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Abstract

In this paper we consider the Convex Quadratic Optimization problem with simultaneous perturbation in the right-hand-side of the constraints and the linear term of the objective function with different parameters. The regions with invariant optimal partitions are investigated as well as the behavior of the optimal value function on the regions. We show that identifying these regions can be done in polynomial time in the output size. An algorithm for identifying all invariancy regions is presented. Some implementation details, as well as a numerical example are discussed.

Keywords: bi-parametric optimization; convex quadratic optimization; interior-point methods; optimal partition; invariancy region

1 Introduction

Let the Bi-Parametric Convex Quadratic Optimization (CQO) problem be given as

$$QP(\Delta b, \Delta c, \epsilon, \lambda) \quad \min \left\{ (c + \lambda \Delta c)^T x + \frac{1}{2} x^T Q x \mid Ax = b + \epsilon \Delta b, x \geq 0 \right\},$$

where $A \in \mathbb{R}^{m \times n}$, $Q \in \mathbb{R}^{n \times n}$ is a symmetric semi-definite matrix, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are fixed data, ϵ and λ are two real parameters, $\Delta b \in \mathbb{R}^m$, $\Delta c \in \mathbb{R}^n$ are the perturbation directions and $x \in \mathbb{R}^n$ is an unknown vector. The dual of problem $QP(\Delta b, \Delta c, \epsilon, \lambda)$ is

$$QD(\Delta b, \Delta c, \epsilon, \lambda) \quad \max \left\{ (b + \epsilon \Delta b)^T y - \frac{1}{2} u^T Q u \mid A^T y + s - Qu = c + \lambda \Delta c, u, s \geq 0 \right\},$$

where $y \in \mathbb{R}^m$, $u \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$ are unknowns. In general, both Δb and Δc are non-zero vectors. Any $x(\epsilon) \geq 0$ satisfying $Ax = b + \epsilon \Delta b$ is called a *primal feasible* solution

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of problem $QP(\Delta b, \Delta c, \epsilon, \lambda)$. Further, a vector $(u(\lambda), y(\lambda), s(\lambda))$ with $u(\lambda), s(\lambda) \geq 0$ is called a *dual feasible* solution of $QD(\Delta b, \Delta c, \epsilon, \lambda)$ if it satisfies $A^T y + s - Qu = c + \lambda \Delta c$. Feasible solutions $x(\epsilon)$ and $(u(\lambda), y(\lambda), s(\lambda))$ of $QP(\Delta b, \Delta c, \epsilon, \lambda)$ and $QD(\Delta b, \Delta c, \epsilon, \lambda)$ are optimal if and only if $Qx(\epsilon) = Qu(\lambda)$ and $x(\epsilon)^T s(\lambda) = 0$ [6]. The equation $x(\epsilon)^T s(\lambda) = 0$ is equivalent to $x_i(\epsilon)s_i(\lambda) = 0$, $i = 1, 2, \dots, n$ and is known as the *complementarity condition*. Let $\mathcal{QP}(\Delta b, \Delta c, \epsilon, \lambda)$ and $\mathcal{QP}^*(\Delta b, \Delta c, \epsilon, \lambda)$ denote the sets of primal feasible and primal optimal solutions of $QP(\Delta b, \Delta c, \epsilon, \lambda)$, respectively. Similar notation is used for the sets of feasible and optimal solutions of $QD(\Delta b, \Delta c, \epsilon, \lambda)$. For the case when $\epsilon = \lambda = 0$, the CQO problem is referred to as the unperturbed problem and we drop Δb , Δc , ϵ and λ in this case. Moreover, when either Δb or Δc is a zero vector, or when $\epsilon = \lambda$, the perturbed CQO problem is referred to as uni-parametric CQO problem. However, we consider the general case when ϵ and λ are not necessarily identical and refer to this problem as the bi-parametric CQO problem.

