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This paper provides sufficient conditions for the optimal value function of a given linear semi-infinite programming problem to depend linearly on the size of the perturbations, when these perturbations involve either the cost coefficients or the right-hand-side function or both, and they are sufficiently small. Two kinds of partitions are considered. The first one concerns the effective domain of the optimal value as a function of the cost coefficients, and consists of maximal regions on which this value function is linear. The second class of partitions considered in the paper concern the index set of the constraints through a suitable extension of the concept of optimal partition from ordinary to semi-infinite linear programming. These partitions provide convex sets, in particular segments, on which the optimal value is a linear function of the size of the perturbations, for the three types of perturbations considered in this paper.

Key words: Sensitivity analysis; linear semi-infinite programming; linear programming; optimal value function

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1. Introduction Given a linear semi-infinite programming (LSIP) problem, we give conditions guaranteeing the linearity of the optimal value function with respect to perturbations provided they are sufficiently small and involve either the cost coefficients or the right-hand-side function or both. The preceding works are, first, a stream of papers on sensitivity analysis in ordinary and parametric linear programming (LP) from an optimal partition perspective ([1], [2], [6], [8], [9], [13], [14], [15], [16], [18], [19], [20], [22], [23]) and, second, the recent paper [10], where conditions are given for the linearity (not only on segments) of the optimal value function of a LSIP problem with respect to (non-simultaneous) perturbations of the cost vector or the RHS function from a duality perspective.

Given a vector $c \in \mathbb{R}^n$, we consider two (possibly infinite) sets of indices, U and V, such that $U \cap V = \emptyset$ and $U \neq \emptyset$, and two functions $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$, where $T := U \cup V$. We associate with the triple $(a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ (the data) a primal nominal problem,

$$\begin{array}{lll} P: & \operatorname{Inf}_{x \in \mathbb{R}^n} & c'x \\ & \text{s.t.} & a_t'x \geq b_t,\ t \in U, \\ & & a_t'x = b_t,\ t \in V, \end{array}$$

which is assumed to be consistent, and its corresponding dual nominal problem in $\mathbb{R}^{(T)}$ (the linear space of *generalized finite sequences*, i.e., the functions $\lambda : T \to \mathbb{R}$ such that $\lambda_t = 0$ for all $t \in T$ except maybe for a finite number of indices),

$$\begin{array}{lll} D: & \operatorname{Sup}_{\lambda \in \mathbb{R}^{(T)}} & \sum_{t \in T} \lambda_t b_t \\ & \text{s.t.} & \sum_{t \in T} \lambda_t a_t = c, \\ & & \lambda_t \geq 0,\ t \in U. \end{array}$$

These problems are called *bounded* when their optimal values are finite. In contrast with LP, in LSIP the boundedness of both problems does not imply their solvability and zero duality gap. We denote by F and F^* (by Λ and Λ^*) the *feasible* and the *optimal sets* of P (of D, respectively). We assume throughout that $\emptyset \neq F \neq \mathbb{R}^n$. In many practical applications T is a compact Hausdorff space and the functions $a_.$ and $b_.$ are continuous on T, in which case P is called *continuous*.

If we replace c by $z \in \mathbb{R}^n$ in P and D we get parametric LSIP problems whose optimal value depends on z, namely

$$P(z): \quad \begin{array}{ll} \operatorname{Inf}_{x\in\mathbb{R}^n} & z'x \\ \text{s.t.} & a_t'x \geq b_t,\ t \in U, \\ & a_t'x = b_t,\ t \in V, \end{array}$$

and

$$D(z): \quad \begin{array}{ll} \operatorname{Sup}_{\lambda\in\mathbb{R}^{(T)}} & \sum\limits_{t\in T} \lambda_t b_t \\ \text{s.t.} & \sum\limits_{t\in T} \lambda_t a_t = z, \\ & \lambda_t \geq 0,\ t \in U. \end{array}$$

We denote the optimal values of $P(z)$ and $D(z)$ by $v^P(z)$ and $v^D(z)$, respectively. Since Sections 4-6 deal with optimal value functions of different parameters, in order to avoid confusion, our notation makes explicit the corresponding argument, i.e., we represent the optimal value functions by $v^P(z)$ and $v^D(z)$, instead of just v^P and v^D, which denote the optimal value of the nominal problems P and D, respectively. With this notation, we have $v^P(c) = v^P$ and $v^D(c) = v^D$, respectively. In [10, Section 2], using duality theory, it is shown that $v^P(z)$ is linear on a certain neighborhood of c if and only if P has a strongly unique optimal solution. It is also proved there, that $v^P(z)$ is linear on a segment emanating from c in the direction of $d \in \mathbb{R}^n \setminus \{0_n\}$ if P and D are solvable, with $v^D = v^P$, and the following problem is also solvable and has zero duality gap:

$$D_d: \quad \begin{array}{ll} \operatorname{Sup}_{\lambda\in\mathbb{R}^{(T)},\mu\in\mathbb{R}} & \sum\limits_{t\in T} \lambda_t b_t + \mu v^P(c) \\ \text{s.t.} & \sum\limits_{t\in T} \lambda_t a_t + \mu c = d, \\ & \lambda_t \geq 0,\ t \in U. \end{array}$$

Alternatively, if we replace b by $w \in \mathbb{R}^T$ in P and D we get parametric LSIP problems whose optimal value depends on w. These perturbed problems are

$$P(w): \quad \begin{array}{ll} \operatorname{Inf}_{x\in\mathbb{R}^n} & c'x \\ \text{s.t.} & a_t'x \geq w_t,\ t \in U, \\ & a_t'x = w_t,\ t \in V, \end{array}$$

and

$$D(w): \quad \begin{array}{ll} \operatorname{Sup}_{\lambda\in\mathbb{R}^{(T)}} & \sum\limits_{t\in T} \lambda_t w_t \\ \text{s.t.} & \sum\limits_{t\in T} \lambda_t a_t = c, \\ & \lambda_t \geq 0,\ t \in U, \end{array}$$

with respective optimal values $v^P(w)$ and $v^D(w)$. Consequently, the optimal values of the nominal problem P and its dual D are $v^P(b) = v^P$ and $v^D(b) = v^D$, respectively. Concerning the perturbations of $b: T \to \mathbb{R}$, we consider the linear space $\mathbb{R}^T$ equipped with the pseudometric $\delta(f,g) := \sup_{t\in T} |f(t) - g(t)|$, for $f, g \in \mathbb{R}^T$ (we may have $\delta(f,g) = +\infty$). The zero-vector in $\mathbb{R}^T$ is denoted by 0_T. In [10, Section 2], using also duality theory, it is shown that, if $v^P(w)$ is linear on a certain neighborhood of b (in the pseudometric space $(\mathbb{R}^T, \delta)$), then D has at most one optimal solution (the converse is true under strong assumptions). Moreover, $v^P(w)$ is linear on a segment emanating from b in the direction of a bounded function $f \in \mathbb{R}^T \setminus \{0_T\}$ if P and D are solvable with the same optimal value, the problem

$$P_f: \quad \begin{array}{ll} \operatorname{Inf}_{x\in\mathbb{R}^n, y\in\mathbb{R}} & c'x + v^P(b)\,y \\ \text{s.t.} & a_t'x + b_t y \geq f_t,\ t \in U, \\ & a_t'x + b_t y = f_t,\ t \in V \end{array}$$

is also solvable and has zero duality gap, and P_f satisfies certain additional condition.

The duality approach used in [10] does not provide conditions for the affinity of the optimal value functions for simultaneous perturbations of c and b. In this paper we exploit a suitable extension (from LP to LSIP) of the concept of optimal partition in order to obtain counterparts of the mentioned results about separate perturbations of c and b, as well as conditions guaranteeing the affinity of the optimal value functions under simultaneous perturbations of c and b. The authors of [12] and [25] have extended the notion of optimal partition from LP to semidefinite programming (SDP) and conic linear programming (CLP), respectively, obtaining sensitivity results for both types of optimization problems. Any SDP problem admits a LSIP reformulation, and any LSIP problem admits a CLP reformulation with infinite dimensional decision space ($\mathbb{R}^n \times \mathbb{R}^U$ for our LSIP problem P), whereas the converse reformulations are generally impossible. Since the decision spaces of the CLP problems considered in [25] are finite dimensional, our results cannot be derived from the theories developed in these two papers.

