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Abstract

This paper extends the Log-robust portfolio management approach to the case with short sales, i.e., the case where the manager can sell shares he does not yet own. We model the continuously compounded rates of return, which have been established in the literature as the true drivers of uncertainty, as uncertain parameters belonging to polyhedral uncertainty sets, and maximize the worst-case portfolio wealth over that set in a one-period setting. The degree of the manager’s aversion to ambiguity is incorporated through a single, intuitive parameter, which determines the size of the uncertainty set. The presence of short-selling requires the development of problem-specific techniques, because the optimization problem is not convex.

In the case where assets are independent, we show that the robust optimization problem can be solved exactly as a series of linear programming problems, and of convex programming problems with one variable; as a result, the approach remains tractable for large numbers of assets. We also provide insights into the structure of the optimal solution. In the case of correlated assets, we develop and test three heuristics. Numerical results suggest that the manager should select the heuristic where correlation is maintained only between assets invested in. In computational experiments, the proposed approach exhibits performance superior to that of the traditional robust approach.

1 Introduction

Portfolio management traditionally assumes the perfect knowledge of the underlying probability distributions of asset prices. It is, however, difficult to estimate such probabilities accurately, and

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over-reliance on wrong guesses might yield dire outcomes for financial companies. For instance, Goldman Sachs announced in August 2007 that “its funds had been hit by moves that its models suggested were 25 standard deviations away from normal, [which] translates into a likelihood of 0.000...0006, where there are 138 zeros before the six” (*The Economist* [26]). In another well-publicized case, a trader for the hedge fund Amaranth Advisors “made some very big bets on natural-gas prices that went spectacularly wrong,” costing his employer \$6.5 billion in September 2006, “more than half of what not long before was \$9 billion it had under management” (*The Economist* [25]). The hedge fund shut down shortly thereafter (*International Herald Tribune* [22]).

Confronted with such levels of uncertainty, financial companies are often willing to decrease their expected profit if it decreases their risk exposure. A natural way to incorporate risk preferences would be to rely on the expected value of utility functions, which are nonincreasing and concave in the portfolio wealth. Unfortunately, utility functions are difficult to articulate in practice. A more intuitive risk criterion is the probability that the portfolio value falls below a given threshold, but it suffers from tractability issues, as computing this probability will in general require multi-variate integration. Such issues are further compounded in dynamic optimization by the rapidly increasing size of the state space as the time horizon increases.

Literature Review. The basics of modern portfolio theory were articulated in the 1950s by Markowitz [20], who first introduced the concept of an efficient frontier, capturing the investor’s trade-off between risk and return. The high sensitivity of the optimal asset allocation to input data such as mean and covariance, documented in Chopra and Ziemba [9], has required the development of robust optimization techniques to protect the portfolio (Goldfarb and Iyengar [15], Erdogan et. al. [12]). Robust optimization was originally developed to address parameter uncertainty in mathematical programming problems; uncertain parameters are assumed to belong to a known, convex uncertainty set, and the decision-maker optimizes the worst-case value of the objective, subject to feasibility for the worst-case value of each constraint. This was first proposed by Soyster in [24]; however, because each parameter was set to its worst-case value, the approach was deemed too conservative for practical implementation. It regained momentum in the mid-1990s when teams of researchers led independently by El-Ghaoui and Lebret [11] and Ben-Tal and Nemirovski [1] considered ellipsoidal uncertainty sets for convex programming problems and derived tractable robust reformulations using strong duality. The reader is referred to Bertsimas and Thiele [5] for a tutorial-level introduction to robust optimization.

In subsequent work, Ben-Tal and Nemirovski [2] and Bertsimas and Sim [4] have applied ro-

bust optimization techniques to stochastic optimization problems where the distributions of the underlying random variables, rather than the value of uncertain parameters, are not known. In particular, they model stock returns as uncertain parameters belonging to known range forecasts, and their goal is to maximize the portfolio’s worst-case return, where the worst case is computed over a set of allowable deviations of the stock returns from their nominal values. This implementation of robust optimization in finance also underlies the work in Bertsimas and Pachamanova [3], Pachamanova [21] and Fabozzi et. al. [13]. As pointed out in Kawa and Thiele [18], however, numerous studies of stock price behavior (see Hull [16] and the references therein) indicate that the true drivers of uncertainty are not the stock returns, but instead the continuously compounded rates of return, which are central to the well-known Lognormal model developed by Black and Scholes [6], where they are assumed to obey a Gaussian distribution. This model has become widely implemented in practice, but neglects the fact that the real distributions have fat tails (Jansen and deVries [17], Cont [10]); hence, managers underestimate the probability of adverse events. In addition, the empirical validity of the Lognormal model among possible distributions remains open to debate (Fama [14], Blattberg and Gonedes [7], Kon [19], Jansen and deVries [17], Richardson and Smith [23], Cont [10]). In particular, Jansen and deVries [17] states: “Numerous articles have investigated the distribution of share prices, and find that the returns are fat-tailed. Nevertheless, there is still controversy about the amount of probability mass in the tails, and hence about the most appropriate distribution to use in modeling returns. This controversy has proven hard to resolve.”

The goal of this paper is to investigate a new approach to portfolio management under uncertainty, based on robust optimization, which is more tightly connected to the available information and managers’ risk tolerance. Our focus here will be on the case with short sales, i.e., the case where amounts invested in a stock can be negative. While the approach without short sales was thoroughly investigated in Kawa and Thiele [18], the optimization problems when short-selling is allowed are not convex, thus precluding the direct use of strong duality to derive tractable, exact reformulations, and requiring new solution techniques.

We believe that robust optimization with polyhedral sets applied to the continuously compounded rates of return gives more relevant results for finance practitioners; in addition, it exhibits performance superior to the traditional robust approach, which focuses on uncertainty for the stock returns, while remaining theoretically insightful and numerically tractable.

Contributions.

(a) In the case of independent assets, we present a tractable reformulation to the robust opti-

mization problem as a series of linear programming problems, and of convex programming problems with one variable. We show diversification is achieved naturally, provide insights into the structure of the optimal solution, and test the approach in numerical experiments.

- (b) We use the previous results to develop three heuristics for the case of correlated assets, where the robust problem cannot be reformulated as a convex programming problem. Numerical results suggest the manager should select the heuristic where correlation is maintained only between assets invested in, and correlation between other stocks (either between a stock invested in and a stock short-sold, or between stocks short-sold) is ignored.

Outline. We analyze the case of independent assets in Section 2. We extend the approach to correlated assets in Section 3. We end each section with numerical experiments. Finally, Section 4 contains concluding remarks. All proofs are in appendix.

2 The Case of Independent Assets

2.1 Problem Setup

We consider a one-period portfolio management problem, where the manager invests his money in various assets in order to maximize his terminal wealth. Short-selling is allowed, i.e., the manager can sell assets he does not own, with the intention of purchasing them later when their price has decreased. From a mathematical perspective, this means that amounts invested in some of the stocks can be negative. In line with industry practice, the total amount of money short-sold cannot exceed a pre-determined fraction of the decision-maker's budget. In this section, we assume that assets are independent; this situation arises for instance when the manager invests in asset classes such as gold, timber or real estate. The extension to correlated assets is described in Section 3.

We use the following notation throughout the paper:

- n : the number of stocks,
- T : the length of the time period,
- $S_i(0)$: the initial (known) value of stock i ,
- $S_i(T)$: the (random) value of stock i at time T ,
- w_0 : the initial wealth of the investor,
- μ_i : the drift of the Lévy process for stock i ,
- σ_i : the infinitesimal standard deviation of the Lévy process for stock i ,
- \tilde{x}_i : the number of shares invested in stock i ,
- x_i : the amount of money invested in stock i .
- p : percentage of the initial wealth short sold.

This is the same notation as in Kawa and Thiele [18], with the addition of the leverage parameter p , which is specific to the case with short-sales.

We model asset prices as having unknown but bounded continuously compounded rates of return. The price of asset i , $i = 1, \dots, n$, at time T obeys the following equation:

$$S_i(T) = S_i(0) \exp \left[\left(\mu_i - \frac{\sigma_i^2}{2} \right) T + \sigma_i \sqrt{T} \tilde{z}_i \right],$$

where the \tilde{z}_i , $i = 1, \dots, n$, are unknown parameters of bounded support $[-c, c]$. (The approach can easily be extended to the case where supports vary depending on the asset, but this does not appear to yield significant improvement, since the range of the continuously compounded rates of return is already asset-specific through the estimation of σ_i .) The transformation $\tilde{z}_i = c z_i$ yields:

$$S_i(T) = S_i(0) \exp \left[\left(\mu_i - \frac{\sigma_i^2}{2} \right) T + \sigma_i \sqrt{T} c z_i \right],$$

with $|z_i| \leq 1$ for all i . The parameters z_i , $i = 1, \dots, n$, are called the *scaled deviations* of the continuously compounded rates of return.

While the decision-maker aims at protecting portfolio wealth from downside risk, he is also reluctant to over-protect the portfolio and miss gain opportunities. Robust optimization aims at optimizing a “reasonable worst-case” of the objective, where the “reasonable worst-case” captures the fact that independent uncertainty drivers will rarely reach their worst case simultaneously, provided that enough uncertainty drivers are present. This fact, which underlies the law of large numbers in statistics, is modeled by the constraint:

$$\sum_{i=1}^n |z_i| \leq \Gamma,$$

which puts a bound on the number of uncertainty drivers that can achieve their worst case. This constraint is called a *budget-of-uncertainty constraint* in the literature. When $\Gamma = 0$, the uncertainty set reduces to the point where all scaled deviations are zero and the approach becomes the nominal model, discarding risk to solely consider nominal returns. When $\Gamma = n$, all scaled deviations are equal to their worst case, so that risk protection is emphasized at the expense of return. Selecting Γ between 0 and n allows the decision-maker to achieve a trade-off between risk and return.

