM supplies the BC and her own market with the same or similar products, which use the same technology and facility. Unlike the classical *capacitated production-inventory problem* or the classical *rationing problem* CM needs to consider the strategic decision of supplying to her own market or not, and if she selects to supply she needs to decide the timing of the product release. Furthermore, our demand pattern follows the life-cycle growth model. We also consider to extend our work by allowing the demand from the local market or demand from the BC be affected by the allocation rule of the CM. It differs us from most of the previous literatures which consider the stationary demand in each period.

The incentives for a firm to outsource a portion of their production or service has been studied by many researchers. Quinn and Hilmer (1994) [29] discuss ways to determine a company's core competencies and which activities are better performed externally. Benson and Ieronimo (1996) [5] discuss the impacts of outsourcing maintenance work on firm performance by comparing Australian firms with Japanese firms operating in Australia. Kamien and Li (1990) [19] formulate a production planning model that explicitly considers subcontracting as a planning tool. They also discuss different subcontracting mechanisms and their costs, concluding a class of subcontracting mechanisms Pareto-dominate other subcontracting mechanisms. Van Mieghem (1999) [24] analyze a competitive two stage stochastic investment game between a manufacturer and a supplier. They discuss the outsourcing conditions for three different contract types. For more recent discussion and survey on subcontracting and outsourcing see Simchi-Levi et al. (2004) [30]. Most of the previous researchers focus on the problems from the point of view of the brand carrier. In contrast, our research will consider the issues faced by the contract manufacturers. With a brand new feature in our research, by considering the strategic decision of entering CM's own market or not, we aim to shed light on these issues.

3 Model Formulation

3.1 Problem Description

Before getting into the details of our model, we would like to present some results from the life-cycle growth model proposed by Bass (1969) [2]. In his model, the adoption of an innovative product by customers is driven by two sources: (1) impact of mass-media influences and (2) word of mouth effect of previous adopters. The demand rate seen at an instantaneous time is given by,

$$\begin{aligned} n(t) &= pm + (q-p)N(t) - \frac{q}{m}N(t)^2 \\ &= [p + \frac{q}{m}N(t)][m - N(t)] \end{aligned}$$

where, $N(t) = \int_0^t n(s)\mathrm{d}s$, is the cumulative demand up to time t, and $n(0) = pm$. The first component in the second line is the probability of purchase given no purchase has been made, and the second component is the number of customers that have not made a purchase yet. There are three parameters determining the shape of demand curve in the Bass model; (1)m: the market potential, (2)p: the coefficient of innovation (mass-media effect), and (3)q: the coefficient of imitation (word of mouth effect). The solution of the above equation is obtained by solving the following non-linear differential equation;

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = pm + (q-p)N(t) - \frac{q}{m}N(t)^2$$

According to this, the solution is given by;

$$\begin{aligned} N(t) &= m\left(\frac{1 - \mathrm{e}^{-(p+q)t}}{1 + (q/p)\mathrm{e}^{-(p+q)t}}\right) \\ n(t) &= m\left(\frac{p(p+q)^2\mathrm{e}^{-(p+q)t}}{(p + q\mathrm{e}^{-(p+q)t})^2}\right) \end{aligned}$$

and maximum value of demand is reached at time t^* is $n(t^*)$, where

$$t^* = -\frac{1}{(p+q)}\ln(\frac{p}{q}) \quad \text{and} \quad n(t^*) = \frac{m(p+q)^2}{4q}$$

While the Bass model was originally defined assuming continuous time, we will use the discrete analogue, in which time and size of the peak sales coincide with the original model. We will use time subscript to indicate the demand for a given period. According to this demand at a given period is;

$$n_t = pm + (q-p)N_{t-1} - \frac{q}{m}N_{t-1}^2$$

where $N_t = \sum_0^t n_t$ and $n_0 = pm$.

Assuming that the demands are determined by the Bass model parameters, we can formally build our model. Suppose a contract manufacturer (CM) produces a product for a brand carrying customer (BC). We consider a discrete time finite horizon $\{1, 2, .., T\}$, which is long enough for the CM to observe increasing and diminishing demand pattern from the BC. We assume that demand from BC, D_t^b, is characterized by the Bass model ($D_t^b = n_t$). We further assume that the demand is stochastic with an i.i.d additive error ξ for each period. According to this, the realization of BC demand is given by, $d_t^b = D_t^b + \xi$, where $D_t^b = n_t$. The pdf and cdf of the error is known and given by $f(\xi)$ and $F(\xi)$, respectively.

As a simplifying assumption, we will not deal with the periodic update of the demand parameters, p, q, and m as the demand is observed. The CM has a fixed known periodic production capacity C, and throughout the planning horizon, the CM has an option of entering a local market with a product that uses the same capacity. The market entrance time τ is also a decision variable for the CM. There is a fixed, one time market entry cost K, and the demand in the local market follows the BC demand with a known integer lag parameter k and a known scale parameter $\gamma \in (0, 1)$. Similarly, we assume demand from the local market, D_t^m is stochastic with an additive error ξ, which has identical distribution as the error in BC's demand, but independent across the periods and independent from the BC's error. Then, the realization of demand from the local market is given by,

$$d_t^m = \gamma D_{t-k}^b + \xi \qquad\qquad \forall t : T \geq t \geq k$$

Figure 1 shows a typical demand relationship between the BC demand and the local market demand, where the parameters are $k = 2, \gamma = 0.8, p = 0.02$, $q = 0.7$, and $m = 1$.

The CM faces the following set of decisions: (1) How much to produce and how much to satisfy the BC demand at each period, and (2) Whether she should enter the local market or not, and when. If she enters, she needs to decide the allocation scheme of the on hand inventory between the BC and the local market. We assume that unsatisfied demand is lost for each customer types. Departing from the classical *rationing problem* literature, we do not impose any priority scheme between the two demand types. Instead, we assume the unit price of the product is driven by the quantity demanded from each of the customer types. For the local market, the unit price, π^m is

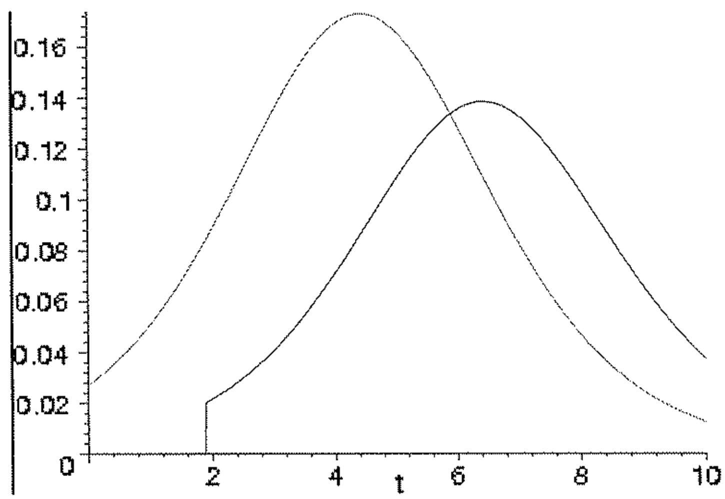

Figure 1: Relationship between the BC demand and the market demand

constant, whereas the unit price for the BC, $\pi_t^b(d_t^b)$, is a function of the quantity demanded, and it is diminishing in d_t^b. Figure 2, shows an instance of revenues from the local market and the BC, when the demand from each is supplied fully. The demand driven price method will clarify the priority of the demands for the planning horizon.

We assume that production and market entry decisions take place at the beginning of each period. The production is instantaneous, and in a given period if the CM decides to enter the local market, she will be able to satisfy the market within that period. After the demand for each type of customers are realized, allocation of the on hand inventory is made, and related costs are incurred at the end of the period. We have three types of costs; positive unit holding cost, h, nonnegative unit production cost, c and nonnegative unit penalty cost, p^b, and p^m for the unsatisfied demand from the BC and the local market, respectively. In order to keep the model meaningful, we further impose following assumptions among the model parameters.

Model Assumptions

1. $\pi_t^b(d_t^b) + p^b > c$ for all possible realizations of d_t^b, and similarly $\pi^m + p^m > c$. This assumption ensures that, it is not optimal to never satisfy current period's demand and accumulate backlogging costs.

2. $\alpha^i(\pi_{t+i}^b(d_{t+i}^b) + p^b) - (\pi_t^b(d_t^b) + p^b) < \sum_{k=0}^{i-1} \alpha^k h$. This assumption ensures that, it is not optimal to cut current period's BC demand to satisfy the future period's BC demand at a higher

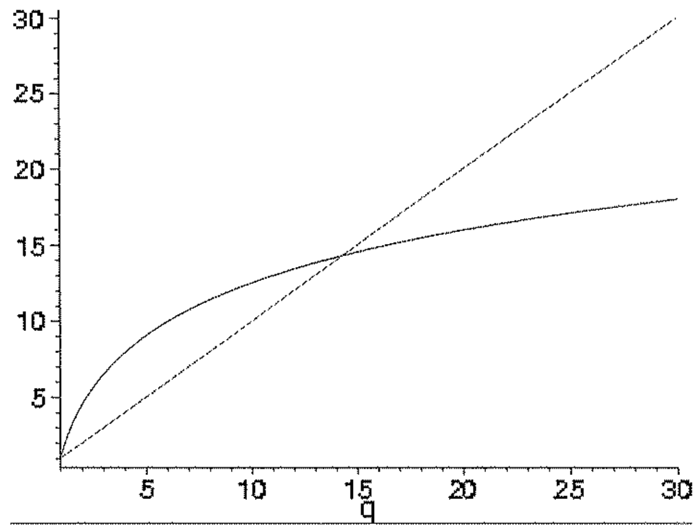

Figure 2: Relationship between revenues

price. The assumption is practically relevant since the CM has no customer other than the BC. Acting in this way, the CM would decrease the trust of BC, endangering the promise of future BC demands. This assumption automatically holds for the LM case, since π^m is constant. Note that we are not imposing any assumptions for the cross benefit case (i.e. cutting current period's BC (LM) demand, to satisfy future LM's (BC's) demand at a higher price). Since, these allocation decisions play major role in the CM's market entry decision.