This paper is structured as follows. Section 2 shows that the domain of any convex homogeneous function can be partitioned into maximal relatively open convex cones where the function is linear, which are called *linearity cones* of the given function. This result generalizes the characterization of the largest open set containing c on which $v^P(z)$ is linear ([10, Theorem 3]), where P is required to have a strongly unique optimal solution, to a wide family of extended functions. Section 3 extends and analyzes the concepts of complementary solution and optimal partition from LP to LSIP. Section 4 examines the linearity of the optimal value functions associated with perturbations of c on convex sets (e.g., on segments emanating from c and on maximal relatively open convex cones containing c) by means of the theory developed in Section 2 (as both optimal value functions are concave, proper and homogeneous in the case of perturbations of c) and Section 3. Sections 5 and 6 give sufficient conditions for the optimal value function to depend linearly on the size of the perturbations when the perturbed data are the RHS function b or both parameters, vector c and function b, respectively. These conditions are expressed in terms of optimal partitions. Finally, Section 7 contains the conclusions.

We finish this introduction by summarizing some basic concepts and results of LSIP theory that will be used throughout. All these results can be easily derived from [11], where $V = \emptyset$. First we introduce some necessary notation.

We consider $\mathbb{R}^n$ equipped with the Euclidean norm in $\mathbb{R}^n$, $\|\cdot\|$. The canonical basis, the zero-vector, and the open unit ball in $\mathbb{R}^n$ will be denoted by $\{e_1, ..., e_n\}$, 0_n, and $B(0_n; 1)$, respectively. For any set X, $|X|$ denotes the *cardinality of* X. If $\emptyset \neq X \subset \mathbb{R}^n$, we denote by $\operatorname{cl} X$, $\operatorname{int} X$, $\operatorname{rint} X$, $\operatorname{conv} X$, $\operatorname{cone} X$, $\operatorname{aff} X$, $\operatorname{span} X$, and X^0 the *closure*, the *interior*, the *relative interior*, the *convex hull*, the *convex conical hull* (of $X \cup \{0_n\}$), the *affine hull*, the *linear hull*, and the *positive polar* of X, respectively. The dimension of a convex set $X \subset \mathbb{R}^n$ will be denoted by $\dim X$. A set $X \subset \mathbb{R}^n$ is relatively open if $\operatorname{rint} X = X$. A vector $y \in \mathbb{R}^n$ is a *feasible direction* at $x \in X$ if there exists $\varepsilon > 0$ such that $x + \varepsilon y \in X$. The *cone of feasible directions* at x will be denoted by $D(X; x)$.

The domain of $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is $\operatorname{dom} f = \{x \in \mathbb{R}^n \mid f(x) \in \mathbb{R}\}$. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called (*positively*) *homogeneous* on a cone $X \subset \operatorname{dom} f$ if $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and $\lambda > 0$. We say that $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *affine* on a nonempty convex set $X \subset \operatorname{dom} f$ if the graph of $f|_X$ is convex and concave, i.e., if there exist $d \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$ such that $f(x) = d'x + \delta$ for all $x \in X$. In particular, if X is a convex cone and f is homogeneous on X, then f is called *linear* on X (i.e., there exists $d \in \mathbb{R}^n$ such that $f(x) = d'x$ for all $x \in X$).

Let problem P be defined by the triple (a, b, c). Its *characteristic cone* is

$$K := \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; -\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in V; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

The generalized Farkas lemma establishes that $u'x \geq \alpha$ for all $x \in F$ if and only if $(u, \alpha) \in \operatorname{cl} K$. Thus $\operatorname{cl} K$ only depends on F whereas Λ depends on K (and so on the constraint system of P). Given $x \in F$, the *set of active indices* at x is $T(x) := \{t \in T \mid a_t'x = b_t\}$. Obviously, $V \subset T(x)$. The *active cone* at x is

$$A(x) := \operatorname{cone}\{a_t,\ t \in T(x); -a_t,\ t \in V\}.$$

It is easy to see that $x \in F^*$ if and only if $c \in D(F; x)^0$ and also that $A(x) \subset D(F; x)^0$ for all $x \in F$. Consequently, if $c \in A(x)$ (the KKT condition) then $x \in F^*$, and the converse statement holds if K is closed.

A point $x^* \in F$ is a *strongly unique* optimal solution if there exists $\alpha > 0$ such that $c'x \geq c'x^* + \alpha \|x - x^*\|$ for all $x \in F$ (in which case $F^* = \{x^*\}$). This happens if and only if $c \in \operatorname{int} D(F; x^*)^0$.

The weak duality theorem establishes that $v^D \leq v^P$. The equality holds if either K is closed or $c \in \operatorname{rint} M$, where $M := \operatorname{cone}\{a_t, t \in T; -a_t, t \in V\}$ is the so-called *first moment cone*. Moreover the first condition entails $\Lambda^* \neq \emptyset$ if $\Lambda \neq \emptyset$ and the second one $F^* \neq \emptyset$.

The set F is bounded if and only if $M = \mathbb{R}^n$ and F^* is bounded if and only if $c \in \operatorname{int} M$. Since M is invariant through the perturbations considered in this paper, if the primal feasible set is bounded, the same is true under arbitrary perturbations of b and sufficiently small perturbations of c. The strong Slater condition (existence of $\overline{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $a_t'\overline{x} \geq b_t + \varepsilon$ for all $t \in U$, and $a_t'\overline{x} = b_t$ for all $t \in V$), together with the linear independence of $\{a_t, t \in V\}$ if $V \neq \emptyset$, guarantees the consistency of the problem obtained by replacing b with $w \in \mathbb{R}^T$ provided $\delta(w, b)$ is sufficiently small. If P is continuous, the strong Slater condition is equivalent to the Slater one (existence of $\overline{x} \in \mathbb{R}^n$ such that $a_t'\overline{x} > b_t$ for all $t \in U$, and $a_t'\overline{x} = b_t$ for all $t \in V$). In the continuous case, under both assumptions, the perturbed problems are solvable and have zero duality gap for sufficiently small perturbations of the data.

2. Linearity cones of convex homogeneous functions In this section we prove that, if f is convex and homogeneous, then there exists a partition of $(\operatorname{dom} f) \setminus \{0_n\}$ into maximal relatively open convex cones on which f is linear.

LEMMA 2.1 *Let C and D be two cones in $\mathbb{R}^n$ such that C is convex, relatively open and $C \cap D \neq \emptyset$. Then $C \subset C + D$.*

Proof: Let $c \in C \cap D$. Given $x \in C$, since $c, x \in C$ and this is relatively open, there exists $\mu > 1$ such that $y := (1-\mu)c + \mu x \in C$. Then $x = \mu^{-1}y + (1-\mu^{-1})c \in C + D$. Hence $C \subset C + D$. □

PROPOSITION 2.1 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex homogeneous function. Let $\{C_i,\ i \in I\}$ be a finite family of relatively open convex cones containing $c \in \mathbb{R}^n \setminus \{0_n\}$ on which f is linear. Then f is linear on $\sum_{i \in I} C_i$.*

Proof: We prove this result by induction on $|I|$. First we prove the statement for $|I| = 2$.

Let $I = \{1, 2\}$. $C_1 + C_2$ is a relatively open convex cone (the three properties are preserved by the sum) and $c = \frac{1}{2}c + \frac{1}{2}c \in C_1 + C_2$.

First we prove that

$$f(c_1 + c_2) = f(c_1) + f(c_2), \ \forall c_1 \in C_1, \forall c_2 \in C_2. \tag{1}$$

Since f is linear on C_i, we can write $f(x) = d_i'x$ for all $x \in C_i$, $i = 1, 2$. By homogeneous convexity,

$$f(c_1 + c_2) \leq f(c_1) + f(c_2) \ \forall c_1 \in C_1, \forall c_2 \in C_2.$$

In order to prove the converse inequality, observe that

$$c = \varepsilon(c_1 + c_2) + \frac{1}{2}(c - 2\varepsilon c_1) + \frac{1}{2}(c - 2\varepsilon c_2) \ \forall \varepsilon \in \mathbb{R}.$$

Take $\varepsilon > 0$ so that $c - 2\varepsilon c_i \in C_i$, $i = 1, 2$. Again by homogeneous convexity, we have

$$\begin{aligned} f(c) &\leq \varepsilon f(c_1 + c_2) + \tfrac{1}{2} f(c - 2\varepsilon c_1) + \tfrac{1}{2} f(c - 2\varepsilon c_2) \\ &= \varepsilon f(c_1 + c_2) + f(c) + \tfrac{1}{2} d_1'(-2\varepsilon c_1) + \tfrac{1}{2} d_2'(-2\varepsilon c_2) \\ &= \varepsilon f(c_1 + c_2) + f(c) - \varepsilon [f(c_1) + f(c_2)], \end{aligned}$$

so that $f(c_1) + f(c_2) \leq f(c_1 + c_2)$.