The manager's goal is to maximize the worst-case value of his portfolio, where the worst case is measured over the uncertainty set defined above, under the constraints that (i) he spends at most w_0 acquiring his portfolio (budget constraint), and (ii) he does not short-sell more than $p w_0$ (short-sales constraint). The Log-robust portfolio management model with short sales can thus be formulated as:

$$\begin{aligned}
\max_x \quad & \min_z \sum_{i=1}^n x_i \exp \left[\left(\mu_i - \frac{\sigma_i^2}{2} \right) T + \sigma_i \sqrt{T} c z_i \right] \\
\text{s.t.} \quad & \sum_{i=1}^n |z_i| \leq \Gamma, \\
& |z_i| \leq 1 \quad \forall i, \\
& \sum_{i=1}^n x_i = w_0, \\
& \sum_{i | x_i < 0} (-x_i) \leq p w_0.
\end{aligned} \tag{1}$$

The tractability of robust optimization hinges on the ability to reformulate max-min problems as large maximization problems; this is traditionally done by invoking strong duality in linear or convex programming to transform the inner minimization problem into an equivalent maximization problem (see, e.g., Bertsimas and Sim [4]). Here, however, the inner minimization problem is not necessarily convex in the scaled deviations, because the amount of money invested in each stock is not constrained to be non-negative, so strong duality for the inner minimization problem does not hold. As a result, the presence of short sales requires the development of new solution techniques and algorithms, beyond those investigated in Kawa and Thiele [18].

We first provide a quick overview of the main results in Log-robust portfolio management without short sales, which will be helpful later in the paper.

Theorem 2.1 (Optimal strategy without short sales (Kawa and Thiele [18]))

(i) The optimal wealth in the Log-robust portfolio management problem without short sales is: $w_0 \exp(F(\Gamma))$, where F is the function defined by:

$$\begin{aligned}
F(\Gamma) = \max_{\eta, \chi, \xi} \quad & \sum_{i=1}^n \chi_i \ln k_i - \eta \Gamma - \sum_{i=1}^n \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sigma_i \sqrt{T} c \chi_i \geq 0, \quad \forall i, \\
& \sum_{i=1}^n \chi_i = 1, \\
& \eta \geq 0, \chi_i, \xi_i \geq 0, \quad \forall i.
\end{aligned} \tag{2}$$

(ii) The optimal amount of money x_i invested at time 0 in stock i is $\chi_i w_0$, for all i , where the χ_i are found by solving Problem (2).

We can further characterize the optimal allocation in the Log-robust portfolio management model, if we assume that the assets are ranked in decreasing order of their nominal return, i.e., $k_1 > \dots > k_n$. We will make this assumption throughout the remainder of the paper.

Theorem 2.2 (Structure of the optimal allocation (Kawas and Thiele [18]))

(i) If the optimal η in Problem (2) is strictly positive, then there exists an index j such that the decision-maker invests only in stocks 1 to j and we have:

$$\chi_i = \begin{cases} \frac{1/\sigma_i}{\left(\sum_{a=1}^j 1/\sigma_a\right)}, & i \leq j, \\ 0, & i > j. \end{cases} \tag{3}$$

In particular, $\chi_i \sigma_i$ is constant for all the assets the manager invests in.

(ii) If the optimal η is zero, then the manager invests all his money into the asset with the highest worst-case return.

Kawas and Thiele also establish in [18] that the degree of diversification of the portfolio, measured by the parameter j , increases with the budget of uncertainty Γ , until the decision-maker becomes so conservative that he prefers investing in the safest asset only, and the number of stocks invested in falls to one.

2.2 The Log-Robust Approach

The remainder of this section will focus on deriving a tractable reformulation of Problem (1) and characterizing the corresponding optimal allocation in the presence of short sales. We first

introduce additional notation.

- z_i^+ : scaled deviation for assets that are not short-sold,
- z_i^- : scaled deviation for assets that are short-sold,
- Γ^+ : budget of uncertainty for assets not short-sold,
- Γ^- : budget of uncertainty for assets short-sold,
- k_i : nominal return without uncertainty of stock i .

Specifically, we have $k_i = \exp\left((\mu_i - \frac{\sigma_i^2}{2})T\right)$ for all i .

We distinguish between assets that are short-sold ($x_i < 0$) and not short-sold ($x_i \geq 0$), allocating a budget of uncertainty (to be optimized) Γ^- and Γ^+ to each group, and reformulate Problem (1) as:

$$\begin{aligned}
& \max_x \quad \min_{\Gamma^+, \Gamma^-} \left(\begin{array}{l} \min_{\tilde{z}^+} \sum_{i|x_i \geq 0}^n x_i k_i \exp(\sigma_i \sqrt{T} c \tilde{z}_i^+) + \min_{\tilde{z}^-} \sum_{i|x_i < 0}^n x_i k_i \exp(\sigma_i \sqrt{T} c \tilde{z}_i^-) \\ \text{s.t.} \quad \sum_{i|x_i \geq 0}^n |\tilde{z}_i^+| \leq \Gamma^+, \quad \quad \quad \text{s.t.} \quad \sum_{i|x_i < 0}^n |\tilde{z}_i^-| \leq \Gamma^-, \\ |\tilde{z}_i^+| \leq 1 \forall i \text{ s.t. } x_i \geq 0. \quad \quad \quad |\tilde{z}_i^-| \leq 1 \forall i \text{ s.t. } x_i < 0. \end{array} \right) \\
& \text{s.t.} \quad \Gamma^+ + \Gamma^- = \Gamma, \\
& \quad \quad \Gamma^+, \Gamma^- \geq 0 \text{ integer.} \\
& \text{s.t.} \quad \sum_{i=1}^n x_i = w_0, \\
& \quad \quad \sum_{i|x_i < 0}^n -x_i \leq p w_0.
\end{aligned} \tag{4}$$

We make the following observations regarding the worst-case uncertainty in the inner minimization problems. These observations allow us to transform the first minimization into a convex programming problem minimizing a differentiable convex function over a polyhedral, bounded set, and, more importantly, the second minimization into a linear programming problem over another bounded polyhedron.

Lemma 2.3 (Worst-Case Uncertainty)

(i) At optimality, $-1 \leq \tilde{z}_i^+ \leq 0$ for all stocks i that are not short-sold (i.e., the worst case is to have returns no higher than their nominal value), and the minimization problem in \tilde{z}_i^+ in

Problem (4) is equivalent to the convex optimization problem over a polyhedral uncertainty set:

$$\begin{aligned}
& \min_{z^+} \quad \sum_{i|x_i \geq 0}^n x_i k_i \exp(-\sigma_i \sqrt{T} c z_i^+) \\
& \text{s.t.} \quad \sum_{i|x_i \geq 0}^n z_i^+ \leq \Gamma^+, \\
& \quad \quad 0 \leq z_i^+ \leq 1 \forall i \text{ s.t. } x_i \geq 0.
\end{aligned} \tag{5}$$

(ii) At optimality, $0 \leq \tilde{z}_i^- \leq 1$ for all stocks that are short-sold (the worst case is to have returns no lower than their nominal value), and the minimization problem in \tilde{z}_i^- in Problem (4) is equivalent to the linear programming problem:

$$\begin{aligned}
& \min_{z^-} \quad \sum_{i|x_i < 0} \left[x_i k_i (1 - z_i^-) + x_i k_i \exp(\sigma_i \sqrt{T} c) z_i^- \right] \\
& \text{s.t.} \quad \sum_{i|x_i < 0}^n z_i^- \leq \Gamma^-, \\
& \quad \quad 0 \leq z_i^- \leq 1, \forall i | x_i < 0.
\end{aligned} \tag{6}$$

In the remainder of this section, we will denote by z_i^+ the absolute value of the scaled deviations of non-short-sold assets, keeping in mind that the true worst-case scaled deviation \tilde{z}_i^+ will be negative. z_i^- will denote the true worst-case scaled deviation \tilde{z}_i .

Reinjecting Lemma 2.3 into Problem (4), and using the fact that $\Gamma^- = \Gamma - \Gamma^+$ yields the following intermediate, convex reformulation when the sets of assets short-sold, $\{i | x_i < 0\}$, and

assets invested in, $\{i \mid x_i \geq 0\}$ are given. (We discuss the choice of these sets later.)

$$\begin{aligned}
\max_x \quad & \min_{\Gamma^+, z^+, z^-} \sum_{i \mid x_i \geq 0}^n x_i k_i \exp(-\sigma_i \sqrt{T} c z_i^+) + \sum_{i \mid x_i < 0}^n x_i k_i \left[\exp(\sigma_i \sqrt{T} c) - 1 \right] z_i^- + \sum_{i \mid x_i < 0}^n x_i k_i \\
\text{s.t.} \quad & \sum_{i \mid x_i \geq 0}^n z_i^+ \leq \Gamma^+, \\
& \sum_{i \mid x_i < 0}^n z_i^- \leq \Gamma - \Gamma^+, \\
& 0 \leq z_i^+ \leq 1, \forall i, \\
& 0 \leq z_i^- \leq 1, \forall i, \\
& 0 \leq \Gamma^+ \leq \Gamma, \\
& \Gamma^+ \text{ integer}, \\
\text{s.t.} \quad & \sum_{i=1}^n x_i = w_0, \\
& \sum_{i \mid x_i < 0}^n -x_i \leq p w_0.
\end{aligned} \tag{7}$$

In what follows, we describe the robust counterpart to Problem (7). Recall that the stocks are ordered in decreasing order of the k_i parameters, i.e., the nominal stock returns without uncertainty.

Theorem 2.4 (Robust Formulation)

- (i) *At optimality, either the manager short-sells the maximum amount allowed, or he does not short-sell at all.*
- (ii) *The optimal wealth is the maximum between the optimal wealth in the no-short-sales model, i.e., $w_0 \exp(F(\Gamma))$, where F is defined in Equation (2), and the convex problem in one variable:*

$$\max_{\theta \geq 0} w_0 \cdot \left(\theta \left[1 + \ln \left(\frac{(1+p)}{\theta} \right) \right] + F_p(\theta, \Gamma) \right), \tag{8}$$

where F_p is defined by:

$$\begin{aligned}
F_p(\theta, \Gamma) = \max_{\eta, \xi, \tilde{\chi}} \quad & \sum_{i|x_i \geq 0} \tilde{\chi}_i \ln k_i - \sum_{i|x_i < 0} \tilde{\chi}_i k_i - \eta \Gamma - \sum_{i=1}^n \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sigma_i \sqrt{T} c \tilde{\chi}_i \geq 0, & \forall i | x_i \geq 0, \\
& \eta + \xi_i - k_i \left[\exp(\sigma_i \sqrt{T} c) - 1 \right] \tilde{\chi}_i \geq 0, & \forall i | x_i < 0, \\
& \sum_{i|x_i \geq 0} \tilde{\chi}_i = \theta, \\
& \sum_{i|x_i < 0} \tilde{\chi}_i = p, \\
& \eta \geq 0, \xi_i \geq 0, \tilde{\chi}_i \geq 0, & \forall i.
\end{aligned} \tag{9}$$

The optimal fraction of money χ_i allocated to asset i is $(1+p)\frac{\tilde{\chi}_i}{\theta}$ if the stock is invested in and $-\tilde{\chi}_i$ if the stock is short-sold.