The decisions that the CM faces throughout the planning horizon can be described as follows: (1) y_t: the inventory level after production at the beginning of period t. I_t being the on hand inventory level before the production decision takes place, the production can be at most the production capacity; $y_t - I_T \leq C$, and the inventory level after the production cannot be less than the beginning onhand inventory. $I_t \leq y_t$. Without loss of generality, we will assume the orders from the BC will be shipped first. (2) x_t: Inventory level below which the CM cannot use the on hand inventory to satisfy the BC demand. $(x_t \leq y_t)$. (3) z_t: Inventory level below which the CM cannot use the on hand inventory to satisfy the local market demand. In other words z_t is the minimum reserved inventory level for the next period. $(0 \leq z_t \leq x_t)$. (4) r_t: Binary decision variable indicating the market entry decision. If it is 1, then the CM enters the local market incurring a fixed cost of K, 0 otherwise. $(\sum_{t=1}^{T} r_t \leq 1)$ Note that, the order of shipments, does not affect the problem structure, since for any given demand, the CM cannot satisfy more than the difference of two inventory levels assigned for that demand. According to this, in any given period, CM will supply $\min\{y_t - x_t, d_t^b\}$ to the BC. After satisfying BC demand, the remaining inventory level is $\max\{x_t, y_t - d_t^b\}$. If the CM does not supply to the local market (i.e. the CM has not made the market entry decision, yet.), then $x_t = z_t$. If CM also supplies to the local market, then CM will deliver $\min\{\max\{x_t, y_t - d_t^b\} - z_t, d_t^m\}$ to the local market, and then the remaining inventory level at the end of period would be $\max\{\max\{y_t - d_t^b, x_t\} - d_t^m, z_t\}$, which is the inventory level at the beginning of the next period.

3.2 Model Formulation

CM's problem is to minimize total expected cost of the planning horizon. There is a periodic discounting factor, $\alpha \in (0, 1)$, of the cash flows. At the end of the planning horizon, the remaining on hand inventory can be salvaged at the production cost, c. Table 3.2 summarizes the notation and description for our problem.

In the light of the above discussion, we can model the CM's problem as follows. In order to evaluate the value of the strategic decision, entering the local market or not, we will have to evaluate two functions in each period:

1. $H_t^1(I_t)$ = minimum expected net discounted cost in periods $t, t+1, ..., T$, given that period t begins with inventory level I_t, and the CM decided to enter the local market at or before the time period t.

2. $H_t^0(I_t)$ = minimum expected net discounted cost in periods $t, t+1, ..., T$, given that period t begins with inventory level I_t, when the CM does not have the market entry option.

$H_t^0(I_t)$ and $H_t^1(I_t)$ are the cost-to-go functions of operational problems. (i.e. how to allocate the fixed capacity when serving one and two customers, respectively) $H_t^0(I_t)$ is found according to the following DP formulation.

$$\begin{aligned} H_t^0(I_t) &= \min_{y_t, x_t} E[g_t^0(I_t, y_t, x_t) + \alpha H_{t+1}^0(I_{t+1})] \\ \text{subject to} &\quad 0 \leq x_t \leq y_t \leq I_t + C, \quad I_t \leq y_t \\ I_{t+1} &= \max\{y_t - d_t^b, x_t\} \end{aligned} \tag{1}$$

where, $g_t^0(I_t, y_t, x_t)$ is the periodic cost function and found by;

$$c(y_t - I_t) - \pi_t^b(d_t^b)\min\{y_t - x_t, d_t^b\} + p^b(y_t - x_t - d_t^b)^- + h\max\{y_t - d_t^b, x_t\}$$

Table 1: Notation and description

Notation	Description
$\pi_t^b(d_t^b)$	Unit selling price to the BC
π^m	Unit selling price to the local market
c	Unit production cost
p^b	Unit penalty cost of not satisfying demand from BC
p^m	Unit penalty cost of not satisfying demand from local market
h	Unit holding cost per time period
I_t	Inventory level at the beginning of period t
C	Maximum periodic production capacity
K	Fixed market entry cost
y_t	Inventory level after production, $I_t + C \geq y_t \geq I_t$
x_t	Minimum inventory level after satisfying the BC's demand, $y_t \geq x_t$
z_t	Minimum inventory level after satisfying local market's demand, $x_t \geq z_t \geq 0$
r_t	Binary decision variable whether enter to local market or not at time t
d_t^b, d_t^m	Realization of demands from BC and local market, respectively
k	Integer lag parameter for the local market demand
γ	Scale parameter for the local market demand
$f(\xi), F(\xi)$	pdf and cdf of random error ξ
α	Periodic discount factor

The first term above is the production cost, the second term is the revenues obtained from BC, the third term is the penalty cost of unsatisfied BC demand, the fourth term is the holding cost of the inventory carried to the next period. Similarly $H_t^1(I_t)$ is found according to the following DP formulation.

$$
\begin{aligned}
H_t^1(I_t) &= \min_{y_t, x_t, z_t} E[g_t^1(I_t, y_t, x_t, z_t) + \alpha H_{t+1}^1(I_{t+1})] \\
\text{subject to} \quad & 0 \leq z_t \leq x_t \leq y_t \leq I_t + C, \quad I_t \leq y_t \\
& I_{t+1} = \max\{\max\{y_t - d_t^b, x_t\} - d_t^m, z_t\}
\end{aligned}
\tag{2}
$$

where, $g_t^1(I_t, y_t, x_t, z_t)$ is the periodic cost function and found by;

$$
\begin{aligned}
& c(y_t - I_t) - \pi_t^b(d_t^b)\min\{y_t - x_t, d_t^b\} + p^b(y_t - x_t - d_t^b)^- - \pi^m\min\{\max\{y_t - d_t^b, x_t\} - z_t, d_t^m\} \\
& + p^m(\max\{y_t - d_t^b, x_t\} - z_t - d_t^m)^- + h\max\{\max\{y_t - d_t^b, x_t\} - d_t^m, z_t\}
\end{aligned}
$$

The first term above is the production cost, the second term is the revenues obtained from BC, the third term is the penalty cost of unsatisfied BC demand, the fourth term is the revenues obtained from the local market, the fifth terms is the penalty cost of unsatisfied LM demand, and the last term is the holding cost of the inventory carried to the next period. The boundary conditions for

Formulations 1 and 2 are $H^0_{T+1}(I_{T+1}) = H^1_{T+1}(I_{T+1}) = -cI_{T+1}$, where I_{T+1} are found from the recursions in 1 and 2, respectively.

The market entry decision depends on the costs before the market entry and after the market entry. If the CM enters the LM at $\tau \in [1, T]$, then during the interval $[1, \tau)$ the CM serves only BC, and during the interval $[\tau, T]$ the CM serves to the BC and LM both. In order to find the optimal market entry time, we need to solve to solve the following DP formulation;

$$\begin{aligned} V_t(I_t) &= \min_{r_t} r_t(K + H^1_t(I_t)) + (1 - r_t) \min_{y_t, x_t} E[g^0_t(I_t, y_t, x_t) + \alpha V_{t+1}(I_{t+1})] \\ &\text{subject to } \sum_{t=1}^{T} r_t \leq 1 \text{ and constraints for 1 and 2} \end{aligned} \tag{3}$$

where, r_t is a binary one time decision. $V_t(I_t)$ is the expected total cost of CM from t to T. To calculate $V_t(I_t)$, at each period, the market entry option is evaluated. Similar to $H^0_t(I_t)$ and $H^1_t(I_t)$, the boundary condition for $V_{T+1}(I_{T+1}) = -cI_{T+1}$. If $r_t = 0$ $\forall t$ then the total cost for CM is $H^0_1(I_1)$, the minimum discounted total expected cost of the planning horizon when CM is only serving the BC. If τ is the market entry time, then for $t > \tau$, $V_t(I_t) = H^1_t(I_t)$, the minimum total expected cost from t to T, when CM serves to LM and BC. For $t = \tau$, $V_t(I_t) = K + H^1_t(I_t)$, the minimum total expected cost from t to T, when CM serves to LM and BC plus the market entry cost. For $t < \tau$ $V_t(I_t) = \sum_{i=t}^{\tau-1} \alpha^{i-t} \min g^0_i(I_i, y_i, x_i) + \alpha^{\tau-t}(K + H^1_\tau(I_\tau))$, the minimum expected total cost from t to τ when CM is serving BC only, plus the market entry cost and the total cost from τ to T when CM is serving BC and LM. When above DP formulation is solved, $V_1(I_1)$ will be the total cost of the problem for a given beginning onhand inventory I_1.

4 Preliminary Results

The optimization problem 3 is quite complex to solve. To gain some insights, we first analyze the structure of two cost-to-go functions separately. (i.e. $H^0_t(I_t)$ and $H^1_t(I_t)$)

Without the capacity constraint, the optimal policy is the base-stock policy for the case when the CM is serving the BC only. (It can be easily shown that $H^0_t(I_t)$ is convex in I_t $\forall t \in \{1..T\}$) With the capacity constraints, the optimal policy is again base-stock type but constrained with the capacity. At each period one brings the on hand inventory to the base stock level as close as possible, if on hand inventory is less than base stock level, and produces nothing, if on hand inventory is more than the base stock level.

However, the analysis of $H_t^1(I_t)$, even without the capacity constraints, is complicated. In this situation, the CM needs to decide the inventory level y_t after production, and the minimum inventory level x_t for LM in each period. For the unlimited capacity case, there is no need to produce more than the demand needed in current period (i.e. $z_t = 0$). Then, the cost function in each period is $H_t^1(I_t)$ in formulation 2 except the upper bound on $y_t \leq I_t + C$ and that $z_t = 0\ \forall t$.