From (1), by the affinity of f on C_1 and C_2, we conclude that f is affine on $C_1 + C_2$, i.e., the statement holds for $|I| = 2$. Now assume that it holds for $|I| - 1$ cones. Select an arbitrary $k \in I$ and let $J = I \setminus \{k\}$. Since $\sum_{i \in J} C_i$ is a relatively open convex cone containing c, f is linear on $\sum_{i \in J} C_i$ by the induction hypothesis. Then, by the same reason, f is linear on $\sum_{i \in I} C_i = C_k + \sum_{i \in J} C_i$. □

Let us illustrate Proposition 2.1 with two simple examples.

EXAMPLE 2.1 *Consider the convex cones $C_1 = \{x \in \mathbb{R}^3 \mid x_1 = 0,\ x_3 > 0\}$ and $C_2 = \{x \in \mathbb{R}^3 \mid x_2 = 0,\ x_3 > 0\}$. They are relatively open and $e_3 \in C_1 \cap C_2$. Thus, any convex homogeneous function $f : \mathbb{R}^3 \to \overline{\mathbb{R}}$ which is linear on both cones, C_1 and C_2, is also linear on $C_I = C_1 + C_2 = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$.*

EXAMPLE 2.2 *The function $f(x) = |x|$ is convex and homogeneous on $\mathbb{R}$, and it is linear on the relatively open convex cones $C_1 = \mathbb{R}_{++}$ and $C_2 = -C_1$, but it is not even linear on its sum $C_1 + C_2 = \mathbb{R}$ because $C_1 \cap C_2 = \emptyset$.*

PROPOSITION 2.2 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex homogeneous function and let $c \in \mathbb{R}^n \setminus \{0_n\}$. Then there exists a largest relatively open convex cone containing c on which f is linear.*

Proof: Let $\mathcal{C} := \{C_i, i \in I\}$ be the class of all relatively open convex cones containing c on which f is linear. We shall prove that $C := \cup_{i \in I} C_i \in \mathcal{C}$ (i.e., C is the maximum of $\mathcal{C}$ with respect to the inclusion).

Since f is linear on cone$\{c\} \setminus \{0_n\}$, this is an element of $\mathcal{C}$ so that $I \neq \emptyset$.

Let us denote with $\mathcal{J}$ the family of all nonempty finite subsets of I. For each $J \in \mathcal{J}$, the sum $C_J := \sum_{i \in J} C_i$ is a relatively open convex cone containing c and so $C_J \in \mathcal{C}$ by Proposition 2.1. Since $\mathcal{C} \subset \{C_J, J \in \mathcal{J}\} \subset \mathcal{C}$, we have $C = \cup_{J \in \mathcal{J}} C_J$. On the other hand, given $\{J, H\} \subset \mathcal{J}$ such that $J \subset H$, by Lemma 2.1,

$$C_J \subset C_H. \tag{2}$$

Now we show that C satisfies all the requirements.

C is a convex cone: The union of cones is a cone. On the other hand, given $x^1, x^2 \in C$, if $x^i \in C_{J_i}$, $i = 1, 2$, taking $J = J_1 \cup J_2 \in \mathcal{J}$, (2) yields $x^i \in C_J$, $i = 1, 2$. Since C_J is convex, we have $[x^1, x^2] \subset C_J \subset C$.

C is relatively open: Let $x \in C$ and let $y \in \operatorname{aff} C$. Then we can write

$$y = \sum_{i=1}^{m} \lambda_i y_i, \; m \in \mathbb{N}, \; \sum_{i=1}^{m} \lambda_i = 1, \text{ and } y_i \in C, i = 1, ..., m.$$

By (2) there exists $J \in \mathcal{J}$ such that $x, y_i \in C_J$, $i = 1, ..., m$. Since C_J is relatively open, there exists $\mu > 1$ such that $\mu x + (1 - \mu) y \in C_J \subset C$. Thus $x \in \operatorname{rint} C$.

f is linear on C: Let $x^1, x^2 \in C$. Let $J \in \mathcal{J}$ such that $x^1, x^2 \in C_J$. Since f is linear on C_J, we have $f\left((1 - \lambda) x^1 + \lambda x^2\right) = (1 - \lambda) f(x^1) + \lambda f(x^2)$ for all $\lambda \in [0, 1]$. □

Given a convex (concave) homogeneous function f, we define the *linearity cone* of f at $z \in (\operatorname{dom} f) \setminus \{0_n\}$ as the largest relatively open convex cone containing z on which f is linear (this definition is correct by Proposition 2.2). We denote it by C_z.

PROPOSITION 2.3 *The linearity cones of a convex (concave) homogeneous function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ constitute a partition of* $(\operatorname{dom} f) \setminus \{0_n\}$.

Proof: We denote by $\mathcal{C}_z$ be the family of all the relatively open convex cones containing $z \in (\operatorname{dom} f) \setminus \{0_n\}$ on which f is linear. Obviously, C_z is the maximum of $\mathcal{C}_z$ with respect to the inclusion.

Let us assume that the statement is not true. Let $z^1, z^2 \in (\operatorname{dom} f) \setminus \{0_n\}$ such that $C_{z^1} \cap C_{z^2} \neq \emptyset$ and $C_{z^1} \neq C_{z^2}$. Take an arbitrary $z \in C_{z^1} \cap C_{z^2}$. Since $C_{z^1}, C_{z^2} \in \mathcal{C}_z$, we have $C_{z^1}, C_{z^2} \subset C_z$, with $C_{z^i} \subsetneq C_z$ for some $i = 1, 2$. Then, C_{z^i} cannot be the linearity cone of f at z^i. □

3. Optimal partitions Let us consider the primal LSIP problem P introduced in Section 1 and its dual problem D. We associate with each primal-dual feasible solution, $(x, \lambda) \in F \times \Lambda$, the *support sets* $\sigma(x) := \{t \in U \mid a_t' x > b_t\}$ and $\sigma(\lambda) := \{t \in U \mid \lambda_t > 0\}$. The pair $(x, \lambda) \in F \times \Lambda$ is called a *complementary solution of the primal-dual problem $P - D$* if $\sigma(x) \cap \sigma(\lambda) = \emptyset$.

The next two results clarify the relationship between optimality and complementary solutions in LSIP, which is more involved than in case of LP.

PROPOSITION 3.1 *The pair $(x, \lambda) \in F \times \Lambda$ is a complementary solution of $P - D$ if and only if it is a primal-dual optimal solution and $v^D = v^P$. In that case, the following statements are true:*
(i) If $\overline{x} \in F$ satisfies $a_t' \overline{x} = b_t$ for all $t \in \sigma(\lambda)$, then $\overline{x} \in F^*$.

(ii) If $\overline{\lambda} \in \Lambda$ satisfies $\overline{\lambda}_t = 0$ for all $t \in \sigma(x)$, then $\overline{\lambda} \in \Lambda^*$.

Proof: Observe that $(x, \lambda) \in F \times \Lambda$ implies that

$$c'x = \sum_{t\in T} \lambda_t a_t' x = \sum_{t\in T} \lambda_t b_t + \sum_{t\in\sigma(x)\cup\sigma(\lambda)} \lambda_t \left(a_t' x - b_t\right), \tag{3}$$

so that $c'x = \sum_{t\in T} \lambda_t b_t$ if and only if $\sigma(x) \cup \sigma(\lambda) = \emptyset$, i.e., (x, λ) is a complementary solution of $P - D$.

Now we assume that (x, λ) is a complementary solution of $P - D$. Then, statements (i) and (ii) also follow from (3), applied to the pairs $(\overline{x}, \lambda), (x, \overline{\lambda}) \in F \times \Lambda$, which gives $v^P \leq c'\overline{x} = \sum_{t\in T} \lambda_t b_t = v^D$ and $v^P = c'x = \sum_{t\in T} \overline{\lambda}_t b_t \leq v^D$, respectively. □

An immediate consequence of Proposition 3.1 is that, if $c \in \operatorname{rint} M$ and K is closed, then there exists a complementary solution of $P - D$.