Corollary 2.5 characterizes the structure of the optimal allocation.

Corollary 2.5 (Optimal Allocation) Assume $\eta > 0$, where η is found by solving Problem (9). If it is optimal to short-sell, there exist indices j and l , $j < l$ such that the manager invests in stocks 1 to j and short-sells stocks l to n .

Specifically, the optimal fraction χ_i of money invested in asset i is given by one of the following two allocations:

Allocation 1.

$$\chi_i = \begin{cases} (1+p) \frac{1/\sigma_i}{\sum_{a=1}^j 1/\sigma_a}, & i \leq j, \\ 0, & j+1 \leq i \leq l-1, \\ -p + \frac{\theta \sqrt{T} c}{\sum_{a=1}^j 1/\sigma_a} \left(\sum_{b=l+1}^n \frac{1}{k_b \left[\exp(\sigma_b \sqrt{T} c) - 1 \right]} \right), & i = l, \\ -\frac{\theta \sqrt{T} c}{k_i \left[\exp(\sigma_i \sqrt{T} c) - 1 \right]} \cdot \frac{1}{\sum_{a=1}^j 1/\sigma_a}, & i > l, \end{cases}$$

with $\chi_l < 0$. Furthermore, $\eta = \frac{\theta \sqrt{T} c}{\sum_{a=1}^j 1/\sigma_a}$.

Allocation 2.

$$\chi_i = \begin{cases} \frac{p}{\sqrt{T}c} \frac{1/\sigma_i}{\sum_{b=l}^n 1/(k_b [\exp(\sigma_b \sqrt{T}c) - 1])}, & i < j, \\ \theta - \frac{p}{\sqrt{T}c} \frac{\sum_{a=1}^{j-1} 1/\sigma_a}{\sum_{b=l}^n 1/(k_b [\exp(\sigma_b \sqrt{T}c) - 1])}, & i = j, \\ 0, & j+1 \leq i \leq l-1, \\ -p \frac{1/(k_i [\exp(\sigma_i \sqrt{T}c) - 1])}{\sum_{b=l}^n 1/(k_b [\exp(\sigma_b \sqrt{T}c) - 1])}, & i \geq l, \end{cases}$$

with $\chi_j > 0$. Furthermore, $\eta = p \frac{1}{\sum_{b=l}^n 1/(k_b [\exp(\sigma_b \sqrt{T}c) - 1])}$.

Corollary 2.6 studies the diversification of the portfolio. The results are similar to those in the no-short-sales case, in the sense that diversification increases until the decision-maker becomes so conservative that he makes his investment decision based solely on the worst-case returns of all the stocks.

Corollary 2.6 (Diversification) *When the assets are independent, the number of the stocks either invested in or short-sold increases in the budget of uncertainty Γ until:*

either Γ reaches its upper bound ($\Gamma = n$),

or the approach achieves complete diversification, i.e., $|\chi_i| > 0$ for all i , for some value of Γ ,

or the shadow price η of the budget-of-uncertainty constraint, which is non-negative and non-increasing with Γ , becomes zero.

Remarks.

- Once η has reached its lower bound of zero, it becomes optimal to invest in the stock with the highest minimum return for all subsequent (higher) values of Γ . We invest w_0 in that stock (no short sales) if the lowest maximum return exceeds the highest minimum return, i.e., there is a possibility that the stock we short-sell will perform better than the stock we invest in. Otherwise, we invest $(1+p)w_0$ in the stock with the highest minimum return, short-selling the stock with the lowest maximum return up to the maximum amount of money allowed.
- Even when complete diversification is achieved ($|\chi_i| > 0$ for all i), the optimal allocation can still evolve as a function of Γ , as two trends compete: (1) the increase of j , pushing

for more stocks invested in, and (2) the decrease of l , pushing for more stocks short-sold. If short sales are not allowed, a dominating first trend results in an increase in the number of stocks invested in, until this number suddenly drops off to one (which corresponds to the asset with the highest worst-case return). A dominating second trend results in an increase, followed by a decrease, in the number of stocks invested in, before that number also drops off to one.

We must now determine the optimal sets of assets short-sold and assets invested in to finalize the methodology.

Algorithm 2.7 *The optimal allocation is found by solving n linear programming problems (9), with $\{i|x_i \geq 0\} = \{1, \dots, j\}$ for Problem j , $j = 1, \dots, n$.*

To see this, we use the fact that the optimal strategy is obtained by ranking the assets in order of decreasing return without uncertainty, k_i (or equivalently, $\ln k_i$). Therefore, to describe the sets of assets invested in and of assets short-sold, we simply have to enumerate the cut-off point between the stocks, which can be either viewed as stock j or l , since the manager neither invests in nor short-sells stocks $j+1, \dots, l-1$. This procedure significantly contributes to the tractability of the approach, because we do not have to resort to binary variables to determine the optimal sets of assets invested in and assets short-sold.

2.3 Numerical Experiments

In this section, we will compare the performance of the Log-robust optimization approach against that of the robust model traditionally used in portfolio management, which we introduced in Kawa and Thiele [18]. We will also investigate the benefits associated with short-selling.

Setup.

The traditional robust model with short sales is given by:

$$\begin{aligned}
& \max_{x, s, q, r} \sum_{i=1}^n (x_i^+ - x_i^-) \exp \left[\left(\mu_i - \frac{\sigma_i^2}{2} \right) T \right] E \left[\exp \left(\sum_{j=1}^n Q_{ij}^{1/2} Z_j \right) \right] - \Gamma s - \sum_{i=1}^n q_i \\
& \text{s.t.} \quad \sum_{i=1}^n (x_i^+ - x_i^-) = w_0, \\
& \quad s + q_i \geq c r_i, \quad \forall i, \\
& \quad -r_i \leq \sum_{k=1}^n M_{ki}^{1/2} (x_k^+ - x_k^-) \leq r_i, \quad \forall i, \\
& \quad \sum_{i=1}^n x_i^- \leq p w_0 \\
& \quad s, q_i, r_i, x_i^+, x_i^- \geq 0, \quad \forall i,
\end{aligned}$$

with $M^{1/2}$ the square root of the covariance matrix of $\exp \left[\left(\mu_i - \frac{\sigma_i^2}{2} \right) T + \sqrt{T} \left(\sum_{j=1}^n Q_{ij}^{1/2} Z_j \right) \right]$.

We consider three models: the traditional robust approach with short sales, and the Log-robust models with and without short-sales. To compute the optimal allocation in each case, we downloaded six months' worth of daily stock price data for 50 stocks from Yahoo! Finance, computed the model parameters using the continuously compounded rates of return $\ln(S_t/S_{t-1})$ and generated 1,000 scenarios for the stock prices with a time horizon of six months. Initial wealth is \$100,000. We take $c = 1.96$, which would correspond to a 95% confidence interval if the demand was Gaussian; experiments not presented in the paper suggest the allocations are not overly sensitive to the choice of the range parameter.

We also investigate the impact of correlation. To achieve this last goal, we replace the off-diagonal elements of the covariance matrix computed above by zeros in these numerical experiments about independent assets; the computational study in Section 3 presents the results with the full (not diagonal) covariance matrix. We also consider two different data sets to make sure our insights do not depend on the stocks selected.

Analysis of optimal solution.

Figure 1 shows the diversification level for the Log-robust and the traditional robust models, regarding the stocks *short-sold*; we note that the Log-robust model results in significantly greater diversification. Specifically, in the traditional robust model, the manager short-sells at most one stock, while the manager short-sells an increasing number of stocks (up to 28 stocks out of 50) in the Log-robust model as Γ increases up to 26. Then we observe a breakpoint, when the manager is so averse to ambiguity that it becomes optimal for him to invest only in the safest stock, i.e.,

the one with the highest minimum return.

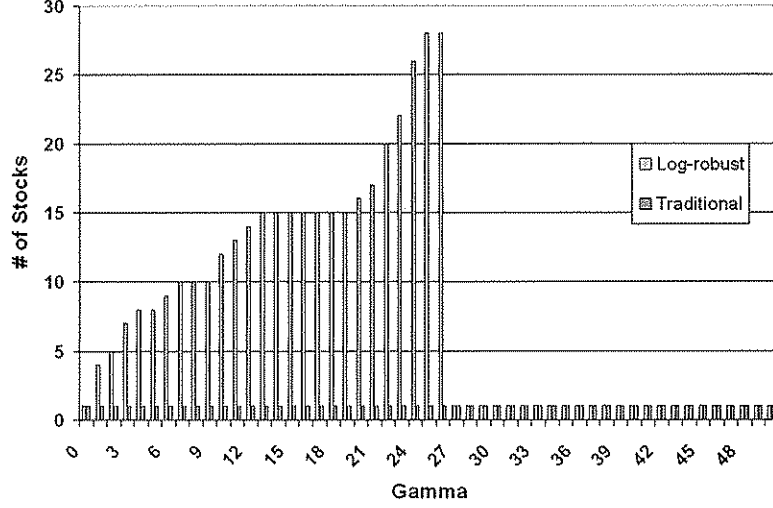


Figure 1: Number of stocks short-sold in optimal portfolio for Γ varying from 0 to 50, for the Log-robust and the Traditional models.

Figure 2, which represents the number of shares either invested in or short-sold, shows that the diversification obtained in the Log-robust model is genuine, in the sense that it is not a result of buying or shorting only a few shares per stock.

Figure 3 shows the diversification in the stocks short-sold for the Log-robust model for two different data sets. We observe similar trends in both cases while noting that the breakpoint occurs much earlier in one data set than in the other.

Figure 4 shows the effect of the leverage parameter p on diversification for the Log-robust model (diversification did not change with p for the traditional robust model). We observe that increasing the amount of money the investor can use to short-sell stocks results in an increase in the number of stocks short-sold. Interestingly, the breakpoint occurs for the same value of the budget of uncertainty.