It is well known [6] that there exist optimal solutions $x^* \in \mathcal{QP}^*$ and $(u^*, y^*, s^*) \in \mathcal{QD}^*$ for which $x^* = u^*$, thus we may denote the dual solution by (x, y, s) for the unperturbed problem QD . If we keep in mind this consideration for the perturbed problems, the primal optimal solution of $QP(\Delta b, \Delta c, \epsilon, \lambda)$ will depend not only on ϵ but also on λ . Analogously, the dual variables y and s in $QD(\Delta b, \Delta c, \epsilon, \lambda)$ will depend on both ϵ and λ . Due to these facts, we denote a primal-dual optimal solution of the perturbed CQO problems by $(x^*(\epsilon, \lambda), y^*(\epsilon, \lambda), s^*(\epsilon, \lambda))$.

The *support* set of a nonnegative vector v is defined as $\sigma(v) = \{i : v_i > 0\}$. Unlike the case of Linear Optimization (LO) problem, where the index set $\{1, 2, \dots, n\}$ can be partitioned into two subsets, the index set for CQO needs to be partitioned into three disjoint subsets as

$$\begin{aligned} \mathcal{B}(\epsilon, \lambda) &= \{i : x_i^*(\epsilon, \lambda) > 0 \text{ for an optimal solution } x^*(\epsilon, \lambda)\}, \\ \mathcal{N}(\epsilon, \lambda) &= \{i : s_i^* > 0 \text{ for an optimal solution } (x^*(\epsilon, \lambda), y^*(\epsilon, \lambda), s^*(\epsilon, \lambda))\}, \\ \mathcal{T}(\epsilon, \lambda) &= \{1, 2, \dots, n\} \setminus (\mathcal{B}(\epsilon, \lambda) \cup \mathcal{N}(\epsilon, \lambda)) \\ &= \{i : x_i^*(\epsilon, \lambda) = s_i^*(\epsilon, \lambda) = 0 \text{ for all primal-dual optimal solutions} \\ &\quad (x^*(\epsilon, \lambda), y^*(\epsilon, \lambda), s^*(\epsilon, \lambda))\}. \end{aligned}$$

We refer to this partition as the *optimal partition* and denote it by $\pi(\epsilon, \lambda) = (\mathcal{B}(\epsilon, \lambda), \mathcal{N}(\epsilon, \lambda), \mathcal{T}(\epsilon, \lambda))$ as well. The optimal partition $\pi(\epsilon, \lambda)$ is unique [3]. A *maximality complementary solution* $(x^*(\epsilon, \lambda), y^*(\epsilon, \lambda), s^*(\epsilon, \lambda))$ is a primal-dual optimal solution of $QP(\Delta b, \Delta c, \epsilon, \lambda)$ and $QD(\Delta b, \Delta c, \epsilon, \lambda)$ for which

$$\begin{aligned} x_i^*(\epsilon, \lambda) &> 0 \text{ if and only if } i \in \mathcal{B}(\epsilon, \lambda), \\ s_i^*(\epsilon, \lambda) &> 0 \text{ if and only if } i \in \mathcal{N}(\epsilon, \lambda). \end{aligned}$$

The existence of maximality complementary optimal solutions is a consequence of the convexity of the optimal solution sets $\mathcal{QP}^*(\Delta b, \Delta c, \epsilon, \lambda)$ and $\mathcal{QD}^*(\Delta b, \Delta c, \epsilon, \lambda)$. Interior Point Methods (IPMs) are widely used to solve CQO problems in polynomial time [14] and sufficiently accurate solutions obtained by an IPM can be used to produce maximally complementary solutions [12]. By knowing a maximality complementary optimal

solution, one can easily identify the optimal partition. If for a given optimal partition $\mathcal{T}(\epsilon, \lambda) = \emptyset$ holds, then any maximally complementary solution is *strictly complementary*. It is worth mentioning that for any primal-dual optimal solution (x^*, y^*, s^*) , the relations $\sigma(x^*(\epsilon, \lambda)) \subseteq \mathcal{B}(\epsilon, \lambda)$ and $\sigma(s^*(\epsilon, \lambda)) \subseteq \mathcal{N}(\epsilon, \lambda)$ hold. Both inclusions hold with equality if and only if the given primal-dual optimal solution is maximally (strictly) complementary.