We explicitly analyze $H_t^1(I_t)$ for the unlimited case starting from the last period T. The underlying cost-to-go function is;

$$\begin{aligned}
H_T^1(I_T) = &- cI_T + \min_{y_T, x_T} cy_T + \int_0^{y_T - x_T} \Bigg[-\pi_T^b(d_T^b)d_T^b - \pi^m \left[\int_0^{y_T - d_T^b} d_T^m \mathrm{d}F(d_T^m) + \int_{y_T - d_T^b}^{\infty} (y_T - d_T^b)\mathrm{d}F(d_T^m) \right] \\
&+ p^m \int_{y_T - d_T^b}^{\infty} (d_T^b + d_T^m - y_T)\mathrm{d}F(d_T^m) + (h - \alpha c) \int_0^{y_T - d_T^b} (y_T - d_T^b - d_T^m)\mathrm{d}F(d_T^m) \Bigg] \mathrm{d}F(d_T^b) \\
&+ \int_{y_T - x_T}^{\infty} \Bigg[-\pi_T^b(d_T^b)(y_T - x_T) + p^b(d_T^b - (y_T - x_T)) - \pi^m \left[\int_0^{x_T} d_T^m \mathrm{d}F(d_T^m) + \int_{x_T}^{\infty} x_T \mathrm{d}F(d_T^m) \right] \\
&+ p^m \int_{x_T}^{\infty} (d_T^m - x_T)\mathrm{d}F(d_T^m) + (h - \alpha c) \int_0^{x_T} (x_T - d_T^m)\mathrm{d}F(d_T^m) \Bigg] \mathrm{d}F(d_T^b)
\end{aligned} \tag{4}$$

subject to

$0 \leq x_T \leq y_T, \quad I_T \leq y_T$

First two line of the above equation is the cost of last period, given that demand from the BC is less than the amount of inventory reserved for it (i.e. $d_T^b \leq y_T - x_T$). Note that, there is no penalty cost for not satisfying the BC demand in this part, since the realized demand does not overshoot the on hand inventory for the BC. Last two lines of Equation 4 represents the cost of last period, given that the demand from the BC is more than on hand inventory reserved for the BC. (i.e. $d_T^b > y_T - x_T$). After combining the terms and making simplifications (see Appendix I), Equation 4 becomes;

$$\begin{aligned}
H_T^1(I_T) = &- cI_T + p^b\mu_T^b + p^m\mu_T^m + \\
&\min_{y_T, x_T} cy_T - (p^b + \pi^b)(y_T - x_T) - (p^m + \pi^m)x_T + (p^b + \pi^b - p^m - \pi^m) \int_0^{y_T - x_T} F(d_T^b)\mathrm{d}d_T^b \\
&+ (p^m + \pi^m + h - \alpha c) \left[\int_0^{y_T - x_T} \int_0^{y_T - d_T^b} F(d_T^m)\mathrm{d}d_T^m \mathrm{d}F(d_T^b) + \int_{y_T - x_T}^{\infty} \int_0^{x_T} F(d_T^m)\mathrm{d}d_T^m \mathrm{d}F(d_T^b) \right]
\end{aligned}$$

where μ_T^b and μ_T^m are the mean levels of the BC and the LM demand at period T. Unfortunately, the function in the minimization part is neither convex nor strictly quasiconvex.(See Appendix I) Even in the uncapacitated case, we are not able to find an analytical solution for $H_t^1(I_t)$.

Inability to find an analytical solution to a dynamic programming problem is quite a common issue. In most cases a numerical solution is necessary. However, the computational requirements for this are often overwhelming, and for many problems a complete solution of the problem by DP is impossible. The reason lies in what Bellman has called the "curse of dimensionality", which refers to an exponential increase of the required computation as the problem's size increases. For $H_t^1(I_t)$ (for the capacitated case), state, control and the disturbance spaces are $(\Re^+)^1, (\Re^+)^3, (\Re^+)^2$, respectively. In a straightforward numerical approach, these spaces are discretized. Taking d discretization points per state axis results in a state space grid with d points. For each of these points, the minimization must be carried out numerically, which involves comparison of d^3 numbers, and to calculate these numbers, one must calculate an expected value of over the disturbance, which is the weighted sum of d^2 numbers. Also, calculation must be done for each of the T stages. Thus, number of computational operations can be as much as Td^6. If $T = 10$ and $d = 100$, then we the number of computations is 10^{13}. Even a computer can perform 1000 operations/sec., then finding a numerical solution would take 317 years.

As indicated by the above discussion, in practice one often has to settle for a suboptimal control scheme that finds a reasonable balance between convenient implementation and adequate performance. For this problem, we propose to apply a modified *Certainty Equivalent Controller* (CEC) to simplify the computational requirements, while keeping the essence of the problem.

CEC is a suboptimal control scheme that is inspired by linear-quadratic control theory. At each stage CEC finds an optimal decision if some or all the uncertain quantities were fixed at some "typical" values. Bertsekas (2000) [6]. In our problem uncertain quantities are the periodic demands of the BC and the LM. "Typical" values for these random quantities would be their expected values, μ_t^b and μ_t^m $\forall t \in \{1..T\}$, for the BC demand and the LM demand, respectively. The CEC approach often performs well in practice and yields near optimal policies. In fact, for the linear quadratic problems where there is no constraint on the selection of decision variables, the CEC produces the optimal policy. However, this is not the case for our problem environment, since we have capacity constraints on our decision variables even though we formulate a quadratic problem.

CEC only considers the expected demand information and uses a deterministic approach to a stochastic problem. In order to minimize the error obtained from the CEC policy, we would like to use a modified version of CEC. In this approach, in order to solve one problem with expected demand information, we would like to solve r problems which have demand sequences sampled from BC's and LM's demand distribution, and take the average of the optimal decisions from each sample. By this we are able to account the stochasticity of demands into the approximation scheme.

In order to find $H_1^0(I_1)$ generate r samples of demand sequence for the horizon. The i^{th} sample would look like, $\{d_{1i}^b, d_{2i}^b, ..., d_{Ti}^b\}$, and solution to the i^{th} problem would look like $\{(y_{1i}, x_{1i}), (y_{2i}, x_{2i}), ..., ($ By averaging over the sample size we obtain the optimal policies as $\{(\bar{y}_1, \bar{x}_1), (\bar{y}_2, \bar{x}_2), ..., (\bar{y}_T, \bar{x}_T)\}$. These policies are applied to the formulation 1 to obtain an approximated cost-to-go function. Similar procedure is applied to approximate the cost to go function $H_t^1(I_t)$. However, in this case r samples of BC and LM demand sequences are generated and solved to obtain $(\bar{y}_t, \bar{x}_t, \bar{z}_t)$ $\forall t$. Once the approximate cost-to-go functions are obtained we can solve the problem 3.

In the subsequent parts, we will find the optimal CEC solutions to the two cost-to-go functions, $H_t^0(I_t)$ and $H_t^1(I_t)$. In order to prevent confusion, we will denote the deterministic cost-to-go functions as $\hat{H}_t^0(I_t)$ and $\hat{H}_t^1(I_t)$, and optimal decisions as $\hat{y}_t, \hat{x}_t, \hat{z}_t$ $\forall t \in \{1..T\}$ that are generated by CEC.

4.1 Optimal CEC Policies

With CEC, our problem reduces to find an optimal policy to two deterministic problems. The first problem is to find an optimal order up to level $(\hat{y}_t)$ and optimal supply to the BC $(\hat{y}_t - \hat{x}_t)$ in a given period t. The second problem is to find an optimal order up to level $(\hat{y}_t)$, optimal supply to the BC $(\hat{y}_t - \hat{x}_t)$, and optimal supply to the LM $(\hat{x}_t - \hat{z}_t)$. Note that, in the first problem (in the second problem), $\hat{x}_t$ $(\hat{z}_t)$ plays a role to determine next periods beginning inventory level.

In the next sections we re-formulate the problem for the two cases. 1. Deterministic problem when the CM supplies to the BC only, and 2. Deterministic problem when the CM supplies to the both BC and LM. As we mentioned above, we will use the same notation except to differentiate from the original functions we add ˆ to the originals. Then for each problem we will provide an exact algorithm that finds the optimal solutions.

4.1.1 Case 1: When the CM Supplies to the BC only

In this section we will provide an exact algorithm to find an optimal solution to $\hat{H}_1^0(I_1)$ given that the initial inventory level is I_1. Without loss of generality, we will use expected demand values, instead of sampling a sequence from the BC demand distribution. Note that when the BC demand is known for a given period, we also know the price for the unit demand. So, for a known sequence of demand $\{\mu_1^b, \mu_2^b, ...\mu_T^b\}$ we can associate a known price vector $\{\pi_1^b, \pi_2^b, ..., \pi_T^b\}$. Let $g_t^0(I_t, x_t, y_t)$ be the periodic cost function.

$$g_t^0(I_t, x_t, y_t) = c(y_t - I_t) - \pi_t^b \min\{y_t - x_t, \mu_t^b\} + p^b \max\{\mu_t^b - (y_t - x_t), 0\} + h\max\{x_t, y_t - \mu_t^b\}$$

Note that $\max\{x_t, y_t - \mu_t^b\} = I_{t+1}$. In order to collect the same stage terms together, let;

$$\begin{aligned}\hat{g}_t^0(I_t, x_t, y_t) &= (\frac{h}{\alpha} - c)I_t + cy_t - \pi^b \min\{y_t - x_t, \mu_t^b\} + p^b \max\{\mu_t^b - (y_t - x_t), 0\} \\ &= (\frac{h}{\alpha} - c)I_t - \pi_t^b \mu_t^b + cy_t + (\pi_t^b + p^b)\max\{\mu_t^b - (y_t - x_t), 0\}\end{aligned}$$

Second equality is obtained by using the relation $\min\{a, b\} = b - \max\{0, b - a\}$ The CEC cost-to-go function $\hat{H}_t^0(I_t)$ can be written as (using the boundary condition $\hat{H}_{T+1}^0(I_{T+1}) = (h/\alpha - c)I_{T+1}$)

$$\begin{aligned}\hat{H}_t^0(I_t) &= \min_{x_t, y_t} \ \hat{g}_t^0(I_t, x_t, y_t) + \hat{H}_{t+1}^0(I_{t+1}) \\ \text{subject to } \ I_{t+1} &= \max\{x_t, y_t - \mu_t^b\}, \ 0 \le x_t \le y_t \le C + I_t, \ I_t \le y_t \ \forall t \in \{1, 2, ..T\}\end{aligned} \tag{5}$$

The initial inventory level at the beginning of horizon I_1 is exogenous and known. Although Problem 5 is a shortest path problem, continuity of the state and decision variables makes it intractable to enumerate the optimal decisions for every possible state variable. In the subsequent parts, we will characterize the structure of optimal decisions, and then propose our algorithm that exactly solves Problem 5.