Figure 5 shows the diversification behavior in relation to the budget of uncertainty for the stocks the manager invests in and the stocks the manager short-sells. Because the stocks are ranked in decreasing order of nominal return, the shaded area in the upper part of the figure corresponds to stocks *invested in*, and the shaded area in the lower part of the figure corresponds to stocks *short-sold*. The blank area between the previous two corresponds to stocks that are neither invested in nor short-sold. For this data set, the manager never achieves complete

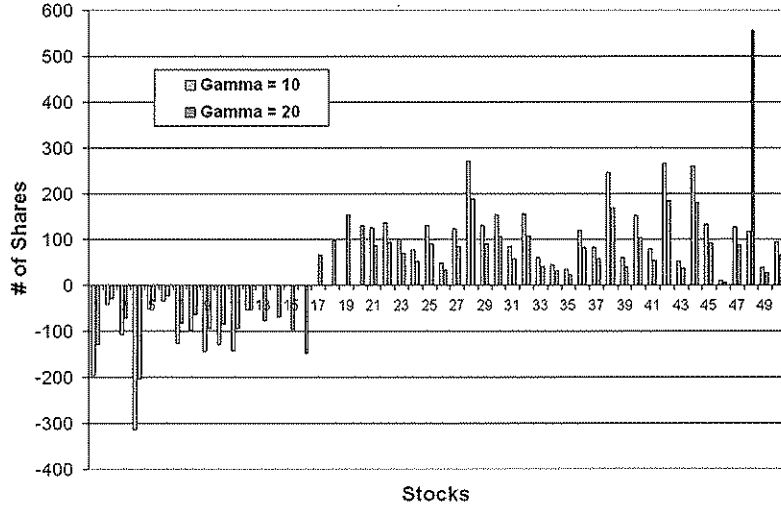


Figure 2: Number of shares of both stocks invested in and short-sold, for $\Gamma = 10$ and $\Gamma = 20$, in the Log-robust model.

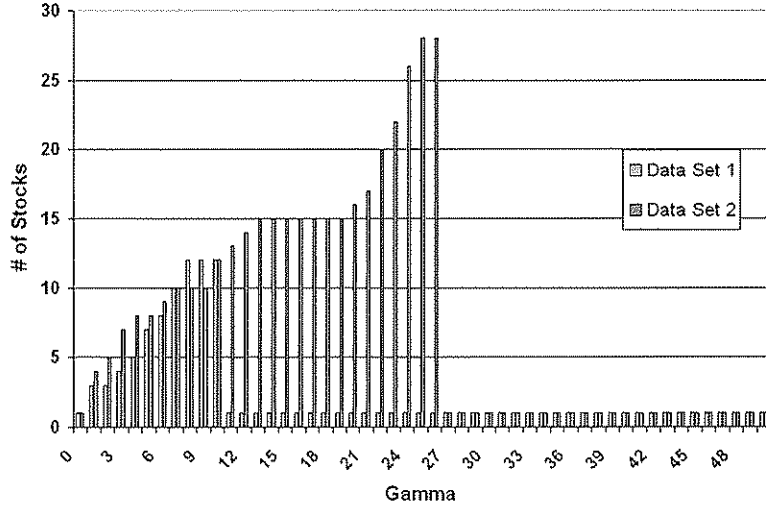


Figure 3: Number of stocks short-sold in optimal portfolio for Γ varying from 0 to 50, for two different data sets for the Log-robust model.

diversification, in the sense that there are stocks he never invests in nor short-sells, for any value of Γ . Diversification increases steadily when Γ increases up to 10, resulting in at most 26 stocks invested in and 12 stocks short-sold, out of 50. Then diversification breaks and the very

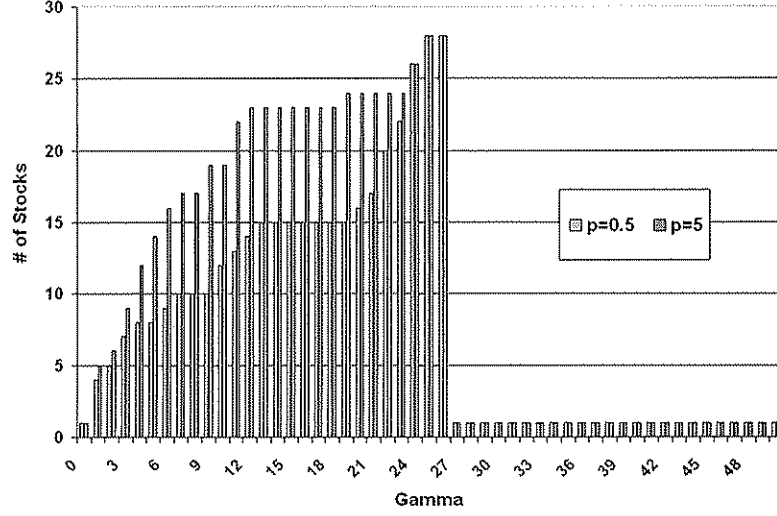


Figure 4: Number of stocks short-sold in optimal portfolio for Γ varying from 0 to 50, for two different values of the leverage parameter p for the Log-robust model.

ambiguity-averse decision-maker invests in one stock only and short-sells one stock only. The stock invested in is the stock with the highest minimum value and the stock short-sold is the stock with the lowest maximum value. The highest minimum value of the former is higher than the lowest maximum value of the latter, which explains why the manager keeps short-selling.

Analysis of performance in simulations.

We now investigate performance of the approach with respect to 99% Conditional Value-at-Risk (cVaR), which is defined as the expected value of the portfolio given that its value falls below the 1% percentile. The decision-maker naturally wants this worst-case portfolio value to be as large as possible, so 99% cVaR is here a *risk-adjusted performance measure*, not a risk measure, and should be maximized here instead of minimized. Hence, it is well-suited to our framework. We also ran all the tests using the 99% VaR, i.e., the value of the 1% of the portfolio; the graphs are omitted in this paper because they are very similar to those obtained in the cVaR case, but all of our conclusions extend to the VaR performance metric.

The simulations were performed using @Risk 5.0 from Palisade Corporation. We consider two cases: (i) the case where the random variables do obey Normal (Gaussian) distribution, in which the only error made by the manager implementing the traditional robust approach is to use symmetric confidence intervals *for the stock prices* themselves rather than for the true drivers

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
MSH	4210	1367	200	133	416	425	703	507	200	374	209	4210	4210	4210	4210	4210
EST	0	303	416	416	339	303	200	319	204	115	191	0	0	0	0	0
AA	0	581	405	287	220	213	102	114	143	132	134	0	0	0	0	0
AFD	0	663	295	309	223	200	170	159	240	124	151	0	0	0	0	0
FCX	0	0	410	226	345	207	195	110	141	141	130	0	0	0	0	0
AU	0	0	427	204	204	204	100	102	121	145	142	0	0	0	0	0
APA	0	0	412	300	245	249	105	120	141	141	130	0	0	0	0	0
E	0	0	0	207	232	200	160	141	134	124	130	0	0	0	0	0
ESV	0	0	0	346	360	233	197	120	150	150	131	0	0	0	0	0
BP	0	0	0	167	220	217	100	104	143	127	135	0	0	0	0	0
OKS	0	0	0	0	156	140	118	101	94	90	100	0	0	0	0	0
TPP	0	0	0	0	224	209	127	151	141	130	130	0	0	0	0	0
EME	0	0	0	0	229	204	123	147	130	132	129	0	0	0	0	0
PMC	0	0	0	0	200	201	212	101	103	101	100	0	0	0	0	0
MT	0	0	0	0	379	410	343	306	270	264	259	0	0	0	0	0
ACN	0	0	0	0	0	315	140	155	145	120	126	0	0	0	0	0
CR	0	0	0	0	0	0	257	204	204	272	236	0	0	0	0	0
LI	0	0	0	0	0	0	124	110	100	90	90	0	0	0	0	0
DAI	0	0	0	0	0	0	201	205	231	217	211	0	0	0	0	0
TRW	0	0	0	0	0	0	0	143	102	103	120	0	0	0	0	0
END	0	0	0	0	0	0	0	317	205	205	231	0	0	0	0	0
FMG	0	0	0	0	0	0	0	404	124	109	107	0	0	0	0	0
BSI	0	0	0	0	0	0	0	0	126	121	110	0	0	0	0	0
EDU	0	0	0	0	0	0	0	0	145	117	143	0	0	0	0	0
RAI	0	0	0	0	0	0	0	0	0	114	112	0	0	0	0	0
OLV	0	0	0	0	0	0	0	0	0	57	12	0	0	0	0	0
HCT	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CD+	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HIC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CL	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CFB	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
RAH	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
AMG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
FFG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
WOL	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
TRW	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
TAC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SIE	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SHC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
MYI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
NDN	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ITC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
DTE	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
DTE	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ATU	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
YEC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ATG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
QED	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
APOD	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ATIR	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 5: Impact of Γ varying from 0 to 50 on stock allocation and diversification for uncorrelated stocks.

of uncertainty, the continuously compounded rates of return, and (ii) the case where the random variables obey a Logistic distribution and hence have “fat tails”, in which case the Lognormal model of stock prices behavior underestimates rare events.

Normal Distribution.

Figure 6 shows the 99% cVaR for the three models (traditional robust and Log-robust model with short-sales, and Log-robust model without short-sales), when assets are uncorrelated and the distribution of the continuously compounded rates of return is Gaussian. We note that the Log-robust optimization approach without short sales beats the traditional robust optimization approach with short sales, for some values of the budget of uncertainty, by up to 3%, further emphasizing the importance of applying robust optimization to the true uncertainty drivers. Short-selling allows the decision-maker to increase the 99% cVaR by 7-11%. The proposed model with short sales also beats the traditional robust model by up to 16%.

Logistic Distribution.