Let $\phi(\epsilon, \lambda)$ denote the *optimal value function* which is defined as:

$$\begin{aligned}\phi(\epsilon, \lambda) &= (c + \lambda \Delta c)^T x^*(\epsilon, \lambda) + \frac{1}{2} x^*(\epsilon, \lambda)^T Q x^*(\epsilon, \lambda) \\ &= (b + \epsilon \Delta b)^T y^*(\epsilon, \lambda) - \frac{1}{2} x^*(\epsilon, \lambda)^T Q x^*(\epsilon, \lambda),\end{aligned}\tag{1}$$

where Δb and Δc are fixed perturbing directions.

In *optimal partition invariancy* sensitivity analysis we aim to identify the range of parameters where the optimal partition remains invariant. The cases when either Δb or Δc is zero has been studied in [4]. The situation when $\epsilon = \lambda$ has been investigated in [8]. In these cases the region of the parameter is an interval of the real line called *invariancy interval*. We refer to the studies mentioned above as *uni-parametric* optimal partition invariancy sensitivity analysis.

Bi-parametric optimal partition based sensitivity analysis has been studied in case of LO in [7]. Bi-parametric active set based sensitivity analysis was developed in [2, 10] for the LO and CQO cases. Earlier studies (see [16, 21, 11, 15, 9] for more details) produced the ideas that are summarized in [7, 2, 10] and we refer the interested reader to consult those publications. Fundamental properties derived in Section 2 overlap with the ones that appeared in [2] with the difference that those are based on the active-set notion and, consequently, require non-degeneracy assumption. While [2] contains theoretical derivations and numerical examples, there is no complete algorithm to do systematic analysis of bi-parametric quadratic optimization problems. In the remainder of this paragraph, we just point out the most recent results from [7] in a nutshell. The crucial difference of the bi-parametric LO case from the CQO case is that in the LO case the invariancy regions are open rectangles while in the CQO case those are open convex polyhedrons. In bi-parametric LO the invariancy regions generate a mesh-like area in \mathbb{R}^2 that simplifies enumeration of the regions. This is not the case for bi-parametric CQO problems that results in a more complicated computational algorithm.

In this paper, we consider the bi-parametric optimal partition invariancy sensitivity analysis for CQO in the general case, when both Δb and Δc are nonzero vectors and parameters ϵ and λ change independently. Let $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ denote the optimal partition for $\epsilon = 0$ and $\lambda = 0$. We are interested in finding all the regions on the “ $\epsilon - \lambda$ ” plane where the optimal partition is invariant, i.e., $\pi(\epsilon, \lambda) = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. We call each of these regions *invariancy region* and denote it by $\mathcal{IR}(\Delta b, \Delta c)$. It is obvious that one of these regions includes the origin $(0, 0)$, and thus, their union is a nonempty set.

The paper is organized as follows. In Section 2 we describe the fundamental properties of the invariancy regions and optimal value function. Section 3 is devoted to the issues arising during the boundary detection of invariancy regions. Algorithmic and implementation issues are addressed in Section 4. We present a numerical example in Section 5 and make some concluding remarks in Section 6.

2 Fundamental Properties

In this section, we prove some fundamental properties of the invariancy region and describe the behavior of the optimal value function in this region. First we prove that this region is a convex set. Analogous result can be found in Theorems 5.4.3 and 5.4.4 in [2] and Theorem 17 in [9].

Lemma 2.1 *The set $\mathcal{IR}(\Delta b, \Delta c)$ is a convex set and its closure is a polyhedron.*

Proof: Let (ϵ_1, λ_1) and (ϵ_2, λ_2) are two arbitrary pairs in $\mathcal{IR}(\Delta b, \Delta c)$. Let $(x^{(1)}, y^{(1)}, s^{(1)})$ and $(x^{(2)}, y^{(2)}, s^{(2)})$ are maximally (strictly) complementary optimal solutions of problems $QP(\Delta b, \Delta c, \epsilon, \lambda)$ and $QD(\Delta b, \Delta c, \epsilon, \lambda)$ at these points. Let (ϵ, λ) be an arbitrary point on the line segment between two points (ϵ_1, λ_1) and (ϵ_2, λ_2) . There is a $\theta \in (0, 1)$ such that:

$$\epsilon = \epsilon_1 + \theta \Delta \epsilon, \quad (2)$$

$$\lambda = \lambda_1 + \theta \Delta \lambda, \quad (3)$$

where $\Delta \epsilon = \epsilon_2 - \epsilon_1$ and $\Delta \lambda = \lambda_2 - \lambda_1$. We define

$$x(\epsilon, \lambda) = x(\epsilon) = \theta x^{(1)} + (1 - \theta)x^{(2)}, \quad (4)$$

$$y(\epsilon, \lambda) = y(\lambda) = \theta y^{(1)} + (1 - \theta)y^{(2)}, \quad (5)$$

$$s(\epsilon, \lambda) = s(\lambda) = \theta s^{(1)} + (1 - \theta)s^{(2)}. \quad (6)$$

It is easy to verify that $x(\epsilon, \lambda)$ is a primal feasible solution and $(x(\epsilon, \lambda), y(\epsilon, \lambda), s(\epsilon, \lambda))$ is a dual feasible solution. On the other hand, $\sigma(x(\epsilon, \lambda)) = \sigma(x^{(1)}) \cup \sigma(x^{(2)}) = \mathcal{B}$ and $\sigma(s(\epsilon, \lambda)) = \sigma(s^{(1)}) \cup \sigma(s^{(2)}) = \mathcal{N}$, that proves the optimality of these solutions for problems $QP(\Delta b, \Delta c, \epsilon, \lambda)$ and $QD(\Delta b, \Delta c, \epsilon, \lambda)$ as well as having the optimal partition $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ at (ϵ, λ) . That completes the proof of the first statement. Having the optimal partition given, the optimality conditions reduce to a linear inequality system when one fixes all x_i variables zero in the $\mathcal{N} \cup \mathcal{T}$ part and the s_i variables zero at the $\mathcal{B} \cup \mathcal{T}$ part, giving a polyhedron [19]. \square

The boundaries between the invariancy regions are line (half-line) segments. The line segment between two adjacent invariancy regions is referred to as *transitional line segment* and the intersection of two transition lines are called *transition points*. Transition points (singleton invariancy regions) and transition lines are called *trivial* invariancy regions. An invariancy region that is neither a singleton nor a transition line is referred to as *non-trivial* invariancy region.

Optimal value function $\phi(\epsilon, \lambda)$ is continuous and piecewise-quadratic. While those results are proven in Theorems 5.5.1 and 5.5.2 in [2], the expression for quadratic function given in the proof below is not derived in [2].

Theorem 2.2 *The optimal value function is a bivariate quadratic function on invariancy region $\mathcal{IR}(\Delta b, \Delta c)$.*

Proof: If the invariancy region is a non-singleton trivial region, then the optimal value function is a univariate quadratic function by Theorem 4.5 in [8]. Let the invariancy

region be a non-trivial region. Further, let (ϵ_1, λ_1) , (ϵ_2, λ_2) and (ϵ_3, λ_3) are three points in general position (not on a line) on the “ $\epsilon - \lambda$ ” plane. We are allowed to make assumption about the general position of points since the invariency region is not a trivial region. Let $(x^*(\epsilon_1), y^*(\lambda_1), s^*(\lambda_1))$, $(x^*(\epsilon_2), y^*(\lambda_2), s^*(\lambda_2))$ and $(x^*(\epsilon_3), y^*(\lambda_3), s^*(\lambda_3))$ be primal-dual optimal solutions at these three points, respectively. Moreover, let (ϵ, λ) be an arbitrary point in the interior of the triangle formed by these three points. Therefore, there are $\theta_1, \theta_2 \in (0, 1)$ with $0 < \theta_1 + \theta_2 < 1$ such that

$$\epsilon = \epsilon_3 - \theta_1 \Delta \epsilon_1 - \theta_2 \Delta \epsilon_2, \quad (7)$$

$$\lambda = \lambda_3 - \theta_1 \Delta \lambda_1 - \theta_2 \Delta \lambda_2, \quad (8)$$