COROLLARY 3.1 *Given a point $\overline{x} \in F$, there exists $\overline{\lambda} \in \Lambda$ such that $(\overline{x}, \overline{\lambda})$ is a complementary solution of $P - D$ if and only if $\overline{x}$ is an optimal solution for some finite subproblem of P.*

Proof: If $(\overline{x}, \overline{\lambda})$ is a complementary solution of $P - D$, by Proposition 3.1, $\left(\sum_{t\in T} \overline{\lambda}_t a_t\right)' \overline{x} = c'\overline{x} = \sum_{t\in T} \overline{\lambda}_t b_t$, so that $\sum_{t\in T} \overline{\lambda}_t \left(a_t'\overline{x} - b_t\right) = 0$, i.e., $c \in A(\overline{x})$. Thus $\overline{x}$ is an optimal solution of the problem obtained by replacing U by $\sigma(\overline{\lambda})$ in P. Replacing in that problem $\{a_t' x = b_t,\ t \in V\}$ by an equivalent finite subsystem, we obtain an equivalent finite subproblem with optimal solution $\overline{x}$.

Conversely, assume that $\overline{x}$ is an optimal solution of the finite subproblem of P obtained by substituting U and V with the finite subsets $\overline{U}$ and $\overline{V}$. Since the KKT condition characterizes optimality in LP, there exists $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ such that $\overline{\lambda}_t = 0$ for all $t \in T \diagdown \left(\overline{U} \cup \overline{V}\right)$, $\overline{\lambda}_t \geq 0$ for all $t \in U$, $\sum_{t\in T} \overline{\lambda}_t \left(a_t'\overline{x} - b_t\right) = 0$, and $c \in \sum_{t\in T} \overline{\lambda}_t a_t$. Then it is easy to show that $(\overline{x}, \overline{\lambda})$ is a complementary solution of $P - D$, again by Proposition 3.1. □

A triple $(B, N, Z) \in \left(2^U\right)^3$ is called an *optimal partition* if there exists a complementary solution (x, λ) such that $B = \sigma(x)$, $N = \sigma(\lambda)$ and $Z = U \diagdown (B \cup N)$ (for the sake of brevity we omit problems and couples of problems when they are implicit in the context). Obviously, the nonempty elements of the *tripartition*[1] (B, N, Z) give a partition of U (similar tripartitions have been used in [2] and [9] in order to extend the optimal partition approach to sensitivity analysis from LP to quadratic programming). We say that a tripartition $(\overline{B}, \overline{N}, \overline{Z})$ is *maximal* if

$$\overline{B} = \bigcup_{x\in F^*} \sigma(x), \qquad \overline{N} = \bigcup_{\lambda\in\Lambda^*} \sigma(\lambda) \quad \text{and} \quad \overline{Z} = U \setminus (\overline{B} \cup \overline{N}).$$

Note that the definition of the maximal partition implies that $B \subset \overline{B}$ and $N \subset \overline{N}$ for every optimal partition (B, N, Z). The uniqueness of the maximal partition is a straightforward consequence of the definition. If there exists an optimal solution pair $\overline{x} \in F^*$ and $\overline{\lambda} \in \Lambda^*$ such that $\sigma(\overline{x}) = \overline{B}$ and $\sigma(\overline{\lambda}) = \overline{N}$, then the maximal partition is called the *maximal optimal partition* and $(\overline{x}, \overline{\lambda})$ a *maximally complementary optimal pair*. As a consequence of Proposition 3.1, if $(\overline{B}, \overline{N}, \overline{Z})$ is an optimal partition such that $\overline{Z} = \emptyset$, then it is a maximal optimal partition. Now, if $(\overline{x}, \overline{\lambda})$ is a complementary solution such that $\overline{B} = \sigma(\overline{x})$ and $\overline{N} = \sigma(\overline{\lambda})$, then $(\overline{x}, \overline{\lambda})$ is called a *strictly complementary solution*. If $(x, \lambda) \in F^* \times \Lambda^*$, by Proposition

[1] The existence of an optimal tripartition for linear complementarity problems was introduced by McLinden [21]. He proved important results concerning such solutions, which was used by Güler and Ye [17] to show that path-following interior point methods generate such a solution (in the limit), and Bonnans and Gonzaga [3] proved that the interior point iterates may converge to the analytic center of the solution set.

3.1, $(\overline{x},\lambda)$ and $(x,\overline{\lambda})$ are complementary solutions, so that $\overline{B}\cap\sigma(\lambda)=\emptyset$ and $\overline{N}\cap\sigma(x)=\emptyset$, i.e., $\sigma(x)\subset\overline{B}$ and $\sigma(\lambda)\subset\overline{N}$.

Next we characterize the existence of maximal optimal partition in the usual case that $V=\emptyset$.

PROPOSITION 3.2 *Let P be such that $V=\emptyset$. Then, the maximal optimal partition exists if and only if $v^D=v^P$, P and D are solvable, and the sets of extreme points and extreme directions of Λ^* are finite.*

Proof: By Proposition 3.1, we can assume that P and D are solvable, with $v^D=v$. Let $\{\lambda^i, i\in I\}$ and $\{\gamma^j, j\in J\}$ be the sets of extreme points and extreme directions of Λ^*, respectively. By Theorem 9.6 in [11], applied to $\Lambda^*=\left\{\lambda\in\mathbb{R}_+^{(T)} \mid \sum_{t\in T}\lambda_t\begin{pmatrix}a_t\\ b_t\end{pmatrix}=\begin{pmatrix}c\\ v\end{pmatrix}\right\}$, we can express

$$\Lambda^*=\operatorname{conv}\{\lambda^i, i\in I\}+\operatorname{cone}\{\gamma^j, j\in J\}.$$

Assume that I and J are finite sets. By the finite dimension of $\mathbb{R}^n$, $\operatorname{rint}F^*\neq\emptyset$. In this case, any $\overline{x}\in\operatorname{rint}F^*$ satisfies $\sigma(x)\subset\sigma(\overline{x})$ for all $x\in F^*$. Concerning Λ^*, $\overline{\lambda}:=\frac{1}{|I|}\sum_{i\in I}\lambda^i+\sum_{j\in J}\gamma^j$ satisfies $\sigma(\lambda)\subset\sigma(\overline{\lambda})$ for $\lambda\in\Lambda^*$.

Conversely, let $\overline{x}\in F^*$ and $\overline{\lambda}\in\Lambda^*$ be such that $\sigma(x)\subset\sigma(\overline{x})$ and $\sigma(\lambda)\subset\sigma(\overline{\lambda})$ for all $(x,\lambda)\in F^*\times\Lambda^*$. Let $\sigma(\gamma^j)=\left\{t\in T\mid \gamma_t^j>0\right\}, j\in J$. By Theorem 9.4 and Corollary 9.4.1 in [11], applied to Λ^*, $\left\{\begin{pmatrix}a_t\\ b_t\end{pmatrix}, t\in\sigma(\lambda^i)\right\}$ is linearly indepedent for all $i\in I$ and $\left\{\begin{pmatrix}a_t\\ b_t\end{pmatrix}, t\in\sigma(\gamma^j)\right\}$ is affinely independent for all $j\in J$, respectively. Since $\bigcup_{i\in I}\sigma(\lambda^i)\subset\sigma(\overline{\lambda})$ and $\bigcup_{j\in J}\sigma(\gamma^j)\subset\sigma(\overline{\lambda})$, a standard algebraic argument yields $|I|\leq\begin{pmatrix}q\\ n\end{pmatrix}$ and $|J|\leq\begin{pmatrix}q\\ n+1\end{pmatrix}$, where $q=\max\left\{n+1,\left|\sigma(\overline{\lambda})\right|\right\}$. □

In many practical applications $V=\emptyset$, K is closed (e.g., P is a continuous problem satisfying the Slater condition), $c\in\operatorname{rint}M$, P is solvable, and D has a unique optimal solution. In that case, according to Proposition 3.2, there exists a maximal optimal partition. The next example illustrates the existence of maximal optimal partitions $(\overline{B},\overline{N},\overline{Z})$ such that $\overline{Z}\neq\emptyset$.