Figure 7 shows the 99% cVaR for the three models, traditional robust and Log-robust model with short-sales, and Log-robust model without short-sales, when assets are uncorrelated and the distribution is Logistic. The curves are similar to the case with Gaussian distribution (Figure 6). The benefit of applying the Log-robust approach with and without short sales, compared to

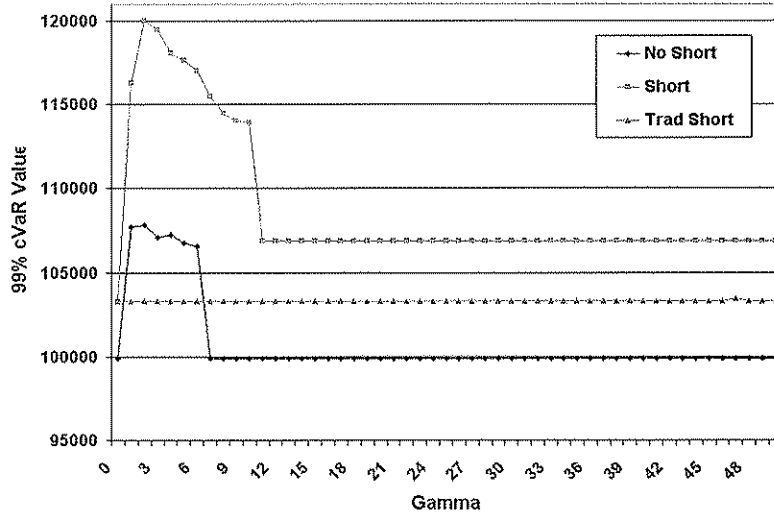


Figure 6: 99% cVaR for the traditional model with short-sales, Log-robust with short-sales, and Log-robust without short-sales, in the uncorrelated case for the Gaussian distribution.

the traditional model, is very pronounced here, as the Log-robust approach without short sales outperforms the traditional model by up to 9%, and underperforms it by at most 1% for other values of Γ .

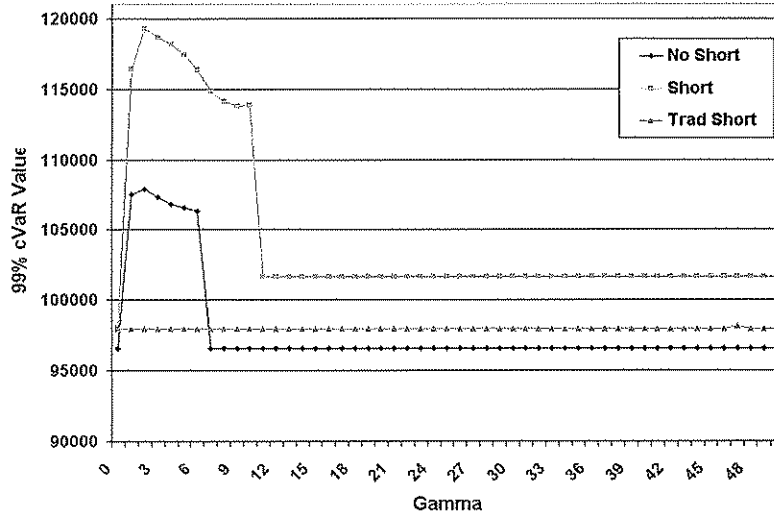


Figure 7: 99% cVaR for the traditional model with short-sales, Log-robust with short-sales, and Log-robust without short-sales, in the uncorrelated case for the Logistic distribution.

3 The Case of Correlated Assets

3.1 Problem Setup

We now extend the approach described in Section 2 to the case of correlated assets. The traditional Log-Normal model of asset prices describes stock prices as follows:

$$\ln \left(\frac{S_i(T)}{S_i(0)} \right) = \left(\mu_i - \frac{\sigma_i^2}{2} \right) T + \sqrt{T} Z_i,$$

where the random vector Z is Normally distributed with mean 0 and covariance matrix Q .

We define:

$$Y = Q^{-\frac{1}{2}} Z,$$

where $Y \sim \mathcal{N}(0, I)$ and $Q^{\frac{1}{2}}$ is the symmetric positive definite matrix square root of Q . In the robust optimization approach, the random variables Z_i are replaced by uncertain parameters ζ_i , which can be expressed as a linear combination of the scaled independent uncertainty drivers:

$$\zeta_i = c \sum_{j=1}^n Q_{ij}^{1/2} \tilde{y}_j,$$

with each component \tilde{y}_i belonging to $[-1, 1]$ and Y_i belonging to $[-c, c]$ for all i ; c is the scale parameter as in Section 2.

Problem (1) becomes:

$$\begin{aligned} \max_x \quad & \min_{\tilde{y}} \sum_{i=1}^n x_i \exp \left[\left(\mu_i - \sigma_i^2/2 \right) T + \sqrt{T} c \left(\sum_{j=1}^n Q_{ij}^{1/2} \tilde{y}_j \right) \right] \\ \text{s.t.} \quad & \sum_{j=1}^n |\tilde{y}_j| \leq \Gamma, \\ & |\tilde{y}_j| \leq 1, \quad \forall j, \\ \text{s.t.} \quad & \sum_{i=1}^n x_i = w_0, \\ & \sum_{i|x_i < 0} (-x_i) \leq p w_0. \end{aligned} \tag{10}$$

Not only do we lose convexity, as in the case for independent assets, but we can no longer isolate the impact of the uncertain parameters \tilde{y}_j on the objective, because of the correlation. In particular, because correlation makes all the stocks affected by all the uncertainty drivers:

- (i) we cannot say whether the worst case is achieved for positive or negative scaled deviations,
- (ii) we can no longer split the problem into two subproblems, one for the uncertain parameters corresponding to stocks we invest in and the other for the uncertain parameters corresponding to stocks we short-sell,
- (iii) we can no longer extract the scaled deviations for the stocks short-sold out of the exponential term in order to go back to a linear programming problem, because other scaled deviations remain inside the exponential.

Therefore, we will consider tractable approximations of Problem (10) instead of solving it exactly.

3.2 Heuristics

This section defines three heuristics. For each heuristic, the master algorithm still ranks the assets in order of decreasing nominal return, as in the case of independent assets, and solves n approximated robust problems, where the set of assets invested in in subproblem j is $\{1, \dots, j\}$. We present numerical experiments in Section 3.3.

Heuristic 1: No correlation for stocks short-sold.

We first investigate a heuristic that discards correlation for the stocks we short-sell. This means that there is neither correlation between stocks short-sold, nor between a stock short-sold and a stock invested in. Decoupling the assets in that manner allows us to use the model for correlated assets without short sales, presented in Kawa and Thiele [18], and the model for independent assets with short sales, developed in Section 2 above. Let \mathbf{R} be the covariance matrix for the stocks we invest in.

Heuristic 1 solves Problem (8), where the function F_p given in Equation (9) is replaced by:

$$\begin{aligned}
F_p(\theta, \Gamma) = \max_{\eta, \xi, \tilde{\chi}} \quad & \sum_{i|x_i \geq 0} \tilde{\chi}_i \ln k_i - \sum_{i|x_i < 0} \tilde{\chi}_i k_i - \eta \Gamma - \sum_{i=1}^n \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sqrt{T}c \sum_{j|x_j \geq 0} R_{ij}^{1/2} \tilde{\chi}_j \geq 0, & \forall i|x_i \geq 0, \\
& \eta + \xi_i + \sqrt{T}c \sum_{j|x_j \geq 0} R_{ij}^{1/2} \tilde{\chi}_j \geq 0, & \forall i|x_i \geq 0, \\
& \eta + \xi_i - k_i \left[\exp(\sigma_i \sqrt{T}c) - 1 \right] \tilde{\chi}_i \geq 0, & \forall i|x_i < 0, \\
& \sum_{i|x_i \geq 0} \tilde{\chi}_i = \theta, \\
& \sum_{i|x_i < 0} \tilde{\chi}_i = p, \\
& \eta \geq 0, \xi_i \geq 0, \tilde{\chi}_i \geq 0, & \forall i.
\end{aligned}$$

Heuristic 2: Approximating the off-diagonal elements by their average.

In this second heuristic, we approximate the off-diagonal elements (of the square root of the covariance matrix) by their average:

$$\exp \left[\sqrt{T}c \left(\sum_{j=1}^n Q_{ij}^{1/2} \tilde{y}_j \right) \right] \approx \exp \left[\sqrt{T}c \left(Q_{ii}^{1/2} \tilde{y}_i + \frac{\sum_{j \neq i} Q_{ij}^{1/2}}{n-1} \left(\sum_{j \neq i} \tilde{y}_j \right) \right) \right],$$

and then approximate $\sum_{j \neq i} \tilde{y}_j$ by $\Gamma - y_i$. This allows us to use the results derived in the independent case (Section 2), with:

$$k'_i = k_i \exp \left[\sqrt{T}c \left(\frac{\sum_{j \neq i} Q_{ij}^{1/2}}{n-1} \Gamma \right) \right],$$

and:

$$\sigma_i = \left| Q_{ii}^{1/2} - \frac{\sum_{j \neq i} Q_{ij}^{1/2}}{n-1} \right|.$$

Heuristic 3: Approximating the off-diagonal elements by a conservative estimate of their worst-case value.

Because the robust optimization approach is inherently a pessimistic approach, it is also reasonable to consider a more conservative approximation of the (squared root of the) covariance matrix than the one proposed in Heuristic 2. Here, we replace $\sum_{j \neq i} Q_{ij}^{1/2} \tilde{y}_j$ by $-L_i$ when asset i is invested in, and $+L_i$ when asset i is short-sold, where the new parameter L_i is defined as the sum of the Γ greatest $|Q_{ij}^{1/2}|$ for $j \neq i$. This decreases, respectively increases, the rate of return of the assets the manager invests in, respectively short-sells. The approximation is more conservative than the robust optimization approach, since the manager selects different values of \tilde{y}_j for different assets i in the heuristic, instead of keeping the same value for \tilde{y}_j throughout.

Therefore, Heuristic 3 reduces to the independent model with $\sigma_i = |Q_{ii}^{1/2}|$ and:

$$k_i^+ = k_i \exp(-\sqrt{T} c L_i), \text{ for } x_i \geq 0,$$

and:

$$k_i^- = k_i \exp(\sqrt{T} c L_i), \text{ for } x_i < 0.$$

3.3 Numerical Experiments

We now repeat the experiments in the case of correlation. The setup remains the same as in Section 2.3. We comment on the allocations given by the three heuristics, compare their performance, and argue that it is better not to incorporate correlation rather than incorporate it in an approximate manner, i.e., the decision-maker should implement Heuristic 1. Then we test the allocation given by Heuristic 1, against the traditional robust model with short sales and the Log-robust model without short sales.

Analysis of optimal solution.

Figure 8 shows the optimal allocation given by Heuristic 1 as a function of the budget of uncertainty. We note that correlation introduces rows of zeros in the allocation for the assets invested in, although the assets are ranked in order of decreasing nominal value as before. There is no row of zeros for the stocks short-sold, because correlation between these stocks has been discarded.