Proposition 4.1. *At period T, optimal value, $\hat{x}_T = 0$*

Proof. The last stage cost-to-go function is;

$$\begin{aligned}\hat{H}_T^0(I_T) &= \min_{y_T, x_T} \hat{g}_T^0(I_T, x_T, y_T) + \alpha H_{T+1}^0(I_{T+1}) \\ &= (\frac{h}{\alpha} - c)I_T - \pi_T^b \mu_T^b + \min_{y_T, x_T} cy_T + (\pi_T^b + p^b)\max\{0, \mu_T^b - (y_T - x_t)\} + \alpha \hat{H}_{T+1}^0(I_{T+1}) \\ & \quad 0 \le x_T \le y_T \le C + I_T, \ I_T \le y_T\end{aligned}$$

Consider only the minimization part of the second line because x_T and y_T have no effect on the first two terms of the second line.

$$
\begin{aligned}
&\min_{y_T,x_T} cy_T + (\pi_T^b + p^b)\max\{0, \mu_T^b - (y_T - x_T)\} + (h - \alpha c)I_{T+1} \\
&\Rightarrow \min_{y_T,x_T} cy_T + \frac{\pi_T^b + p^b}{2}\Big(|\mu_T^b - (y_T - x_T)| + (\mu_T^b - (y_T - x_T))\Big) \\
&+ \frac{h - \alpha c}{2}\Big(|x_T - y_T + \mu_T^b| + (x_T + y_T - \mu_T^b)\Big) \\
&\Rightarrow \min_{y_T,x_T} cy_T + \frac{\pi_T^b + p^b + h - \alpha c}{2}\Big(|\mu_T^b - (y_T - x_T)| + x_T\Big) + \frac{\pi_T^b + p^b - (h - \alpha c)}{2}(\mu_T^b - y_T)
\end{aligned}
\tag{6}
$$

The second and third lines follow from the definition $\max\{a, b\}=1/2(|b - a| + (a + b))$, and the last line is obtained from simple algebra. Suppose $\tilde{x}_T$ and $\hat{y}_T$ be the optimal solution to the above minimization problem satisfying $0 < \tilde{x}_T \leq \hat{y}_T \leq C + I_T$, and let $\tilde{H}_T(I_T)$ be the cost obtained from 6 using $\tilde{x}_T$ and $\hat{y}_T$.

There can be two cases regarding the BC demand μ_T^b. 1. $\mu_T^b > C + I_T$ and 2. $\mu_T^b \leq C + I_T$.

Case 1. $\mu_T^b > C+I_T$. Then $\mu_T^b+\tilde{x}_T-\hat{y}_T > 0$, since $\hat{y}_T \leq C+I_T$, and from simple algebra the cost $\tilde{H}_T(I_T)$ becomes $(h/\alpha - c)I_T - \pi_T^b\mu_T^b + c\hat{y}_T + (\pi_T^b + p^b)(\mu_T^b - \hat{y}_T) + (\pi_T^b + p^b + h - \alpha c)\tilde{x}_T$. Note that $\pi_T^b + p^b + h - \alpha c$ is strictly positive using our model assumption 1. Let $\hat{H}_T(I_T)$ is the cost function obtained from $\hat{x}_T = 0$ and $\hat{y}_T$. Then $\hat{H}_T(I_T) < \tilde{H}_T$, thus $\tilde{x}_T > 0$ cannot be optimal.

Case 2. $\mu_T^b \leq C+I_T$. We also need to analyze this case in two subcases. 2.1. $\mu_T^b \leq C+I_T < \mu_T^b+\tilde{x}_T$ and 2.2. $\mu_T^b < \mu_T^b + \tilde{x}_T \leq C + I_T$.

Case 2.1 exactly follows the same logic with Case 1, since $\mu_T^b + \tilde{x}_T - \hat{y}_T > 0$.

Case 2.2: In this part, depending on the value of $\hat{y}_T$, $\mu_T^b + \tilde{x}_T - \hat{y}_T$ is either positive, 0, or negative. When it is positive Case 1. applies. When it is 0, $\tilde{H}_T(I_T) = (h/\alpha - c)I_T - \pi_T^b\mu_T^b + c\hat{y}_T + 0.5(\pi_T^b + p^b + h - \alpha c)\tilde{x}_T + 0.5(\pi_T^b + p^b - (h - \alpha c))(\mu_T^b - \hat{y}_T)$. Since, $(\pi_T^b + p^b + h - \alpha c)$ is strictly positive $\tilde{x}_T > 0$ cannot be optimal. When $\mu_T^b + \tilde{x}_T - \hat{y}_T$ is negative, the absolute value in 6 comes out as $-\mu_T^b - \tilde{x}_T + \hat{y}_T$ and after simplification $\tilde{H}_T(I_T) = (h/\alpha - c)I_T - \pi_T^b\mu_T^b + (h - \alpha c)(\hat{y}_T - \mu_T^b) + c\hat{y}_T$. Note that,

$$
\begin{aligned}
\tilde{H}_T(I_T) &> (h/\alpha - c)I_T - \pi_T^b\mu_T^b + (h - \alpha c)(\mu_T^b + \tilde{x}_T - \mu_T^b) + c(\mu_T^b + \tilde{x}_T) \\
&= (h/\alpha - c)I_T - \pi_T^b\mu_T^b + c\mu_T^b + (h + c - \alpha c)\tilde{x}_T
\end{aligned}
$$

The inequality comes from replacing $\hat{y}_T$ with $\mu_T^b + \tilde{x}_T$. (Since $h + c - \alpha c > 0$ and $\hat{y}_T > \mu_T^b + \tilde{x}_T$)

Let $\hat{H}_T(I_T)$ be the cost function obtained from $\hat{y}_T$ and $\hat{x}_T = 0$. Then

$$\begin{aligned}
\tilde{H}_T(I_T) &> (h/\alpha - c)I_T - \pi_T^b \mu_T^b + c\mu_T^b + (h + c - \alpha c)\tilde{x}_T \\
&> (h/\alpha - c)I_T - \pi_T^b \mu_T^b + c\mu_T^b + (h + c - \alpha c)\hat{x}_T \\
&= (h/\alpha - c)I_T - (\pi_T^b - c)\mu_T^b = \hat{H}_T(I_T)
\end{aligned}$$

Thus, $\tilde{x}_T > 0$ cannot be optimal.

Corollary 4.1. *At period T, optimal value, $\hat{y}_T = max\{I_T, min\{\mu_T^b, C + I_T\}\}$*

Proof. Using Proposition 4.1, we insert $\hat{x}_T = 0$ into first line of Equation 6. The resulting function is

$$\min_{y_T} cy_T + (\pi_T^b + p^b)\max\{\mu_T^b - y_T, 0\} + (h - \alpha c)\max\{y_T - \mu_T^b, 0\}$$

$$\Rightarrow \min_{y_T} cy_T + \frac{\pi_T^b + p^b}{2}(|y_T - \mu_T^b| + (\mu_T^b - y_T)) + \frac{h - \alpha c}{2}(|\mu_T^b - y_T| + (y_T - \mu_T^b))$$

When $\mu_T^b > C + I_T$, above function becomes $cy_T + (\pi_T^b + p^b)(\mu_T^b - y_T)$. Since $\pi_T^b + p^b > c$ from model assumption, $\hat{y}_T = C + I_T$ is optimal. When $\mu_T^b \leq C + I_T$, $\hat{y}_T = \max\{I_T, \mu_T^b\}$ is optimal. (There is no negative production, if the demand is less than beginning on hand inventory) Thus $\hat{y}_T = \max\{I_T, \min\{\mu_T^b, C + I_T\}\}$. Then the optimal cost-to-go function $\hat{H}_T^0(I_T)$ becomes;

$$\hat{H}_T^0(I_T) = \begin{cases} (h/\alpha - c)I_T + p^b\mu_T^b - (\pi_T^b + p^b - c)(C + I_T) & \text{if } \mu_T^b > C + I_T, \\ (h/\alpha - c)I_T - (\pi_T^b - c)\mu_T^b & \text{if } I_T \leq \mu_T^b \leq C + I_T, \\ (h/\alpha + h - \alpha c)(I_T - \mu_T^b) - \pi_T^b\mu_T^b & \text{if } \mu_T^b < I_T < C + I_T, \end{cases}$$

Proposition 4.2. *Let j be the smallest period in $\{1, 2, .., T\}$ such that $\mu_i^b \leq C + I_i$ $\forall i \in \{j, j + 1, ..., T\}$, then $\hat{x}_i = 0$ $\forall i \in \{j - 1, j, ..., T\}$.*

Proof. Let, $j = T$. Then from Proposition 4.1 $\hat{x}_T = 0$, and from Corollary 4.1 $\hat{y}_T = \max\{I_T, \mu_T^b\}$. We need to show, $\hat{x}_{T-1} = 0$. The cost to go function at period $T - 1$ is given by;

$$\begin{aligned}
\hat{H}_{T-1}^0(I_{T-1}) &= \min_{y_{T-1}, x_{T-1}} \hat{g}_{T-1}^0(I_{T-1}, x_{T-1}, y_{T-1}) + \alpha\hat{H}_T^0(I_T) \\
&= (\frac{h}{\alpha} - c)I_{T-1} - \pi_{T-1}^b\mu_{T-1}^b + \min_{y_{T-1}, x_{T-1}} cy_{T-1} \\
&+ (\pi_{T-1}^b + p^b)\max\{0, \mu_{T-1}^b - (y_{T-1} - x_{T-1})\} + (h - \alpha c)I_T - \alpha(\pi_T^b - c)\mu_T^b
\end{aligned}$$

Second equality is obtained by replacing $\hat{H}_T^0(I_T)$'s value from Corollary 4.1. (WLOG with the second case) Since the last term above is constant, it can be excluded from the minimization. Then, minimization in the second line has the same structure as the first line of Equation 6, thus we can apply proposition 4.1 and find $\hat{x}_{T-1} = 0$. (If we had used the third case for $\hat{H}_T^0(I_T)$, after removing the constant terms, we would have $(h + \alpha h - \alpha^2 c)I_T$ instead of $(h - \alpha c)I_T$, and we could still use Proposition 4.1). In order to keep simple, from now on we will do our analysis assuming $\mu_t^b > I_t$.