EXAMPLE 3.1 *Consider the problem P in $\mathbb{R}^2$ such that $T=\{-2,-1,0,1,...\}$, the objective function is the null one, and the constraints are $tx_1\geq-1$, for $t=1,2,..,$ $-x_1\geq0$ $(t=0)$, $x_2\geq0$ $(t=-1)$, and $-x_2\geq-1$ $(t=-2)$. We have $F^*=\{0\}\times[0,1]$ and $\Lambda^*=\{0_T\}$. It is easy to show that $(T\diagdown\{0\},\emptyset,\{0\})$ is the maximal optimal partition.*

The next example shows that the assumption on the finiteness of the sets of extreme points and extreme directions of Λ^* in Proposition 3.2 is not superfluous.

EXAMPLE 3.2 *Consider the following LSIP problem:*

$$\begin{array}{llll} P: & \mathit{Inf}_{x\in\mathbb{R}^2} & x_2 & \\ & s.t. & -x_1+x_2\geq0, & (t=1)\\ & & x_1+x_2\geq0, & (t=2)\\ & & x_2\geq0, & t=3,4,...\end{array}$$

Obviously, $v^D=v^P=0$, with $F^*=\{0_2\}$. For $r\in\mathbb{N}$ we denote by $\lambda^r:\mathbb{N}\to\mathbb{R}$ the function such that $\lambda_r^r=1$ and $\lambda_t^r=0$ for all $t\neq r$. Since $\Lambda^*=\Lambda=\operatorname{conv}\left\{\frac{\lambda^1+\lambda^2}{2},\lambda^3,\lambda^4,...\right\}$, $\bigcup_{\lambda\in\Lambda^*}\sigma(\lambda)=T$ and so the maximal partition $(\emptyset,T,\emptyset)$ cannot be optimal.

Concerning the optimality tests based on statements (i) and (ii) of Proposition 3.1, observe that, if (B,N,Z) is an optimal partition of P and its maximal optimal partition $(\overline{B},\overline{N},\overline{Z})$ exists, then

$$\sigma(x^*)\cap\overline{N}=\emptyset\Rightarrow\sigma(x^*)\cap N=\emptyset\Rightarrow x^*\in F^*,\ \text{for all } x^*\in F$$

and

$$\sigma(\lambda^*)\cap\overline{B}=\emptyset\Rightarrow\sigma(\lambda^*)\cap B=\emptyset\Rightarrow\lambda^*\in\Lambda^*,\ \text{for all } \lambda^*\in\Lambda.$$

4. Perturbing c The perturbed problems of P and D to be considered in this section are $P(z)$ and $D(z)$ as defined in Section 1.

LEMMA 4.1 *Let $\{(c^i, \lambda^i), i \in I\} \subset \mathbb{R}^n \times \mathbb{R}^{(T)}$ and $\overline{x} \in \mathbb{R}^n$ be such that $(\overline{x}, \lambda^i)$ is a complementary solution of $P(c^i) - D(c^i)$ for all $i \in I$. Then $P(z)$ and $D(z)$ are solvable and*

$$v^P(z) = v^D(z) = \overline{x}'z \text{ for all } z \in \operatorname{conv}\{c^i, i \in I\}. \tag{4}$$

Proof: Let $z \in \operatorname{conv}\{c^i, i \in I\}$. Then there exists $\mu \in \mathbb{R}_+^{(I)}$ such that

$$z = \sum_{i \in I} \mu_i c^i \text{ and } \sum_{i \in I} \mu_i = 1.$$

Since the feasible set is the same for $P(z)$ and for all $P(c^i)$, $i \in I$, $\overline{x}$ is a feasible solution of $P(z)$.

It is easy to prove that $\lambda^z := \sum_{i \in I} \mu_i \lambda^i \in \mathbb{R}^{(T)}$. Since $\sigma(\lambda^z) \subset \cup_{i \in I} \sigma(\lambda^i)$ and $\sigma(\overline{x}) \cap \sigma(\lambda^i) = \emptyset$ for all $i \in I$, we have $\sigma(\overline{x}) \cap \sigma(\lambda^z) = \emptyset$, i.e., $(\overline{x}, \lambda^z)$ is a complementary solution of $P(z)$. Then, applying Proposition 3.1 to $P(z)$, we conclude that $v^P(z) = v^D(z) = z'\overline{x}$. □

PROPOSITION 4.1 *Let $\{c^i, i \in I\} \subset \mathbb{R}^n$ be such that there exists a common optimal partition for the family of problems $\{P(c^i), i \in I\}$. Then $v^P(z) = v^D(z)$ is linear on $\operatorname{conv}\{c^i, i \in I\}$.*

Proof: Let (B, N, Z) be an optimal partition for $P(c^i)$, for all $i \in I$. Let (x^i, λ^i) be a primal-dual optimal solution of $P(c^i) - D(c^i)$, $i \in I$. Select $j \in I$ arbitrarily and let $\overline{x} = x^j$. Then, by Proposition 3.1, $(\overline{x}, \lambda^i)$ is a complementary solution of $P(c^i) - D(c^i)$, for all $i \in I$. Applying Lemma 4.1, $P(z)$ and $D(z)$ are solvable and $v^P(z) = v^D(z) = z'\overline{x}$ for all $z \in \operatorname{conv}\{c^i, i \in I\}$. □

Under the assumption of Proposition 4.1, if $c \in \operatorname{int}\operatorname{conv}\{c^i, i \in I\}$ (e.g., if all the problems $P(c^i)$ have the same maximal optimal partition), then P has a strongly unique optimal solution. This is the case if there exists a common optimal partition for all the problems $P(z)$, such that z belongs to a certain neighborhood of c. In fact, the next example shows that the linearity of $v^P(z) = v^D(z)$ on a neighborhood of c does not entail the existence of a set $\{c^i, i \in I\}$ as in Proposition 4.1.

EXAMPLE 4.1 *Let us consider the LSIP problem with index set $\mathbb{Z}$*

$$\begin{array}{lll} P: & Inf_{x \in \mathbb{R}^2} & x_1 + x_2 \\ & s.t. & tx_1 \geq -1, \quad t = 1, 2, 3, \ldots, \\ & & -tx_2 \geq -1, \quad t = 0, -1, -2, \ldots. \end{array}$$

Since the characteristic cone is $K = \{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 < 0\} \cup \{0_3\}$, $F = \mathbb{R}_+^2$, 0_2 is the strongly unique solution of P and $v^P(z) = 0$ for all $z \in \mathbb{R}_+^2$ (the effective domain of $v^P(z)$). Given $z \in \mathbb{R}_+^2$, since $v^D(z) \leq v^P(z) = 0$ and the sequence $\{\lambda^r\} \subset \mathbb{R}_+^{(\mathbb{Z})}$ such that

$$\lambda_t^r = \begin{cases} \frac{z_1}{r}, & t = r, \\ \frac{z_2}{r}, & t = -r, \\ 0, & otherwise, \end{cases}$$

is feasible for $D(z)$ and satisfies $\sum_{t \in \mathbb{Z}} \lambda_t^r b_t = -\frac{z_1 + z_2}{r} \to 0$ as $r \to \infty$, we have also $v^D(z) = 0$ for all $z \in \mathbb{R}_+^2$ although $D(z)$ is only solvable when $z = 0_2$. Thus no complementary solution exists for $D(z)$ if $z \neq 0_2$. It is easy to see that the maximal optimal partition of $P(0_2)$ is $(\mathbb{Z}, \emptyset, \emptyset)$.