Figure 9 shows the allocations for the three heuristics when Γ is equal to 5. We observe that Heuristic 1 is the least diversified, since the manager would neither invest in nor short-sell asset 7 to 28; furthermore, he would invest a large amount of shares in asset 29. Heuristic 2 invests

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
MMM	1210	1403	1010	609	520	474	272	267	309	266	295	4219	4219	4219	4219
SYT	0	993	267	245	190	161	164	140	125	102	99	0	0	0	0
ALA	0	717	416	296	240	231	154	136	120	115	102	0	0	0	0
APD	0	672	510	264	291	269	241	222	187	164	144	0	0	0	0
FCX	0	0	655	328	221	222	207	227	168	126	116	0	0	0	0
ADJ	0	0	550	362	293	269	219	199	181	156	144	0	0	0	0
APA	0	0	0	558	120	97	193	96	85	76	63	0	0	0	0
E	0	0	0	0	256	241	231	241	262	187	140	0	0	0	0
ESV	0	0	0	0	210	226	201	157	164	146	149	135	0	0	0
HP	0	0	0	0	353	264	241	243	219	210	192	184	0	0	0
ONS	0	0	0	0	156	169	131	123	106	76	69	0	0	0	0
IPP	0	0	0	0	0	21	223	225	213	160	163	0	0	0	0
EPS	0	0	0	0	231	214	105	155	126	99	71	0	0	0	0
HMC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
NI	0	0	0	0	312	217	160	154	443	414	360	364	0	0	0
NOX	0	0	0	0	0	0	0	22	162	143	113	0	0	0	0
CR	0	0	0	0	315	213	160	155	494	459	420	421	0	0	0
LE	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
DAI	0	0	0	0	0	0	0	0	253	476	476	0	0	0	0
DDV	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
DKD	0	0	0	0	0	0	412	260	306	232	227	0	0	0	0
FMK	0	0	0	0	0	1270	922	206	340	740	600	0	0	0	0
BR	0	0	0	0	0	0	0	0	0	0	121	0	0	0	0
DDO	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
RAI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CLX	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HST	0	0	0	0	0	0	276	205	181	162	143	0	0	0	0
CBV	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HHZ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CL	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CFB	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
RAH	0	0	0	0	0	0	0	0	0	263	263	0	0	0	0
AMQ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
FFO	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HOLX	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
TRB	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
TAC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
STE	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
RYT	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
YEDN	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
HC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
DTI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
GRS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ABU	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
VEC	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ATO	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
QCD	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
APDQ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ATIR	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 8: Impact of Γ varying from 0 to 50 on stock allocation and diversification for correlated stocks.

in or short-sells asset classes in the middle of the graph, but does not short-sell assets with low nominal return, which are short-sold by both Heuristics 1 and 3, specifically, assets 1 to 4. Figure

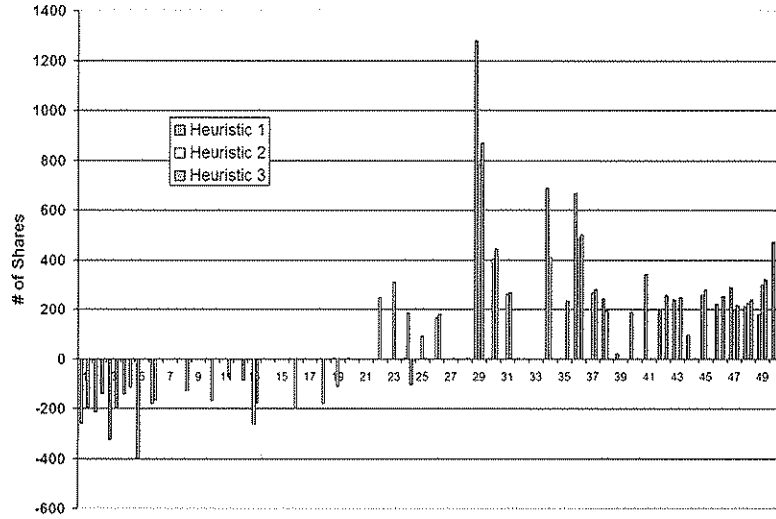


Figure 9: Allocation for the three heuristics, $\Gamma = 5$.

10 shows the allocations for the three heuristics when Γ is equal to 10. The difference between Heuristic 2 and Heuristics 1 and 3 is even starker here: it has become optimal to short-sell only one stock and invest in only one stock with Heuristic 2, while the other two heuristics remain diversified, and generally invest similar (but not equal) amounts in the same stocks. Finally,

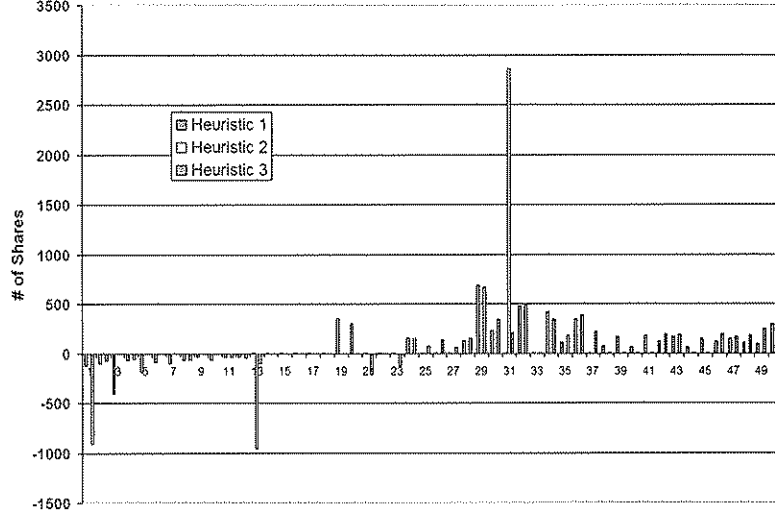


Figure 10: Allocation for the three heuristics, $\Gamma = 10$.

Figure 11 shows the allocations for the three heuristics when Γ is equal to 20. At that stage, it has become optimal for all three allocations to short-sell only one stock and invest in only one stock. We observe that Heuristics 2 and 3 yield the same allocation, while Heuristic 1 differs significantly from the other two, short-selling a stock with lower nominal return and investing in a stock with higher nominal return.

This analysis of the three allocations suggests that Heuristic 1 is going to perform better in experiments. This is indeed what we observe in simulations, as described below.

Comparison of the three heuristics.

As in the case of independent assets, we use Conditional Value-at-Risk (cVaR) as our performance metric.

Normal Distribution.

Figure 12 compares the three heuristics when the distribution is Normal. Heuristic 1 outperforms the other two, and Heuristic 3 outperforms Heuristic 2, for all values of Γ . Heuristic 1 most outperforms the other two when the decision-maker is most conservative, and makes money for all

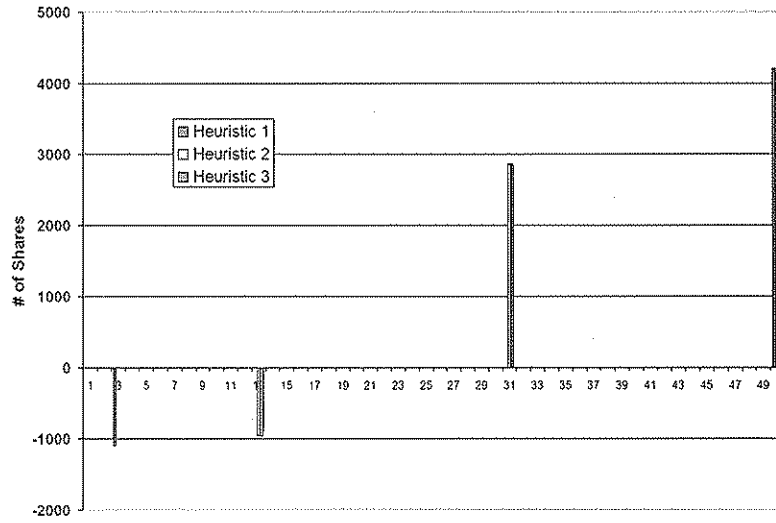


Figure 11: Allocation for the three heuristics, $\Gamma = 20$.

values of Γ , in the sense that 99% cVaR is higher than the initial budget of \$100,000. Heuristic 3 makes money for Γ from 0 to 11. Heuristic 2 loses money except for Γ equal to 4 and 5.

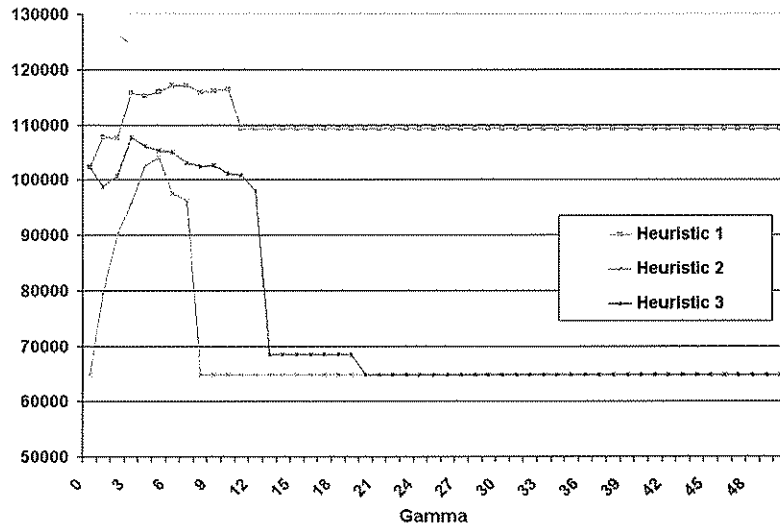


Figure 12: Comparison of the three heuristics with Normal distribution using cVaR.

Logistic Distribution.

Figure 13 compares the three heuristics when the distribution is Logistic. The curves are very

similar to the case where the distribution is Normal and Heuristic 1 again significantly outperforms the other two, while Heuristic 3 outperforms Heuristic 2 for all values of Γ . This suggests that incorporating correlation in an approximated way can hurt the performance of the portfolio, and that it is better not to incorporate correlation at all for the stocks short-sold.