Let $j = T-1$, then from above discussion we know that $\hat{x}_{T-1} = \hat{x}_T = 0$ and $\hat{y}_T = \mu_T$. Using the fact that $\mu_{T-1}^b \leq C + I_{T-1}$ following the same logic as in Corollary 4.1, $\hat{y}_{T-1} = \mu_{T-1}^b$. If we write the cost-to-go function for $T-2$, we see that after excluding the constant terms, the minimization part has the same structure as the first line of Equation 6, and thus we can conclude that $\hat{x}_{T-2} = 0$.

Continuing in this fashion we can show that $\hat{x}_i = 0$, $\forall i \in \{j-1, j, ..., T\}$ and $\hat{y}_i = \mu_i^b$ $\forall i \in \{j, j+1, ..., T\}$.

Corollary 4.2. *Cost-to-go function in period $j-1$ is*

$$\hat{H}_{j-1}^0(I_{j-1}) = \alpha \sum_{i=j}^{T} -\alpha^{i-j}(\pi_i^b - c)\mu_i^b +$$
$$\begin{cases} (\frac{h}{\alpha} - c)I_{j-1} + p^b\mu_{j-1}^b - (\pi_{j-1}^b + p^b - c)(C + I_{j-1}) & \text{if } \mu_{j-1}^b > C + I_{j-1}, \\ (\frac{h}{\alpha} - c)I_{j-1} - (\pi_{j-1}^b - c)\mu_{j-1}^b & \text{if } \mu_{j-1}^b \leq C + I_{j-1} \end{cases}$$

Proof. From Proposition 4.2 we know that $\hat{x}_i = 0$ $\forall i \in \{j-1, j+1, ..., T\}$, and $\hat{y}_i = \mu_i^b$ $\forall i \in \{j, j+1, ..., T\}$. Thus, $I_i = 0$ $\forall i \in \{j, j+1, ..., T\}$. The first term is the cost-to-go function at period j, $\hat{H}_j^0(I_j = 0)$. Since, $\hat{H}_j^0(0)$ is constant, the minimization at period $j-1$ has the same structure as the one in the proof of Corollary 4.1. Thus, $\hat{y}_{j-1} = \min\{\mu_{j-1}^b, C + I_{j-1}\}$. Then, $\hat{H}_{j-1}^0(I_{j-1})$ is straightforward.

Proposition 4.3. *It is never optimal to carry inventory to the next periods unless current period's demand is fully satisfied.*

Proof. This proposition is a direct result of our 2nd model assumption. The cost of producing one product (by cutting current period's supply) and carrying it for i periods is, $\pi_t^b + p^b + c + \sum_{j=0}^{i-1} \alpha^j h$. The benefit of being able to satisfy one more unit demand at i periods later is (assuming capacity is binding at $t+i$), $\alpha^i(\pi_{t+i}^b + p^b)$. Using the relation in our model assumption we obviously see that cost > benefit, hence it is not optimal to cut current period's demand and produce for the next periods.

Corollary 4.3. *If $\mu_t^b > C + I_t$ for any t, then $\hat{y}_t = C + I_t$ and $\hat{x}_t = 0$.*

Proof. $\hat{x}_t = 0$ is a direct result of Proposition 4.3(i.e. not to cut the BC supply by $x_t > 0$), and $\hat{y}_t = C + I_t$ is a result of cost minimization in period t.

Proposition 4.4. *Suppose at current period $\mu_t^b < C + I_t$ and $\mu_{t+i} > C + I_{t+i}$ for some $i \in \{1, 2, ..., T - t\}$. If total cost (sum of costs from t to T) is reduced by carrying inventory for i periods, then $c + \sum_{j=0}^{i-1} \alpha^j h < (\pi_{t+i}^b + p^b)\alpha^i$ must hold.*

Proof. $c + \sum_{j=0}^{i-1} \alpha^j h$ is the cost of producing one more unit product today (without cutting supply of the current demand) and carrying it for i periods. $(\pi_{t+i}^b + p^b)\alpha^i$ is the benefit obtained from satisfying one more unit of product i periods later. If $c + \sum_{j=0}^{i-1} \alpha^j h \geq (\pi_{t+i}^b + p^b)\alpha^i$ then carrying inventory for i periods will not reduce the total cost at t.

Now, we can propose our algorithm to find the optimal CEC solutions for the CM's BC problem, for a given beginning inventory I_1.

The BC Algorithm

Step 1: For all t, assign $y_t = \max\{I_t, \min\{\mu_t^b, C + I_t\}\}$ and $x_t = \max\{y_t - \mu_t^b, 0\}$.
Step 2: Find all t such that $\mu_t^b < C + I_t$ and $\mu_{t+i_t}^b > C + I_{t+i_t}$ for some $i_t \in \{1, 2, ..., T - t\}$. If there is no such t end the algorithm, else put them in a list and go to Step 3.
Step 3 If the list is empty end the algorithm, else select the largest t, delete from list and go to Step 4.
Step 4: Among the all i_t's of t, select the one as i_t^*=argmax$\{(\pi_{t+i_t}^b + p^b)\alpha^{i_t} - c - \sum_{j=0}^{i_t-1} \alpha^j h\}$. If the value in argmax is positive, then increase x_j $(\forall j \in \{t, ..., t + i_t - 1\})$ by $\min\{\mu_{t+i_t} - (y_{t+i_t} - x_{t+i_t}), C + I_t - y_t\}$ and increase y_j $(\forall j \in \{t, ..., t + i_t\})$ by $\min\{\mu_{t+i_t} - (y_{t+i_t} - x_{t+i_t}), C + I_t - y_t\}$. If the value in argmax is negative, then go to Step 3. Repeat step 4 until either $y_t = C + I_t$ becomes true, or until the value in argmax becomes negative. In both cases go to step 3.

Theorem 4.4. *the BC algorithm terminates after finite number of steps, and at termination values of y_t and x_t are optimal. (i.e. $\hat{y}_t, \hat{x}_t$)*

Proof. Algorithm ends in finite number of steps, since the size of the list can be at most $T - 1$. For each t in the list, one checks at most $T - t$ values (the values in argmax function). Once, production capacity is reached for t, or there is no improvement in the cost by carrying inventory, then one never considers t again. Thus, after checking all the improvements that can be made, algorithm will end in the finite number of steps.

To prove second part of the theorem, we will first prove that if at *Step 2* no t is found then *Step 1* produces optimal results. If we cannot find a t that satisfies the condition at *Step 2*, then either all $\mu_t^b > C + I_t$ or all $\mu_t^b < C + I_t$. If all $\mu_t^b > C + I_t$ then *Step 1* assigns $y_t = C + I_t$ and $x_t = 0\ \forall t$ (Note that I_1 is supplied to the demand at first period, thus $I_t = 0\ \forall t > 1$). then, from Corollary 4.3, y_t and x_t are optimal. (i.e. $\hat{y}_t$ and $\hat{x}_t$) If all $\mu_t^b < C + I_t$, then *Step 1* assigns $y_t = \max\{I_t, \mu_t^b\}$ and $x_t = y_t - \mu_t^b, 0$. If initial inventory $I_1 \leq \mu_1^b$ then all $y_t = \mu_t^b$ and $x_t = 0$. Then using Proposition 4.2 (using j=1), y_t and x_t are optimal. If initial inventory $I_1 > \mu_1^b$ then no extra production is made until all beginning inventory is used to satisfy demand. One can never reduce costs acting otherwise, since all the demands can be satisfied from that period, and carrying would incur extra positive holding cost. Thus, if there is no t found in *Step 2*, algorithm ends with the optimal $\hat{y}_t$ and $\hat{x}_t$.

Now, suppose in *Step 2*, some t is found. These are the only t's that might improve the overall cost-to-go function by carrying inventory to the future. By selecting the largest t from the list at *Step 3*, we are attempting to find the optimal cost-to-go function at t. Since, there is no other periods larger than t that can produce more than their demand, the cost-to-go functions at periods greater than t are optimal. At *Step 4* one chooses (if exists) best improvement period $(t + i_t)$ for t among the all future periods of t. If the function in argmax is positive for the i_t^*, then production at t is increased by the minimum of what is needed to supply the demand at $t + i_t^*$ and how much more can be produced at t. Then this amount is carried until $t + i_t$, by adjusting the y and x levels for periods $[t, t + t + i_t]$. When the capacity becomes binding for t, then t cannot improve the cost-to-go function further. (Note that from Proposition 4.3, decreasing period t's supply for μ_t^b, to carry inventory to the next periods, would increase costs). Thus when capacity is binding $\hat{H}_t^0(I_t)$ is optimal. When no i_t is found, then from Proposition 4.4, cost-to-go function can not be reduced by carrying inventory. Thus it is optimal.

Going backwards, the algorithm produces the optimal cost-to-go functions for each t in the list. Thus, at termination we have the optimal cost-go-functions for each period. $\hat{H}_1^0(I_1)$ is the optimal cost of the CEC problem.