COROLLARY 4.1 *Given $d \in \mathbb{R}^n$, if there exists $\varepsilon > 0$ such that $P(c + \varepsilon d)$ and P have a common optimal partition, then $v^P(z) = v^D(z)$ is linear on $[c, c + \varepsilon d]$.*

Proof: Apply Proposition 4.1 to $\{c^1, c^2\}$, where $c^1 := c$ and $c^2 := c + \varepsilon d$. □

EXAMPLE 4.2 *Consider the primal LSIP problem*

$$\begin{array}{lll} P: & Inf_{x\in\mathbb{R}^2} & c'x \\ & s.t. & -(\cos t)\,x_1 - (\sin t)\,x_2 \geq -1, \;\; t \in \left[0, \frac{\pi}{2}\right], \\ & & x_1 \geq 0 \; (t=2), \;\; x_2 \geq 0 \; (t=3). \end{array}$$

for three different cost vectors:

(a) $c = (1,1)'$. *If* $z \in \mathbb{R}^2_{++}$, *there exists a unique complementary solution of* $P(z) - D(z) : \left(0_2, \overline{\lambda}\right)$, *where*

$$\overline{\lambda}_t = \begin{cases} z_1, & t = 2, \\ z_2, & t = 3, \\ 0, & otherwise. \end{cases}$$

Since $\left(\left[0, \frac{\pi}{2}\right], \{2,3\}, \emptyset\right)$ *is a common optimal (actually maximal) partition for* $\left\{P(z), z \in \mathbb{R}^2_{++}\right\}$, $v^P(z) = v^D(z)$ *is linear on* $\mathbb{R}^2_{++}$ *by Proposition 4.1. In fact,* $v^P(z) = v^D(z) = 0$ *for all* $z \in \mathbb{R}^2_{++}$ *(Figure 1 represents the graph of* $v^P(z) = v^D(z)$*).*

(b) $c = (1,0)'$. $P(c)$ *has a maximal optimal partition* $\left(\left[0, \frac{\pi}{2}\right] \cup \{3\}, \{2\}, \emptyset\right)$, *and two other optimal partitions. If* $d \notin \operatorname{cone}\{c\}$ *and* $\varepsilon > 0$ *is sufficiently small,* $z := c + \varepsilon d$ *satisfies* $z_1 > 0$ *and either* $z_2 > 0$ *(in which case the maximal partition of* $P(z)$ *is* $\left(\left[0, \frac{\pi}{2}\right], \{2,3\}, \emptyset\right)$, *as in (a)) or or* $z_2 < 0$. *In this case the unique complementary solution is* $\left((0,1), \overline{\lambda}\right)$, *where*

$$\overline{\lambda}_t = \begin{cases} -z_2, & t = \frac{\pi}{2}, \\ z_1, & t = 2, \\ 0, & otherwise. \end{cases}$$

Thus the maximal optimal partition of $P(z)$ *is* $\left(\left[0, \frac{\pi}{2}\right[\cup \{3\}, \left\{\frac{\pi}{2}, 2\right\}, \emptyset\right)$. *This implies that, for any* $d \in \mathbb{R}^2$, *there exists* $\varepsilon > 0$ *such that* $v^P(z) = v^D(z)$ *is linear on* $[c, c + \varepsilon d]$.

(c) $c = (-1,-1)'$. *The unique complementary solution is* $\left(x^0, \lambda^0\right)$ *such that* $x^0 = \frac{1}{\sqrt{2}}(1,1)'$ *and*

$$\lambda^0_t = \begin{cases} \sqrt{2}, & t = \frac{\pi}{4}, \\ 0, & otherwise, \end{cases}$$

so that the maximal optimal partition of $P(-1,-1)$ *is* $(B, N, \emptyset)$, *where* $B = \left\{\left[0, \frac{\pi}{2}\right] \setminus \left\{\frac{\pi}{4}\right\}\right\} \cup \{2,3\}$ *and* $N = \left\{\frac{\pi}{4}\right\}$. *Given an arbitrary* $d \in \mathbb{R}^2$, $c + \rho d \in \mathbb{R}^2_{--}$ *if* ρ *is sufficiently small. For such a* ρ, *the optimal set of* $P(c+\rho d)$ *is* $F^*(c+\rho d) = \{x^\rho\}$, *where* $x^\rho = -\frac{c+\rho d}{\|c+\rho d\|} \in \mathbb{R}^2_{++}$. *There exists a unique* $\alpha \in \left]0, \frac{\pi}{2}\right[$ *(depending on* ρ*) such that* $x^\rho = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$. *Obviously,* $\sigma(x^\rho) = \left\{\left[0, \frac{\pi}{2}\right] \setminus \{\alpha\}\right\} \cup \{2,3\}$. *Similarly, the optimal set of* $D(c+\rho d)$ *is* $\Lambda^*(c+\rho d) = \{\lambda^\rho\}$, *where*

$$\lambda^\rho_t = \begin{cases} \|c+\rho d\|, & t = \alpha, \\ 0, & otherwise. \end{cases}$$

Thus $\sigma(x^\rho) = B$ *and* $\sigma(x^\rho) = N$ *if and only if* $d \in \operatorname{span}\{c\}$. *Observe that, given* $d \in \mathbb{R}^2$, *there exists* $\varepsilon > 0$ *such that* $v^P(z) = v^D(z)$ *is linear on* $[c, c+\varepsilon d]$ *if and only if* $d \in \operatorname{span}\{c\}$.

Figure 1 shows the existence of a partition of $\left(\operatorname{dom} v^P(z)\right) \setminus \{0_2\} = \mathbb{R}^2 \setminus \{0_2\}$ in relatively open convex cones on which $v^P(z)$ is linear. In fact, since the hypograph of $v^P(z)$ is the convex cone $\operatorname{cl} K$ ([11, Theorem 8.1]), $v^P(z)$ is a concave, proper, upper semi-continuous homogeneous function and, according to Proposition 2.3, $\left\{C^P_z, z \in \left(\operatorname{dom} v^P(z)\right) \setminus \{0_n\}\right\}$, where C^P_z denotes the linearity cone of $v^P(z)$ at z, is a partition of $\left(\operatorname{dom} v^P(z)\right) \setminus \{0_n\}$ in maximal regions of linearity.

In the particular case of Example 4.2, the partition associated with $v^P(z)$ has infinitely many elements, i.e.,
$C^P_{(1,1)} = \mathbb{R}^2_{++}$, $C^P_{(-1,-1)} = \operatorname{cone}\{(-1,-1)\} \setminus \{0_2\}$, $C^P_{(1,0)} = \operatorname{cone}\{(1,0)\} \setminus \{0_2\}$.
Observe that $\left\{C^P_z, z \in \mathbb{R}^2 \setminus \{0_2\}\right\}$ is a partition of $\mathbb{R}^2 \setminus \{0_2\}$, such that

$$\dim C^P_z = \begin{cases} 1, & z \in \left[\mathbb{R}^2_{-} \cup (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)\right] \setminus 0_2, \\ 2, & \text{otherwise.} \end{cases}$$

Concerning $v^D(z)$, it is also concave, proper and homogeneous. We denote by $\left\{C^D_z, z \in M \setminus \{0_n\}\right\}$ the corresponding partition. In Example 4.2, $v^D(z) = v^P(z)$, so that both functions have the same partition. This is not true in general, as the following example shows.

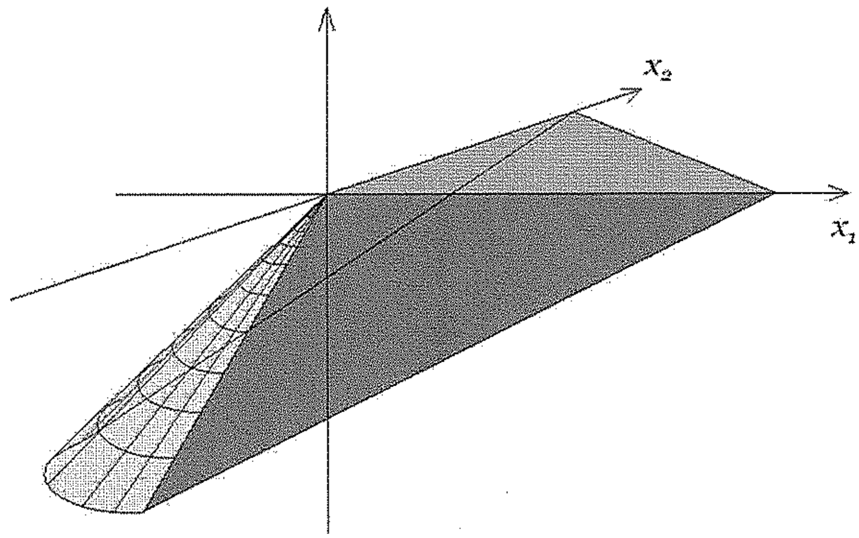

Figure 1: Graph of the primal optimal value function.

EXAMPLE 4.3 *Take* $n = 3$, $T = \left\{t \in \mathbb{R}^3 \mid t_1 + t_2 + t_3 = 1, t_i > 0, i = 1,2,3\right\} \cup \{(1,1,0)\}$, *and the constraints* $t_1x_1 + t_2x_2 + t_3x_3 \geq 0$ *for all* $t \neq (1,1,0)$ *and* $x_1 + x_2 \geq -1$ *otherwise. Then the linearity cones of* $v^P(z)$ *are the seven faces of* $\operatorname{dom} v^P(z) = \mathbb{R}^3_+$ *different from* $\{0_3\}$ *whereas* $v^D(z)$ *has only two linearity cones,* $\mathbb{R}^3_{++}$ *and* $\operatorname{cone}\{(1,1,0)\} \setminus \{0_3\}$.