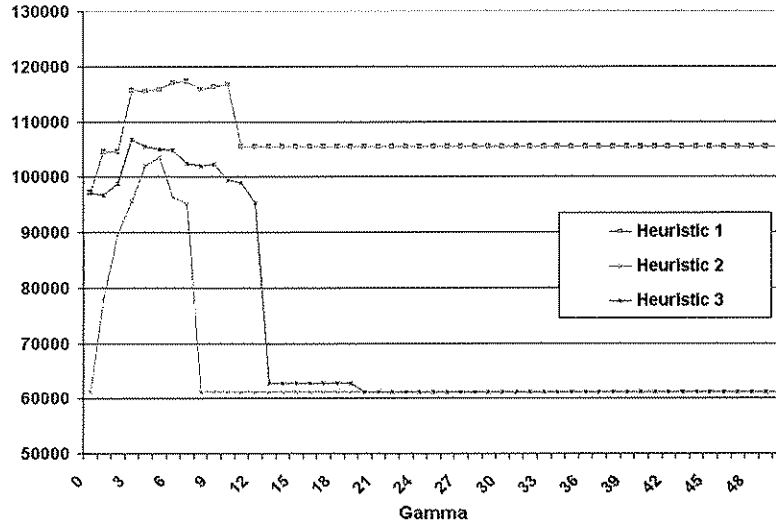


Figure 13: Comparison of the three heuristics with Logistic distribution using cVaR.

Analysis of performance in simulations.

Due to Heuristic 1's dominance, we only test that heuristic against the allocations obtained in the traditional robust model with short sales and the Log-robust model without short sales.

Normal Distribution.

Figure 14 shows the 99% cVaR for the three models. Notice that the Log-robust model with no short sales beats the traditional with short sales for $\Gamma > 0$, and that the Log-robust model with short-sales beats both, by up to 15% compared to the traditional model and up to 10% compared to the Log-robust model without short sales. The main lesson from this graph is that it is much more beneficial for the decision-maker to incorporate uncertainty at the level of the true drivers of uncertainty rather than to allow short sales, although incorporating both is of course best. Correlation seems to improve the performance of the proposed approach, especially the Log-robust model without short sales, which beats the traditional approach more consistently than before.

Logistic Distribution.

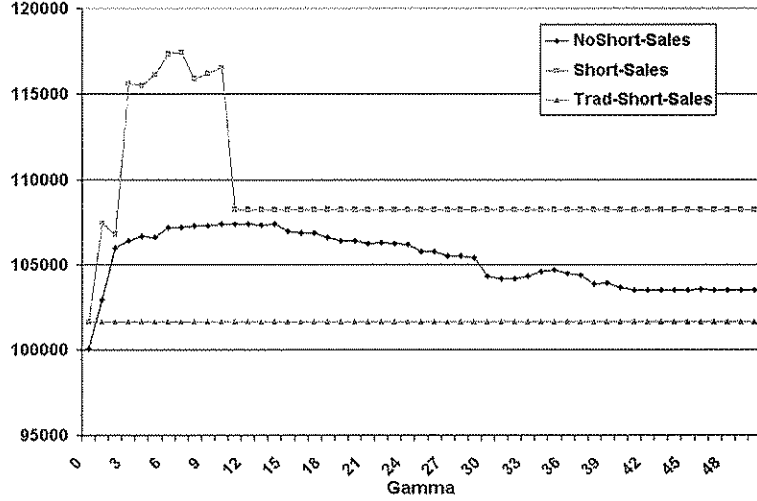


Figure 14: 99% cVaR for the traditional model with short-sales, Log-robust with short-sales, and Log-robust without short-sales, for Gaussian distribution.

Figure 15 shows the 99% cVaR using a Logistic distribution for the simulated random variables. While the curves exhibit trends similar to the ones in the Gaussian case (Figure 14) and our broad conclusions still hold, the most notable feature of this set of experiments is that the Log-robust model without short sales beats the Log-robust model with short sales for Γ between 11 and 24; however, maximum cVaR is achieved for Γ equal to 6 and 7, in line with the rule of thumb presented in Bertsimas and Sim [4], which suggests a budget of uncertainty equal to the square root of the number of uncertain parameters.

4 Conclusions

We have extended the Log-robust portfolio management approach to the case with short sales. When assets are independent, it is optimal for the manager to either short-sell as much as he can, or not short-sell at all; we have also derived tractable, equivalent reformulations, and provided optimal allocations. When assets are correlated, we have compared three heuristics, which we have tested in simulations. Our numerical results suggest that the manager should implement the heuristic where only the assets invested in are correlated with each other, and other correlations (between stocks invested in and stocks short-sold, or between stocks short-sold) are discarded. Computational experiments are encouraging. In particular, they indicate that it is critical for

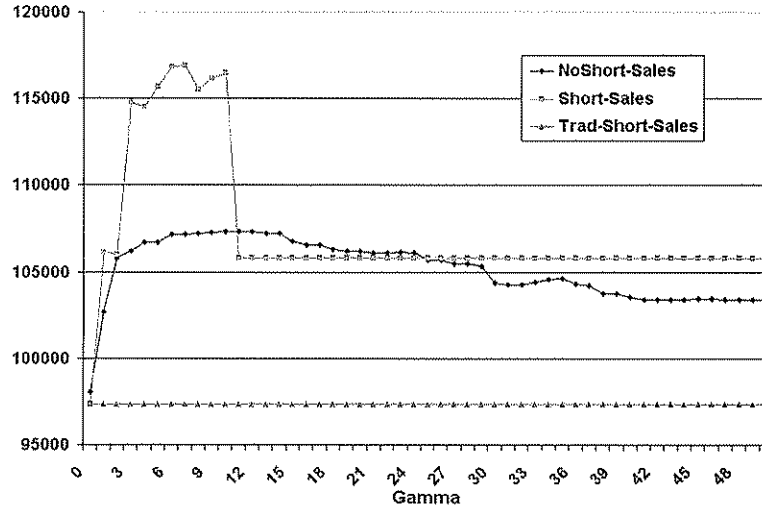


Figure 15: 99% cVaR, for the traditional with short-sales, Log-robust with short-sales, and Log-robust without short-sales, for Logistic distribution.

the decision-maker to apply robust optimization at the level of the true uncertainty drivers – the continuously compounded rates of return – rather than at the level of the stock returns, as has been traditionally done in the literature.

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A Proofs

A.1 Proof of Lemma 2.3

(i) Follows immediately from the sign of the x_i 's and the fact that the exponential function is increasing in its argument.

(ii) An argument similar to (i) yields the formulation, for the stocks short-sold:

$$\begin{aligned}
 \min_{z^-} \quad & \sum_{i|x_i < 0}^n x_i k_i \exp(\sigma_i \sqrt{T} c z_i^-) \\
 \text{s.t.} \quad & \sum_{i|x_i < 0}^n z_i^- \leq \Gamma^-, \\
 & 0 \leq z_i^- \leq 1, \forall i \text{ s.t. } x_i < 0.
 \end{aligned} \tag{11}$$

It follows from the concavity of the objective (due to $x_i < 0$ for all stocks short-sold), which is minimized over a polyhedron, that the optimal solution is achieved at one of the corner points of the feasible set. In particular, because Γ^- is integer, the corner points satisfy $z_i^- \in \{0, 1\}$ for all i such that stock i is short-sold. The gross return of stock i can therefore be rewritten as k_i if $z_i^- = 0$ and $k_i \exp(\sigma_i \sqrt{T} c)$ if $z_i^- = 1$. This yields Problem (6) immediately. \square

A.2 Proof of Theorem 2.4

The inner minimization in Problem (7) is a convex problem. To reformulate it in a tractable manner, we invoke strong duality in convex programming (Boyd and Vandenberghe [8]) and rewrite the inner minimization problem as:

$$\begin{aligned}
& \max_{\eta^+, \eta^-, \xi^+, \xi^-} \min_{z^-, z^+, \Gamma^+} \sum_{i|x_i \geq 0}^n x_i k_i \exp(-\sigma_i \sqrt{T} c z_i^+) + \sum_{i|x_i \leq 0}^n x_i k_i \left[\exp(\sigma_i \sqrt{T} c) - 1 \right] z_i^- + \sum_{i|x_i < 0}^n x_i k_i \\
& \quad + \eta^+ \left(\sum_{i|x_i \geq 0}^n z_i^+ - \Gamma^+ \right) + \eta^- \left(\sum_{i|x_i < 0}^n z_i^- - (\Gamma - \Gamma^+) \right) \\
& \quad + \sum_{i|x_i \geq 0} [(z_i^+ - 1)\xi_i^{+1} - \xi_i^{+0} z_i^+] + \sum_{i|x_i < 0} [(z_i^- - 1)\xi_i^{-1} - \xi_i^{-0} z_i^-] \\
& \text{s.t.} \quad \xi_i^{+1}, \xi_i^{+0} \geq 0, \forall i | x_i \geq 0, \\
& \quad \xi_i^{-1}, \xi_i^{-0} \geq 0, \forall i | x_i < 0, \\
& \quad \eta^+, \eta^- \geq 0,
\end{aligned} \tag{12}$$

Note that we are able to discard the constraint $0 \leq \Gamma^+ \leq \Gamma$ because it will be naturally enforced in the optimal solution, as there would be infeasibility of the primal problem otherwise. Since Γ^+ is unconstrained in Problem (12), we have $\eta^+ = \eta^-$, which we denote η . We then differentiate with respect to z_i^- and z_i^+ , which yields at optimality:

$$z_i^+ = -\frac{1}{\sigma_i \sqrt{T} c} \ln \left(\frac{\eta + \xi_i^{+1} - \xi_i^{+0}}{x_i k_i \sigma_i \sqrt{T} c} \right), \forall i | x_i \geq 0,$$

and:

$$x_i k_i \left(1 - \exp(\sigma_i \sqrt{T} c) \right) = \eta + \xi_i^{-1} - \xi_i^{-0}, \forall i | x_i < 0.$$

Injecting these equations into Problem (12) leads to:

$$\begin{aligned}
\max_{x, \eta, \xi} \quad & \sum_{i|x_i \geq 0} \left(\frac{\eta + \xi_i^{+1} - \xi_i^{+0}}{\sigma_i \sqrt{T} c} \right) \left[1 - \ln \left(\frac{\eta + \xi_i^{+1} - \xi_i^{+0}}{x_i k_i \sigma_i \sqrt{T} c} \right) \right] + \sum_{i|x_i < 0} x_i k_i \\
& - \sum_{i|x_i \geq 0}^n \xi_i^{+1} - \sum_{i|x_i < 0} \xi_i^{-1} - \eta \Gamma \\
\text{s.t.} \quad & x_i k_i \left(\exp(\sigma_i \sqrt{T} c) - 1 \right) + \eta + \xi_i^{-1} - \xi_i^{-0} = 0 \quad \forall i | x_i < 0, \\
& \xi_i^{+1}, \xi_i^{+0} \geq 0, \quad \forall i | x_i \geq 0, \\
& \xi_i^{-1}, \xi_i^{-0} \geq 0, \quad \forall i | x_i < 0, \\
& \eta \geq 0, \\
& \sum_{i=1} x_i = w_0, \\
& \sum_{i|x_i < 0} (-x_i) \leq p w_0.
\end{aligned} \tag{13}$$