4.1.2 Case 2: When the CM Supplies to the both BC and LM

Similar to the derivation of $\hat{g}_t^0(I_t)$ we will use following periodic cost function for the CM.

$$\begin{aligned}
\hat{g}_t^1(I_t, z_t, x_t, y_t) &= (\frac{h}{\alpha} - c)I_t + cy_t - \pi_T^b \min\{y_t - x_t, \mu_t^b\} + p^b \max\{\mu_t^b - (y_t - x_t), 0\} \\
&- \pi^m \min\{\max\{y_t - \mu_t^b, x_t\} - z_t, \mu_t^m\} + p^m \max\{\mu_t^m + z_t - \max\{y_t - \mu_t^b, x_t\}, 0\} \\
&= (\frac{h}{\alpha} - c)I_t - \pi_t^b \mu_t^b - \pi^m \mu_t^m + cy_t + (\pi_t^b + p^b)\max\{\mu_t^b - (y_t - x_t), 0\} \\
&+ (\pi^m + p^m)\max\{\mu_t^m + z_t - \max\{y_t - \mu_t^b, x_t\}, 0\}
\end{aligned}$$

The boundary condition is $\hat{H}_{T+1}^1(I_{T+1}) = (h/\alpha - c)I_{T+1}$ where I_{T+1} is obtained from the recursive formula $I_{t+1} = \max\{z_t, \max\{y_t - \mu_t^b, x_t\} - \mu_t^m\}$. Then being analogous to Problem 5, CEC cost-to-go function;

$$\begin{aligned}
\hat{H}_t^1(I_t) &= \min_{y_t, x_t, z_t} \hat{g}_t^1(I_t, z_t, x_t, y_t) + \alpha \hat{H}_{t+1}^1(I_{t+1}) \\
\text{subject to } \ I_{t+1} &= \max\{z_t, \max\{y_t - \mu_t^b, x_t\} - \mu_t^m\}, \\
0 \leq z_t &\leq x_t \leq y_t \leq C + I_t, \ \forall t \in \{1, 2, ..T\}
\end{aligned} \tag{7}$$

Using the similar arguments we did above for the BC problem, we will analyze the structure of the cost-to-go function for BC and LM problem.

Proposition 4.5. *At period T, optimal value $\hat{z}_T = 0$.*

Proof. We will follow an analogous procedure as the proof of Proposition 4.1. After excluding the constant terms, the minimization problem to find the last stage cost-to-go function can be written as;

$$\begin{aligned}
&\min_{y_T, x_T, z_T} cy_T + (\pi_T^b + p^b)\max\{\mu_t^b - (y_T - x_T), 0\} + (\pi^m + p^m)\max\{\mu_T^m + z_T - \max\{y_T - \mu_T^b, x_T\}, 0\} \\
&+ (h - \alpha c)\max\{z_T, \max\{y_T - \mu_T^b, x_T\} - \mu_T^m\} \\
\Rightarrow &\min_{y_T, x_T, z_T} cy_t + \frac{\pi_T^b + p^b}{2}\Big(|\mu_T^b + x_T - y_T| + (\mu_T^b + x_T - y_T)\Big) + 0.5(\pi^m + p^m + h - \alpha c)z_T \\
&\frac{\pi^m + p^m + h - \alpha c}{2}\left|\mu_T^m + z_T - \frac{1}{2}\Big(|\mu_T^b + x_T - y_T| + (x_T + y_T - \mu_T^b)\Big)\right| + \\
&\frac{\pi^m + p^m - (h - \alpha c)}{2}\Big(\mu_T^m - \frac{1}{2}(|\mu_T^b + x_T - y_T| + (x_T + y_T - \mu_T^b))\Big)
\end{aligned} \tag{8}$$

Third, fourth and fifth lines are obtained from the definition of max and min functions. Let $\hat{y}_T$, $\hat{x}_T$ $\hat{z}_T$ be the optimal solution to the minimization problem 8 satisfying $0 < \tilde{z}_T \leq \hat{x}_T \leq \hat{y}_T \leq$

$C + I_T$, and $\tilde{H}_T^1(I_T)$ is the cost obtained from this solution. As in Proposition 4.1 we will analyze $\tilde{H}_T^1(I_T)$ in two cases; 1. $\mu_T^b + \mu_T^m > C + I_T$ and 2. $\mu_T^b + \mu_T^m \leq C + I_T$

Case 1 $\mu_T^b + \mu_T^m > C + I_T$. Then, depending on the value of $\hat{x}_T$ and $\hat{y}_T$ the outer absolute value in the fourth line simplifies to either $|\mu_T^m + \tilde{z}_T - \hat{x}_T|$ (if, $\hat{y}_T \leq \mu_T^b + \hat{x}_T$) or, $|\mu_T^m + \tilde{z}_T + \mu_T^b - \hat{y}_T|$ (if, $\hat{y}_T > \mu_T^b + \hat{x}_T$). In the first case;

$$\mu_T^m + \tilde{z}_T - \hat{x}_T \geq \mu_T^m + \tilde{z}_T + \mu_T^b - \hat{y}_T > C + I_T + \tilde{z}_T - \hat{y}_T > 0$$

The first, inequality is obtained from using $\hat{y}_T \leq \mu_T^b + \hat{x}_T$, second equality is obtained using the fact that $\mu_T^b + \mu_T^m > C + I_T$ and the last inequality is obtained using $\tilde{z}_T > 0$ and $\hat{y}_T \leq C + I_T$. Thus in either case the function in the outer absolute value (on the fourth line) is positive. Since $\pi^m + p^m + h - \alpha c > 0$ (from our first model assumption) $\tilde{z}_T > 0$ cannot be optimal.

Case 2 $\mu_T^b + \mu_T^m \leq C + I_T$. From the Case 1 discussion, it is sufficient to analyze the sign of $\mu_T^m + \tilde{z}_T + \mu_T^b - \hat{y}_T$. Depending on the value of $\hat{y}_T$, $\mu_T^m + \tilde{z}_T + \mu_T^b - \hat{y}_T$ can be positive, 0, or negative. When it is positive Case 1 applies. When it is 0, $\tilde{H}_T^1(I_t)$ becomes $c\hat{y}_T + 0.5(\pi^m + p^m + h - \alpha c)\tilde{z}_T + 0.5(\pi^m + p^m - h + \alpha c)(\mu_T^m + \mu_T^b - \hat{y}_T)$. Since $(\pi^m + p^m + h - \alpha c) > 0$ $\tilde{z}_T > 0$ can not be optimal. When $\mu_T^m + \tilde{z}_T + \mu_T^b - \hat{y}_T$ is negative, then $\tilde{H}_T^1(I_T)$ becomes, $c\hat{y}_T + (h - \alpha c)(\hat{y}_T - \mu_T^m - \mu_T^b)$. Since $h + c - \alpha c > 0$ we can use $\hat{y}_T > \mu_T^b + \mu_T^m + \tilde{z}_T$ to have;

$$\tilde{H}_T^1(I_T) > c(\mu_T^b + \mu_T^m) + (h + c - \alpha c)\tilde{z}_T$$

Thus, $\tilde{z}_T > 0$ cannot be optimal.

Corollary 4.5. *At T, $\hat{y}_T = max\{I_T, min\{\mu_T^b + \mu_T^m, C + I_T\}\}$, and for $\mu_T^b + \mu_T^m > I_T$*

$$\hat{x}_T = \begin{cases} max\{\hat{y}_T - \mu_T^b, 0\} & \text{if, } \pi_T^b + p^b \geq \pi^m + p^m \\ min\{\mu_T^m, \hat{y}_T\} & \text{if, } \pi_T^b + p^b < \pi^m + p^m \end{cases}$$

for $\mu_T^b + \mu_T^m \leq I_T$, $\hat{x}_T = \hat{y}_T - \mu_T^b$.

Proof. Due to the similarity with Corollary 4.1, when $\mu_T^b + \mu_T^m < I_T$, $\hat{y}_T = I_T$ and $\hat{x}_T = \hat{y}_T - \mu_T^b$. For the other cases, we will analyze the cost-to-go function (with $\hat{z}_T = 0$), in four categories:

1. $\mu_T^b > C + I_T$ and $\mu_T^m > C + I_T$
2. $\mu_T^b \leq C + I_T$ and $\mu_T^m > C + I_T$
3. $\mu_T^b > C + I_T$ and $\mu_T^m \leq C + I_T$

4. $\mu_T^b + \mu_T^m \leq C + I_T$

Case 1 $\mu_T^b > C + I_T$ and $\mu_T^m > C + I_T$

$$\min_{y_T, x_T} cy_T + (\pi_T^b + p^b)\max\{\mu_T^b + x_T - y_T, 0\} + (\pi^m + p^m)\max\{\mu_T^m - \max\{y_T - \mu_T^b, x_T\}, 0\}$$
$$(h - \alpha c)\max\{0, \max\{y_T - \mu_T^b, x_T\} - \mu_T^m\}$$
$$\Rightarrow \min_{y_T, x_T} cy_T + (\pi_T^b + p^b)(\mu_T^b + x_T - y_T) + (\pi^m + p^m)(\mu_T^m - x_T)$$
$$\Rightarrow (\pi_T^b + p^b)\mu_T^b + (\pi_T^m + p^m)\mu_T^m + \min_{y_T, x_T} cy_T - (\pi_T^b + p^b)y_T + (\pi_T^b + p^b - \pi_T^m - p^m)x_T$$

Third line is obtained by the fact that $\mu_T^b + x_T - y_T > 0$ (since $\mu_T^b > C + I_t \geq y_T$ and $x_T \geq 0$), $\max\{y_T - \mu_T^b, x_T\} = x_T$ (since $y_T - \mu_T^b < 0$) leading to $\max\{\mu_T^m - x_T, 0\} = \mu_T^m - x_T$ (since $\mu_T^m > C + I_T \geq y_T \geq x_T$). The $h - \alpha c$ term becomes 0, since $x_T - \mu_T^m < 0$.

For the minimization problem in the last line, since $\pi_T^b + p^b > c$, the minimization leads to $\hat{y}_T = C + I_T$, and if $\pi_T^b + p^b > \pi^m - p^m$, $\hat{x}_T = 0$, else $\hat{x}_T = \hat{y}_T = C + I_T$.