PROPOSITION 4.2 *Let* $c \neq 0_n$. *If* $d \in \operatorname{span} C_c^P$ ($d \in \operatorname{span} C_c^D$), *then there exists* $\varepsilon > 0$ *such that* $v^P(z)$ ($v^D(z)$, *respectively) is linear on* $[c, c + \varepsilon d]$.

Proof: If $d \in \operatorname{span} C_c^P$, then there exists $\varepsilon > 0$ such that $[c, c+\varepsilon d] \subset C_c^P$. Since $v^P(z)$ is linear on C_c^P the conclusion is immediate (the proof is the same for $v^D(z)$). □

5. Perturbing *b* The perturbed problems in this section are the parametric problems $P(w)$ and $D(w)$ defined in Section 1. Observe that now $v^P(w), v^D(w) : \mathbb{R}^T \to \overline{\mathbb{R}}$, so that we cannot expect simple counterparts for the results in Section 4 unless $|T| < \infty$. In fact, in LP, $v^P(w), v^D(w) : \mathbb{R}^{|T|} \to \overline{\mathbb{R}}$ are ordinary homogeneous convex functions, so that Proposition 4.2 applies (observe that the parameter is now the gradient of the objective function of D, as in Section 4, but exchanging the roles of the problems). In such a case, if there exists $x^* \in F^*$ such that $\{a_t, t \in T(x^*)\}$ is a basis of $\mathbb{R}^n$, then $v^P(w) = c'x(w)$ in a certain neighborhood of b, where $x(w)$ is the unique solution of the system $\{a_t'x = w_t, t \in T(x^*)\}$ (by Cramer's rule). Then $\dim C_b^P = |T|$ and $v^P(w)$ is linear on a certain neighborhood of b.

If T is infinite, then the first difficulty comes from the fact that the perturbations of w affect the feasible set of the primal problem and possibly its consistency and the second from the infinite dimension of $\mathbb{R}^T$ which does not allow us to use Proposition 2.3.

LEMMA 5.1 *Let* $\left\{\left(b^i, x^i\right), i \in I\right\} \subset \mathbb{R}^T \times \mathbb{R}^n$ *and* $\overline{\lambda} \in \mathbb{R}^{(T)}$ *be such that* $\left(x^i, \overline{\lambda}\right)$ *is a complementary solution of* $P\left(b^i\right) - D\left(b^i\right)$ *for all* $i \in I$. *Then* $P(w)$ *and* $D(w)$ *are solvable and*

$$v^P(w) = v^D(w) = \sum_{t \in T} \overline{\lambda}_t w_t \quad \textit{for all } w \in \operatorname{conv}\left\{b^i, i \in I\right\}. \tag{5}$$

Proof: Let $w = \sum_{i\in I} \mu_i b^i$, with $\sum_{i \in I} \mu_i = 1$ and $\mu \in \mathbb{R}_+^{(I)}$.

It is easy to prove that $x^w := \sum_{i\in I} \mu_i x^i$ is a feasible solution of $P(w)$. On the other hand, if $t \in U$ satisfies $a_t'x^w > w_t$, i.e., $\sum_{i\in I} \mu_i \left(a_t'x^i - b_t^i\right) > 0$, then there exists $j \in I$ such that $\mu_j \left(a_t'x^j - b_t^j\right) > 0$

so that $a_t' x^j - b_t^j > 0$. Since $\left(x^j, \overline{\lambda}\right)$ is a complementary solution of $P\left(b^j\right)$, we must have $\overline{\lambda}_t = 0$. We have shown that the primal-dual feasible solution $\left(x^w, \overline{\lambda}\right)$ of $P(w)$ is a complementary solution of that problem. Applying Proposition 3.1 we get the aimed conclusion. □

PROPOSITION 5.1 *Let* conv $\left\{b^i, i \in I\right\}$ *be such that all the problems* $P\left(b^i\right)$, $i \in I$, *have the same optimal partition. Then* $v^P(w) = v^D(w)$ *is linear on* conv $\left\{b^i, i \in I\right\}$.

Proof: It is a straightforward consequence of Lemma 5.1. □

In particular, if $b \in \operatorname{int} \operatorname{conv}\left\{b^i, i \in I\right\}$ (e.g., the maximal partition is the same for all the problems $P(w)$ such that w belongs to a certain neighborhood of b), then D has a unique optimal solution. We can have $v^P(w) = v^D(w)$ linear (or even constant) on a certain neighborhood of b such that no optimal partition exists on that neighborhood.

EXAMPLE 5.1 *(Example 4.1 revisited) Let* $w \in \mathbb{R}^T$ *be such that*

$$\delta(w, b) = \sup_{t \in T} |w_t + 1| < 1.$$

It is easy to see that $-2 < w_t < 0$ *for all* $t \in T$. *Thus* $P(w)$ *and* P *have the same characteristic cone*

$$K = \left\{x \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 < 0\right\} \cup \{0_3\},$$

in which case

$$v^P(w) = \sup\{\gamma \in \mathbb{R} \mid (1,1,\gamma) \in \operatorname{cl} K\} = 0$$

and

$$v^D(w) = \sup\{\gamma \in \mathbb{R} \mid (1,1,\gamma) \in K\} = 0.$$

Since $0 \notin \{\gamma \in \mathbb{R} \mid (1,1,\gamma) \in K\}$, $D(w)$ *is not solvable and so* $P(w)$ *has no complementary solution.*

COROLLARY 5.1 *Given* $d \in \mathbb{R}^T$, *if there exists* $\varepsilon > 0$ *such that* $P(b + \varepsilon d)$ *has the same optimal partition as* P, *then* $v^P(w) = v^D(w)$ *is linear on* $[b, b + \varepsilon d]$.

Proof: It follows from Lemma 5.1. □

Let us mention that the recent paper [5] provides an upper bound for $v^D(b) - v^D(w)$ when $D(b)$ is consistent and $P(w)$ is also consistent in some neighborhood of b.

6. Perturbing c and b The main advantage of the optimal partition approach is that it allows to study the simultaneous perturbation of cost and RHS coefficients. We denote by (z, w) the result of perturbing the vector (c, b) (called *rim data* in the LP literature [16]). To do this we consider the parametric problem

$$\begin{array}{lll} P(z,w): & \operatorname{Inf}_{x \in \mathbb{R}^n} & z'x \\ & \text{s.t.} & a_t' x \geq w_t, \ t \in U, \\ & & a_t' x = w_t, \ t \in V, \end{array}$$

and its corresponding dual

$$\begin{array}{lll} D(z,w): & \operatorname{Sup}_{\lambda \in \mathbb{R}^{(T)}} & \sum_{t \in T} \lambda_t w_t \\ & \text{s.t.} & \sum_{t \in T} \lambda_t a_t = z, \\ & & \lambda_t \geq 0, \ t \in U. \end{array}$$

In order to describe the behavior of the value functions of these problems, we define a class of functions after giving a brief motivation. Let L be a linear space and let $\varphi : L^2 \rightarrow \mathbb{R}$ be a bilinear form on L. Let $C = \operatorname{conv}\{v_i, i \in I\} \subset L$ and let $q_{ij} := \varphi(v_i, v_j)$, $(i,j) \in I^2$. Then any $v \in C$ can be expressed as

$$v = \sum_{i \in I} \mu_i v_i, \ \sum_{i \in I} \mu_i = 1, \ \mu \in \mathbb{R}_+^{(I)}. \tag{6}$$

Then we have

$$\varphi(v,v)=\sum_{i,j\in I}\mu_i\mu_j q_{ij}. \tag{7}$$

Accordingly, given $q:C\to\mathbb{R}$, where $C=\operatorname{conv}\{v_i,i\in I\}\subset L$, we say that q is *quadratic* on C if there exist real numbers q_{ij}, $i,j\in I$, such that (7) holds for all $v\in C$ satisfying (6).