We rewrite the first group of constraints as $x_i k_i \left(\exp(\sigma_i \sqrt{T} c) - 1 \right) + \eta + \xi_i^{-1} \geq 0$ because $\xi_i^{-0} \geq 0$. As in Kawas and Thiele [18], we use the change of variables $\chi_i = \frac{\eta + \xi_i^{+1} - \xi_i^{+0}}{\sigma_i \sqrt{T} c}$ to replace ξ_i^{+0} and optimize over the non-negative x_i with w^+ the amount invested in stocks we do not short-sell, using a straightforward Lagrange approach (dualizing the equality constraint). We find that the optimal amount invested in asset i is $x_i = \chi_i w^+ \left(\sum_{j|x_j \geq 0} \chi_j \right)^{-1}$. We parametrize by $\sum_{i|x_i \geq 0} \chi_i = \theta$. Problem (13) becomes:

$$\begin{aligned}
\max \quad & \theta \left(1 + \ln \left(\frac{w^+}{\theta} \right) \right) + \tilde{F}_p(\theta, w^+, \Gamma), \\
\text{s.t.} \quad & w_0 \leq w^+ \leq (1 + p) w_0, \\
& \theta \geq 0,
\end{aligned} \tag{14}$$

where \tilde{F}_p is defined by (after concatenating the vector of ξ_i^{+1} for $x_i \geq 0$ and ξ_i^{-1} for $x_i < 0$ into one vector ξ):

$$\begin{aligned}
\tilde{F}_p(\theta, w^+, \Gamma) = \max_{\eta, \xi, \chi, x} \quad & \sum_{i|x_i \geq 0} \chi_i \ln k_i + \sum_{i|x_i < 0} x_i k_i - \eta \Gamma - \sum_{i=1}^n \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sigma_i \sqrt{T} c \chi_i \geq 0, & \forall i | x_i \geq 0, \\
& \eta + \xi_i + k_i [\exp(\sigma_i \sqrt{T} c) - 1] x_i \geq 0, & \forall i | x_i < 0, \\
& \sum_{i|x_i \geq 0} \chi_i = \theta, \\
& \sum_{i|x_i < 0} (-x_i) = w^+ - w_0, \\
& \eta \geq 0, \xi_i \geq 0, & \forall i, \\
& \chi_i \geq 0, & \forall i | x_i \geq 0.
\end{aligned} \tag{15}$$

We then optimize Problem (14) with respect to θ . The objective function is concave as the sum of two concave functions. Differentiating yields $\theta = w^+ \exp(\partial \tilde{F}_p)$ at optimality, where $\partial \tilde{F}_p$ is a subgradient of the piecewise linear function \tilde{F}_p at θ . In particular, ∂F does not depend on θ (using strong duality, we define F as a minimization problem where θ only appears in the objective, so we can theoretically enumerate the extreme points of this dual feasible set, which do not depend on θ , to find the optimal solution.)

Reinjecting into Problem (14), we obtain a linear expression in w^+ after noting that $\tilde{F}_p(w^+ \exp(\partial \tilde{F}_p), w^+, \Gamma) = w^+ \exp(\partial \tilde{F}_p) \partial \tilde{F}_p + k$ where k is a constant. It follows that the maximum is reached at one of the extremities of $[w_0, (1+p)w_0]$. Problem (9) follows immediately when $w^+ = (1+p)w_0$ is optimal, where we redefine F_p to drop the dependency in w^+ .

To conclude the proof, we need to check that the resulting Γ^- is indeed integer, since this is key to Lemma 2.3. To do this, we observe that Problem (9) always has an optimal solution, so that strong duality holds. We compute the dual, with η and ξ as the primal variables, when the $\tilde{\chi}_i$ are set to their optimal values $\tilde{\chi}_i^*$. This yields an objective of:

$$\begin{aligned}
\sum_{i|x_i \geq 0} \tilde{\chi}_i^* \ln k_i - \sum_{i|x_i < 0} \tilde{\chi}_i^* k_i + \min \quad & \left(- \sum_{i|x_i \geq 0} \sigma_i \sqrt{T} c \tilde{\chi}_i^* z_i - \sum_{i|x_i < 0} k_i [\exp(\sigma_i \sqrt{T} c) - 1] \tilde{\chi}_i^* z_i \right) \\
\text{s.t.} \quad & \sum_{i=1}^n z_i \leq \Gamma, \\
& 0 \leq z_i \leq 1, \forall i.
\end{aligned}$$

This is a linear programming problem and the optimal solution will be achieved at one of the corner points, i.e., the components of the optimal z are all either 0 or 1. In particular, $\sum_{i|x_i < 0} z_i$, which is equal to Γ^- , is integer. \square

A.3 Proof of Corollary 2.5

Problem (9) is a linear programming problem; hence, it allows for constraint splitting, i.e., the dualization of only some of the constraints. We choose to dualize the budget constraints $\sum_{i|x_i \geq 0} \tilde{\chi}_i = \theta$ and $\sum_{i|x_i < 0} \tilde{\chi}_i = p$, introducing the dual variables v^+ and v^- , respectively, and then note that $\xi_i = \max(0, \sigma_i \sqrt{T} c \tilde{\chi}_i - \eta, k_i [\exp(\sigma_i \sqrt{T} c) - 1] \tilde{\chi}_i - \eta)$. The problem becomes:

$$\begin{aligned} \min_{v^+, v^-} \max_{\eta, \chi^+, \chi^-} & \sum_{i|x_i \geq 0} \tilde{\chi}_i \ln k_i - \sum_{i|x_i < 0} \tilde{\chi}_i k_i - \eta \Gamma \\ & - \sum_{i=1}^n \max \left(0, \sigma_i \sqrt{T} c \tilde{\chi}_i - \eta, k_i [\exp(\sigma_i \sqrt{T} c) - 1] \tilde{\chi}_i - \eta \right) \\ & - v^+ \left(\sum_{i|x_i \geq 0} \tilde{\chi}_i - \theta \right) - v^- \left(\sum_{i|x_i < 0} \tilde{\chi}_i - p \right). \end{aligned} \quad (16)$$

Assume η is a given, arbitrary positive number. If $\tilde{\chi}_i > 0$ and $\sigma_i \sqrt{T} c \tilde{\chi}_i > \eta$, so that $\xi_i = \sigma_i \sqrt{T} c \tilde{\chi}_i - \eta > 0$, then we must have $\ln k_i - \sigma_i \sqrt{T} c - v^+ = 0$ to prevent unboundedness (a strict less-than-zero inequality would make $\tilde{\chi}_i$ hit its lower bound, a contradiction.) Then if $\sigma_i \sqrt{T} c \tilde{\chi}_i \leq \eta$, we have $\tilde{\chi}_i = 0$ if $\ln k_i < v^+$ and $\tilde{\chi}_i = \eta / (\sigma_i \sqrt{T} c)$ if $\ln k_i > v^+$. If there exists j such that $\sigma_j \sqrt{T} c \tilde{\chi}_j > \eta$, then we use the fact that the budget constraint for assets not short-sold must hold in order to determine that $\tilde{\chi}_j$. Otherwise, we have $v^+ = \ln k_j$ for some j , again because the budget constraint must hold, and that $\tilde{\chi}_j$ is such that $\sum_{i=1}^n \tilde{\chi}_i = \theta$. A similar argument holds for $\tilde{\chi}_i$ when $x_i < 0$, leading to the following allocation as a function of η :

$$\chi_i = \begin{cases} \frac{\eta}{\sigma_i \sqrt{T} c}, & i < j, \\ (1 + p) - \sum_{a=1}^{j-1} \frac{\eta}{\sigma_a \sqrt{T} c}, & i = j, \\ 0, & j + 1 \leq i \leq l - 1, \\ -p + \sum_{a=l+1}^n \frac{\eta}{k_a [\exp(\sigma_a \sqrt{T} c) - 1]}, & i = l, \\ -\frac{\eta}{k_i [\exp(\sigma_i \sqrt{T} c) - 1]}, & i > l. \end{cases}$$

where χ_i is the fraction allocated to stock i . It is equal to $(1+p)\tilde{\chi}_i/\theta$ if the stock is invested in, and $-\tilde{\chi}_i$ if the stock is short-sold.

Problem (16) can then be rewritten as a piecewise linear, convex problem as a function of η (note that the j and l indices have a hidden dependence in η , since we must have $\chi_j > 0$ and $\chi_l < 0$, so that j decreases and l increases as η increases). Hence, the optimal η is achieved at one of the breakpoints, which is when either χ_j or χ_l become zero (which means that either j or l would be relabeled as $j-1$ or $l+1$, respectively, to respect the fact that $\chi_j > 0$ and $\chi_l < 0$ by definition, and leads to the two possible allocations stated in the theorem), or when $\eta = 0$, i.e., the manager does not use all of his budget of uncertainty Γ . In the case where $\eta = 0$, χ_j becomes χ_1 and χ_l becomes χ_n , which remains of the form advertised.

Note that $j < l$ because if $j \geq l$, there would exist an index i such that the manager both invests in and short-sells stock i , and he would strictly increase his portfolio wealth by allocating his money away from such a stock i , and that would contradict the optimality assumption. \square

A.4 Proof of Corollary 2.6

The optimal η is non-increasing with Γ , because F_p is a convex, non-increasing function of Γ with slope $-\eta$. If the optimal allocation is given by Allocation 1 in Corollary 2.5, it follows from $\eta = \frac{\theta \sqrt{T} c}{\sum_{a=1}^j 1/\sigma_a}$ that j increases, i.e., more stocks are invested in. As the $\sum_{a=1}^j 1/\sigma_a$ term becomes bigger but the $1/\sigma_i$ in Allocation 1 for the assets short-sold does not change, the amounts of money invested in short-sold assets from the previously determined stock l to stock n decrease. Since the amount of money short-sold must equal $p w_0$, we conclude that l is non-increasing, so that more stocks are short-sold. A similar argument holds if the optimal allocation is given by Allocation 2. This trend continues until $\Gamma = n$ or complete diversification is achieved. \square