Case 2 $\mu_T^b \leq C + I_T$ and $\mu_T^m > C + I_T$

$$\min_{y_T, x_T} cy_T + (\pi_T^b + p^b)\max\{\mu_T^b + x_T - y_T, 0\} + (\pi^m + p^m)\max\{\mu_T^m - \max\{y_T - \mu_T^b, x_T\}, 0\}$$
$$(h - \alpha c)\max\{0, \max\{y_T - \mu_T^b, x_T\} - \mu_T^m\}$$
$$\Rightarrow \min_{y_T, x_T} cy_T + (\pi_T^b + p^b)\max\{\mu_T^b + x_T - y_T, 0\} + (\pi^m + p^m)(\mu_T^m - \min\{\mu_T^m, \max\{y_T - \mu_T^b, x_T\}\})$$
$$(h - \alpha c)(\max\{y_T - \mu_T^b, x_T\} - \min\{\mu_T^m, \max\{y_T - \mu_T^b, x_T\}\})$$
$$\Rightarrow (\pi_T^m + p^m)\mu_T^m + \min_{y_T, x_T} cy_T - (\pi^m + p^m)\max\{y_T - \mu_T^b, x_T\} + (\pi_T^b + p^b)\max\{\mu_T^b + x_T - y_T, 0\}$$
$$\Rightarrow \min_{y_T, x_T} cy_T - (\pi^m + p^m)\max\{y_T - \mu_T^b, x_T\} - (\pi_T^b + p^b)\min\{y_T - \mu_T^b, x_T\} + (\pi_T^b + p^b)x_T$$

Third line is obtained by using the relation $\max\{0, b - a\} = b - \min\{a, b\}$ with $b = \mu_T^m$ and $a = \max\{y_T - \mu_T^b, x_T\}$ for the $(\pi^m + p^m)$ term, and with $a = \mu_T^m$ and $b = \max\{y_T - \mu_T^b, x_T\}$ for the $(h - \alpha c)$ term. $(h - \alpha c)$ term is canceled since $\min\{\mu_T^m, \max\{y_T - \mu_T^b, x_T\}\} = \max\{y_T - \mu_T^b, x_T\}$. Last relation is obtained (after omitting the constant term) by using the relation $\max\{0, b - a\} = b - \min\{a, b\}$ with $b = x_T$ and $a = y_T - \mu_T^b$. If $\pi^m + p^m > \pi_T^b + p^b$ then last relation is minimized at, $\hat{x}_T = C + I_T$ and $\hat{y}_T = C + I_T$, else the solution is $\hat{y}_T = C + I_T$ and $\hat{x}_T \in [0, \hat{y}_T - \mu_T^b]$.

Case 3 $\mu_T^b > C + I_T$ and $\mu_T^m \leq C + I_T$

$$\min_{y_T, x_T} cy_T + (\pi_T^b + p^b)\max\{\mu_T^b + x_T - y_T, 0\} + (\pi^m + p^m)\max\{\mu_T^m - \max\{y_T - \mu_T^b, x_T\}, 0\}$$
$$(h - \alpha c)\max\{0, \max\{y_T - \mu_T^b, x_T\} - \mu_T^m\}$$
$$\Rightarrow \min_{y_T, x_T} cy_T + (\pi_T^b + p^b)(\mu_T^b + x_T - y_T) + (\pi^m + p^m)\max\{\mu_T^m - x_T, 0\} + (h - \alpha c)\max\{0, x_T - \mu_T^m\}$$
$$\Rightarrow (\pi_T^b + p^b)\mu_T^b + (\pi_T^m + p^m)\mu_T^m + \min_{y_T, x_T} cy_T - (\pi_T^b + p^b)y_T + (\pi_T^b + p^b + h - \alpha c)x_T$$
$$- (\pi^m + p^m + h - \alpha c)\min\{x_T, \mu_T^m\}$$

Above function is minimized at $\hat{y}_T = C + I_T$ and $\hat{x}_T = 0$, (if $\pi_T^b + p^b > \pi^m + p^m$) or $\hat{x}_T = \mu_T^m$ (if $\pi_T^b + p^b < \pi^m + p^m$).

Case 4 $\mu_T^b + \mu_T^m \leq C + I_T$. This part is straightforward. $\hat{y}_T = \mu_T^b + \mu_T^m$ and $\hat{x}_T \in [0, \hat{y}_T - \mu_T^b]$. There is no need to produce more, since cost of carrying unit inventory $h + c - \alpha c > 0$.

If we combine all the solutions together we obtain;

Proposition 4.6. *Let j be the smallest period in $\{1, 2, .., T\}$ such that $\mu_i^b + \mu_i^m \leq C + I_i$ $\forall i \in \{j, j+1, ..., T\}$, then $\hat{z}_i = 0$ $\forall i \in \{j-1, j, ..., T\}$.*

Proof. The proof has the same analogy as the proof of Proposition 4.2. At the end of the proof we have the result $\hat{y}_i = \mu_i^b + \mu_i^m$ $\forall i \in \{j, j+1, ..., T\}$, $\hat{x}_i = \hat{y}_i - \mu_i^b = \mu_i^m$ $\forall i \in \{j, j+1, ..., T\}$ and $\hat{z}_i = \hat{x}_i - \mu_i^m = 0$ $\forall i \in \{j-1, j, ..., T\}$ are optimal.

Differing from the BC Problem, in this case when the capacity is binding at a given period, it might be still optimal to carry inventory and increase the supply of the future period demands. These are the cross benefit cases. i.e. Cutting BC's (or LM's) demand today to satisfy more LM (or BC) demand tomorrow. We will show this in the following Proposition.

Proposition 4.7. *Let (i, j), $(j > i)$ be a pair of two periods satisfying $\mu_i^b + \mu_i^m > C + I_i$, $\mu_j^b + \mu_j^m > C + I_j$. Assume an allocation satisfying $y_i - x_i < \mu_i^b$, $x_i - z_i < \mu_i^m$ and $y_j - x_j < \mu_j^b$, $x_j - z_j < \mu_j^m$. Then a better allocation can be made if $(\pi_i^b + p^b) + \sum_{k=0}^{i-j-1} \alpha^k h < (\pi^m + p^m)\alpha^{j-i}$ or $(\pi^m + p^m) + \sum_{k=0}^{i-j-1} \alpha^k h < (\pi_j^b + p^b)\alpha^{j-i}$ holds.*

Proof. We will only show the case, cutting BC supply at period i to increase LM supply at period j improves cost. The other way follows exactly the same logic. The unit cost of cutting BC demand is $\pi_i^b + p^b$, and cost carrying unit inventory for $i - j$ periods is $\sum_{k=0}^{i-j-1} \alpha^k h$. Note that production cost is not considered since, no extra production is made at i. Current benefit of supplying one more unit to the LM at period j is $(\pi^m + p^m)\alpha^{j-i}$. If the relation in the proposition holds, then one can reduce $y_i - x_i$ (without affecting the LM supply at period i) and increase $x_j - z_j$ (without affecting the BC supply at period j) and reduce the overall cost.

Before proposing our algorithm we need to show a similar argument as in Proposition 4.4. However, the logic of the proof is exactly same, thus without showing it here, we will say that argument in Proposition 4.4 holds for LM and BC problem, too.

Below we propose our algorithm to solve CM's BC and LM Problem. To improve the initial solution, we will not only check the periods where there is available production capacity, but also the periods where the capacity is binding. For the periods in which the capacity is binding, by cutting low priced customer's demand and reserving inventory, the CM might be better off when she satisfies a higher priced customer's demand in the future. We call this action *Switching*. After, switching the algorithm goes as the BC algorithm, by checking the periods where there is available production capacity.

The BC and LM Algorithm

Step 1 For all t, assign $y_t = \max\{I_t, \min\{\mu_t^b + \mu_t^m, C + I_t\}\}$. If $\mu_t^b + \mu_t^m < I_t$, assign

$$x_t = \begin{cases} \max\{y_t - \mu_t^b, 0\} & \text{if, } \pi_t^b + p^b \geq \pi^m + p^m \\ \min\{\mu_t^m, y_t\} & \text{if, } \pi_t^b + p^b < \pi^m + p^m \end{cases}$$

else, assign $x_t = y - \mu_t^b$. For all t, assign $z_t = \max\{x_t - \mu_t^m, 0\}$

Step 2 Switching. Find all t such that $\mu_t^b + \mu_t^m > C + I_t$ and $\mu_{t+i_t}^b > y_{t+i_t} - x_{t+i_t}$ or $\mu_{t+i_t}^m > x_{t+i_t} - z_{t+i_t}$ for some $i_t \in \{1, 2, ..T - t\}$. If no such t is found go to *Step 3*, else put them in a list and go to *Step 2.1*.