PROPOSITION 6.1 *Let* $\{(c^i,b^i),i\in I\}\subset\mathbb{R}^n\times\mathbb{R}^T$ *be such that there exists a common optimal partition for the family of problems* $P(c^i,b^i)$, $i\in I$. *Then* $P(z,w)$ *and* $D(z,w)$ *are solvable,* $v^P(z,w)=v^D(z,w)$ *on* $\operatorname{conv}\{c^i,i\in I\}\times\operatorname{conv}\{b^i,i\in I\}$ *and* $v^P(z,w)$ *is quadratic on* $\operatorname{conv}\{(c^i,b^i),i\in I\}$.
Moreover, if $(c,b)\in\operatorname{conv}\{c^i,i\in I\}\times\operatorname{conv}\{b^i,i\in I\}$, *then* $v^P(z,b)$ *and* $v^P(c,w)$ *are linear on* $\operatorname{conv}\{c^i,i\in I\}$ *and* $\operatorname{conv}\{b^i,i\in I\}$, *respectively.*

Proof: Let (B,N,Z) be a common optimal partition of $P(c^i,b^i)$ for all $i\in I$. Let $(z,w)\in\operatorname{conv}\{c^i,i\in I\}\times\operatorname{conv}\{b^i,i\in I\}$. Then we can write

$$z=\sum_{i\in I}\delta_i c^i,\ w=\sum_{i\in I}\gamma_i b^i,\ \sum_{i\in I}\delta_i=\sum_{i\in I}\gamma_i=1,\ \delta,\gamma\in\mathbb{R}_+^{(T)}. \tag{8}$$

Let $(x^i,\lambda^i)\in\mathbb{R}^n\times\mathbb{R}^{(T)}$ be a complementary solution of $P(c^i,b^i)-D(c^i,b^i)$, $i\in I$, corresponding to (B,N,Z). We prove that $\overline{x}:=\sum_{i\in I}\gamma_i x^i$ and $\overline{\lambda}:=\sum_{i\in I}\delta_i\lambda^i$ constitute a complementary solution of $P(z,w)$.

Since $a_t'x^i\geq b_t^i$ for all $t\in U$ and $a_t'x^i=b_t^i$ for all $t\in V$, we have $a_t'\overline{x}\geq w_t$ for all $t\in U$ and $a_t'\overline{x}=w_t$ for all $t\in V$, i.e., $\overline{x}$ is a feasible solution of $P(z,w)$.

On the other hand, $\lambda_t^i\geq 0$ for all $t\in U$ and all $i\in I$ entails $\overline{\lambda}_t\geq 0$ for all $t\in U$, whereas $\sum_{t\epsilon T}\lambda_t^i a_t=c^i$ for all $i\in I$ implies $\sum_{t\epsilon T}\overline{\lambda}_t a_t=z$.

We have shown that $(\overline{x},\overline{\lambda})$ is a primal-dual feasible solution. Moreover, if $t\in U$ satisfies $a_t'\overline{x}>w_t$, i.e., $\sum_{i\in I}\gamma_i(a_t'x^i-b_t^i)>0$, then there exists $j\in I$ such that $a_t'x^j>b_t^j$. Thus, by the assumption on the optimal partition of the family of problems, $t\in B$ and so $\lambda_t^i=0$ for all $i\in I$. Hence $\overline{\lambda}_t=0$ and $(\overline{x},\overline{\lambda})$ turns out to be a complementary solution of $P(z,w)$. Then, by applying Proposition 3.1 to $P(z,w)$, we have that $P(z,w)$ and $D(z,w)$ are solvable and $v^P(z,w)=v^D(z,w)$. Since $(\overline{x},\overline{\lambda})$ is a primal-dual optimal solution, we have

$$v^P(z,w)=\overline{x}'z=\sum_{t\epsilon T}\overline{\lambda}_t w_t=v^D(z,w). \tag{9}$$

Let $q_{ij}=(c^i)'x^j$, $i,j\in I$ and let $C:=\operatorname{conv}\{(c^i,b^i),i\in I\}$. Let $(z,w)=\sum_{i\in I}\mu_i(c^i,b^i)$, $\sum_{i\in I}\mu_i=1$ and $\mu\in\mathbb{R}_+^{(T)}$. Then, since we can take $\delta_i=\gamma_i=\mu_i$ in (8), (9) yields

$$v^P(z,w)=\left(\sum_{j\in I}\mu_j x^j\right)'\left(\sum_{i\in I}\mu_i c^i\right)=\sum_{i,j\in I}\mu_i\mu_j q_{ij}.$$

Now assume that $(c,b)\in\operatorname{conv}\{c^i,i\in I\}\times\operatorname{conv}\{b^i,i\in I\}$.

Let $b=\sum_{i\in I}\gamma_i b^i$, with $\sum_{i\in I}\gamma_i=1$, $\gamma\in\mathbb{R}_+^{(T)}$. Then $\overline{x}:=\sum_{i\in I}\gamma_i x^i$ is constant and (9) yields $v^P(z,b)=z'\overline{x}$ for all $z\in\operatorname{conv}\{c^i,i\in I\}$. Similarly, $v^P(c,w)=\sum_{t\epsilon T}\overline{\lambda}_t w_t$ if $w\in\operatorname{conv}\{b^i,i\in I\}$, with $\overline{\lambda}$ fixed, and this is an affine function of w. □

Obviously, if $(c,b)\in\operatorname{int}\operatorname{conv}\{(c^i,b^i),i\in I\}$, then $v^P(z,w)=v^D(z,w)$ is quadratic on a neighborhood of (c,b). In particular, if problems $P(z,w)$ have a common optimal partition when (z,w) ranges on a certain neighborhood of (c,b), then we can assert that P has a strongly unique solution and D has a unique solution. In Example 4.1, $v^P(c,w)=v^D(c,w)=0$ for all (c,w) such that $\delta(w,b)<1$ and $\|z-c\|<1$. Nevertheless, the only perturbed problems which have optimal partition are of the form $P(0_n,w)$, so that the condition in Proposition 6.1 fails to hold.

COROLLARY 6.1 *Given* $(d, f) \in \mathbb{R}^n \times \mathbb{R}^T$, *if there exists* $\varepsilon > 0$ *such that the problem* $P((c, b) + \varepsilon(d, f))$ *has the same maximal optimal partition as* P, *then* $v^P(z, w) = v^D(z, w)$ *is quadratic on the interval* $[(c, b), (c, b) + \varepsilon(d, f)]$. *Moreover,* $v^P(z, b)$ *(*$v^P(c, w)$*) is an affine function of* z *on* $[c, c + \varepsilon d]$ *(of* w *on* $[b, b + \varepsilon f]$, *respectively).*

Proof: It is an immediate consequence of Proposition 6.1. □

7. Conclusions In this paper we examine the linearity of the primal and the dual optimal value functions, which can be different in LSIP, relative to perturbations of the cost vector, the RHS vector or both, on convex subsets of their domain. The new results on sensitivity analysis in LSIP in Sections 4-6 have been obtained by means of two different partition approaches whose fundamentals are developed in Sections 2 and 3:

(i) Partition of the domain of the optimal value functions in maximal relatively open convex cones, where they are linear (the so-called linearity cones). The partition corresponding to the primal optimal value function only depends on the primal feasible set, whereas the one corresponding to the dual optimal value function depends on the constraints. The advantage of this approach is that it provides a significant insight into the behavior of the optimal value functions. The inconveniences are: first, that this approach only applies to perturbations of c; and second, that computing linearity cones may be a difficult task in practice.

(ii) Optimal partitions of the index set of the inequality constraints. The advantage of this approach is that it yields sufficient conditions for the linearity of the optimal value functions for a variety of convex sets for the three types of perturbations considered in this paper. The multiplicity of optimal partitions and the possible lack of a maximal partition in LSIP is the main difficulty when checking these sufficient conditions in practice (at least in comparison with LP).

Duality theory provides a third approach to sensitivity analysis in LSIP, as sketched at the beginning of Section 1, which is valid for perturbation of b or c, but not both. The main inconvenience of this approach is that it only provides affinity tests for the optimal value functions on segments, and its main advantage consists of the fact that these tests also provide directional derivatives in the direction of the corresponding segment extending Gauvin's formulae [7].

Sensitivity analysis in LSIP can also be approached from a nonlinear perspective, obtaining bounds for either the optimal value functions or their directional derivatives in terms of the admissible perturbations. For instance, a lower bound for the dual optimal value under perturbations of b, and an upper bound for the directional derivative of the primal optimal value function under arbitrary perturbation can be found in [5] and [4], respectively. The main inconvenience of this approach is that it provides inaccurate information on the variation of the optimal value functions, and its main advantage is that, in general, this type of results can be applied under weaker conditions on P.

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