Step 2.1 If the list is empty go to *Step 3*, else pick the largest t, delete from the list. While,

$y_t - x_t > 0$ or $x_t - z_t > 0$ find i_t^* according to;

$$i_t^m = \text{argmax}\{(\pi^m + p^m)\alpha^i - (\pi_t^b + p^b + \sum_{j=0}^{i-1} \alpha^j h)\} \quad \forall i \in \{1, 2, ...T - t\}$$

$$\text{subject to } y_t - x_t > 0 \text{ and } \mu_{t+i}^m - (x_{t+i} - z_{t+i}) > 0$$

$$i_t^b = \text{argmax}\{(\pi_{t+i}^b + p^b)\alpha^i - (\pi^m + p^m + \sum_{j=0}^{i-1} \alpha^j h)\} \quad \forall i \in \{1, 2, ...T - t\}$$

$$\text{subject to } x_t - z_t > 0 \text{ and } \mu_{t+i}^b - (y_{t+i} - x_{t+i}) > 0$$

$$i_t^* = \text{argmax}\{\max\{(\pi^m + p^m)\alpha^{i_t^m} - (\pi_t^b + p^b + \sum_{j=0}^{i_t^m - 1} \alpha^j h), (\pi_{t+i_t^b}^b + p^b)\alpha^{i_t^b} - (\pi^m + p^m + \sum_{j=0}^{i_t^b - 1} \alpha^j h)\}\}$$

If the function in the last argmax is negative or no i_t^* is found, then go to the beginning of *Step 2.1*, else do the following; If the function in the last argmax is maximized by the first component, then increase y_j, $(\forall j \in \{t + 1, ..., t + i_t\})$, x_j $(\forall j \in \{t, ..., t + i_t\})$ and z_j $(\forall j \in t, ..., t + i_t - 1)$ by $\min\{y_t - x_t, \mu_{t+i_t}^m - (x_{t+i_t} - z_{t+i_t})\}$. If function in the last argmax is maximized by the second component, then increase y_j, $(\forall j \in \{t + 1, ..., t + i_t\})$, x_j $(\forall j \in \{t + 1, ..., t + i_t - 1\})$ and z_j $(\forall j \in t, ..., t + i_t - 1)$ by $\min\{x_t - z_t, \mu_{t+i_t}^b - (y_{t+i_t} - x_{t+i_t})\}$

Step 3 Find all t's such that $\mu_t^b + \mu_t^m < C + I_t$ and $\mu_{t+i_t}^b + \mu_{t+i_t}^m > C + I_{i_t}$ for some $i_t \in \{1, 2, ..., T - t\}$. If there is no such t, end the algorithm, else put them in a list and go to *Step 4*.

Step 4 If the list is empty, end the algorithm else, pick the largest t, delete from the list and find i_t^* as follows

$$i_t^b = \text{argmax}\{\alpha^i(\pi_{t+i}^b + p^b) - c - \sum_{j=0}^{i-1} \alpha^j h\} \quad \forall i \in \{1, 2, ..., T - t\}$$

$$\text{subject to } \mu_{t+i}^b - (y_{t+i} - x_{t+i}) > 0$$

$$i_t^m = \text{argmax}\{\alpha^i(\pi_{t+i}^m + p^m) - c - \sum_{j=0}^{i-1} \alpha^j h\} \quad \forall i \in \{1, 2, ..., T - t\}$$

$$\text{subject to } \mu_{t+i}^m - (x_{t+i} - z_{t+i}) > 0$$

$$i_t^* = \text{argmax}\{\max\{\alpha^{i_t^b}(\pi_{t+i_t^b}^b + p^b) - c - \sum_{j=0}^{i_t^b - 1} \alpha^j h, \alpha^{i_t^m}(\pi_{t+i_t^m}^m + p^m) - c - \sum_{j=0}^{i_t^m - 1} \alpha^j h\}\}$$

If the function in the last argmax is negative or no i_t^* is found, then go to beginning of *Step 4*, else do the following; If the function in the last argmax is maximized by the first component, then increase y_j, $(\forall j \in \{t, ..., t + i_t\})$, x_j $(\forall j \in \{t, ..., t + i_t - 1\})$ and z_j $(\forall j \in t, ..., t + i_t - 1)$

by $\min\{\mu^b_{t+i_t} - (y_{t+i_t} - x_{t+i_t}), C + I_t - y_t\}$. If the function in the last argmax is maximized by the second component, then increase y_j, $(\forall j \in \{t, ..., t + i_t\})$, x_j $(\forall j \in \{t, ..., t + i_t - 1\})$ and z_j $(\forall j \in t, ..., t + i_t - 1)$ by $\min\{\mu^m_{t+i_t} - (x_{t+i_t} - z_{t+i_t}), C + I_t - y_t\}$.

Theorem 4.6. *The BC and LM algorithm terminates after finite number of steps, and at termination values of y_t, x_t and z_t are optimal. (i.e. $\hat{y}_t, \hat{x}_t, \hat{z}_t$)*

Proof. In *Step 2*, the size of the list can be at most $T - 1$. For every t in the list, the algorithm searches for a better allocation than the one assigned in *Step 1*. For a given t, the result of a search can be either negative (when no i^*_t is found or cost cannot be improved for i^*_t), or positive (when i^*_t improves overall cost). In the first case, t is removed and is not considered in this list, again. In the second case, a better allocation is made (by reducing current allocation of one customer and increasing future allocation of the other customer). After the allocation, either current allocation of the customer becomes 0, or future allocation of the customer is at its demanded quantity. In the latter case, algorithm searches for the next best i^*_t. Again, if the search for i^*_t is positive, a better allocation is made. At some point, algorithm will stop the search for next best i^*_t, since current allocation for the customer will hit 0, and in this case other t's in the list will be considered. After switching step is done, the algorithm will search for cost improvements by producing more, and this step proceeds exactly same as the improvement step in the BC Algorithm. Hence, the algorithm will end after finite number of steps.

To prove the optimality of the allocation, first we will prove that, if no t is found in *Step 3*, then the allocation is optimal. If the algorithm cannot find any t at *Step 3*, then either $\mu^b_t + \mu^m_t < C + I_t$ $\forall t$, or $\mu^b_t + \mu^m_t > C + I_t$ $\forall t$. In the first case, then *Step 1* assigns $y_t = \max\{I_t, \mu^b_t + \mu^m_t\}$, $x_t = y_t - \mu^b_t$ and $z_t = \max\{x_t - \mu^m_t, 0\}$. If initial inventory $I_1 \leq \mu^b_1 + \mu^m_1$, then $y_t = \mu^b_t + \mu^m_t$, $x_t = y_t - \mu^b_T = \mu^m_t$ and $z_t = 0$. Then using Proposition 4.6, (using j=1), y_t, x_t and z_t are optimal. If initial inventory $I_1 > \mu^b_1 + \mu^m_1$, then algorithm imposes no production until all beginning inventory is used to satisfy demand. One can never reduce costs acting otherwise, since all the demands can be satisfied from that period, and carrying would incur extra positive holding cost. Thus, the allocation at *Step 1* is optimal, when $\mu^b_t + \mu^m_t < C + I_t$ $\forall t$. The algorithm does not find any t at *Step 2*, since all the demands have been satisfied. In the second case, $(\mu^b_t + \mu^m_t > C + I_t$ $\forall t)$ *Step 1* assigns all

$y_t = C + I_t$, $z_t = 0$, and selects x_t depending on the profitability of the customers. However, this allocation may or may not be optimal. *Step 2*, searches for a better allocation. In this case, all the periods (except the terminal period) will be in the list. Because, the total supply is less than the total demand at each period. By going backwards, *Step 2* checks whether cost can be decreased by cutting the supply of one customer at that period and increasing the supply of the other customer for all the future periods. If no such period is found then the current allocation is optimal for that period. If a period is found, then a better allocation is made. This procedure repeats itself until no improvement can be found for that period. Since, every period is checked, *Step 2* produces optimal cost-to-go functions for each period. When *Step 2* ends no t is found at Step 3, since *Step 2* does not reduce production level of a given period. (Still each period use its capacity in full).

For all the other demand cases, every step of the algorithm will be executed. After the allocation at the initial step, the algorithm will search for switching. *Step 2* checks only the periods that use its capacity in full. Allocation after switching is at least as good as the allocation made in *Step 1*. Since, the periods that have available capacity haven't been checked, the algorithm performs this at *Step 3*. Hence, all of the periods are checked by the algorithm, starting from the largest periods.

5 Future Agenda

In the previous section, we proposed two algorithms to solve the CEC approximation to problems 1 and 2. (Namely problems 5 and 7, respectively). Solutions to the CEC problems provide us the optimal allocation scheme in the deterministic environment. We aim to approximate the true cost of the original problems 5 and 7, by simply inserting these solutions to the original cost-to-go functions. However the strategic market entry decision depends simultaneously on the cost of the CM serving BC (the costs before the market entry) and the cost of CM serving BC and LM (the costs after the market entry). Currently, both algorithms proceed independently. In order to approximate the cost of our main problem 3, we need to combine the two algorithms. After combining the two algorithms, we will be able to approximate the cost-to-go function $V_t(I_t)$. Let $\hat{V}_t(I_t)$ be the approximation of the cost-to-go function $V_t(I_t)$. Then, by evaluating $\hat{V}_1(I_1)$ for all possible values of market entry points, we can select the one that gives us the minimum $\hat{V}_1(I_1)$. This means we need to make $T - k + 2$ evaluations. (Remember that k is the lag between the BC and the LM demands) One evaluation for each period in $\{k, k+1, ..., T\}$ plus one evaluation for no market entry decision.

In the previous section, we also discussed the quality of the CEC approximation and proposed a modification to reflect the stochastic environment of the original problem. Once, we are able to combine the two algorithms, we can generate numerous samples from the BC and LM demand distributions, and solve the CEC problems for each sample. Then, averaging these sample solutions to hopefully get a better cost approximation for $V_1(I_1)$.

After having developed the theoretical background described above, our aim is to generalize the theory for a wide range of settings, and combine them into a strategic planning tool. We plan to direct our future research into following areas;

1. *The characterization of the decision patterns under different demand types.*
   A significant factor of the above decision framework is the modeling of market demand. We will consider short life-cycle products follows as life-cycle growth model, which exhibits a demand life-cycle of growth, maturity, and decline. These types of models are suitable to describe high technology products such as mobile phones and micro chips. Meade and Islam (1998) document 29 different growth curves found in the literature, which provide a rich set of demand modeling options. By investigating the optimal decisions under these demand type setting, we hope to get some insight from the aspect of CM, about how she should react to the market environment.

2. *The impact of the contracting type between the CM and the BC on the CM's decisions.*
   Most of literatures study the outsourcing problem from the aspect of brand carrying customer. In this research, we are taking the view of CM to study the influence of outsourcing. As we have seen in the literature, different contracts affect the decisions a lot. Thus, it is quite useful to study the impact of different contracting types on the CM's decisions under our framework. A wide variety of contract types could be investigated, for instance, minimum quantity, quantity flexible, buy-back, etc. We might even consider the CM has the option to outsource part of her production to a smaller contractor.

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