where $\Delta \epsilon_1 = \epsilon_3 - \epsilon_1$, $\Delta \epsilon_2 = \epsilon_3 - \epsilon_2$, $\Delta \lambda_1 = \lambda_3 - \lambda_1$ and $\Delta \lambda_2 = \lambda_3 - \lambda_2$. Let us define

$$x^*(\epsilon, \lambda) = x^{(3)} - \theta_1 \Delta x^{(1)} - \theta_2 \Delta x^{(2)}, \quad (9)$$

$$y^*(\epsilon, \lambda) = y^{(3)} - \theta_1 \Delta y^{(1)} - \theta_2 \Delta y^{(2)},$$

$$s^*(\epsilon, \lambda) = s^{(3)} - \theta_1 \Delta s^{(1)} - \theta_2 \Delta s^{(2)},$$

where $\Delta x^{(1)} = x^{(3)} - x^{(1)}$, $\Delta x^{(2)} = x^{(3)} - x^{(2)}$, $\Delta y^{(1)} = y^{(3)} - y^{(1)}$, $\Delta y^{(2)} = y^{(3)} - y^{(2)}$, $\Delta s^{(1)} = s^{(3)} - s^{(1)}$ and $\Delta s^{(2)} = s^{(3)} - s^{(2)}$. It is easy to verify that $(x^*(\epsilon), y^*(\lambda), s^*(\lambda))$ is a primal-dual optimal solution of problems $QP(\Delta b, \Delta c, \epsilon, \lambda)$ and $QD(\Delta b, \Delta c, \epsilon, \lambda)$. Substituting (8) and (9) in (1) gives

$$\begin{aligned} \phi(\epsilon, \lambda) &= (b + \epsilon \Delta b)^T y^*(\epsilon, \lambda) - \frac{1}{2} x^*(\epsilon, \lambda)^T Q x^*(\epsilon, \lambda) \\ &= a_0 + a_1 \theta_1 + a_2 \theta_2 + a_3 \theta_1 \theta_2 + a_4 \theta_1^2 + a_5 \theta_2^2, \end{aligned} \quad (10)$$

where

$$\begin{aligned} a_0 &= b^T y^{(3)} + \epsilon_3 \Delta b^T y^{(3)} - \frac{1}{2} (x^{(3)})^T Q x^{(3)}, \\ a_1 &= -b^T \Delta y^{(1)} - \epsilon_3 \Delta b^T y^{(1)} - \Delta \epsilon_1 \Delta b^T y^{(3)} + (x^{(3)})^T Q \Delta x^{(1)}, \\ a_2 &= -b^T \Delta y^{(2)} - \epsilon_3 \Delta b^T y^{(2)} - \Delta \epsilon_2 \Delta b^T y^{(3)} + (x^{(3)})^T Q \Delta x^{(2)}, \\ a_3 &= \Delta \epsilon_1 \Delta b^T \Delta y^{(2)} + \Delta \epsilon_2 \Delta b^T \Delta y^{(1)} - (\Delta x^{(2)})^T Q \Delta x^{(2)}, \\ a_4 &= \Delta \epsilon_1 \Delta b^T \Delta y^{(1)} - \frac{1}{2} (\Delta x^{(1)})^T Q \Delta x^{(1)}, \\ a_5 &= \Delta \epsilon_2 \Delta b^T \Delta y^{(2)} - \frac{1}{2} (\Delta x^{(2)})^T Q \Delta x^{(2)}. \end{aligned} \quad (11)$$

On the other hand, solving equations (8) and (7) for θ_1 and θ_2 gives

$$\theta_1 = \alpha_1 + \beta_1 \epsilon + \gamma_1 \lambda, \quad (12)$$

$$\theta_2 = \alpha_2 + \beta_2 \epsilon + \gamma_2 \lambda, \quad (13)$$

where

$$\begin{aligned}
\alpha_1 &= \frac{\epsilon_3 \Delta \lambda_2 - \lambda_3 \Delta \epsilon_2}{\Delta \epsilon_1 \Delta \lambda_2 - \Delta \epsilon_2 \Delta \lambda_1}, \quad \alpha_2 = \frac{\lambda_3 (\Delta \epsilon_1 + \Delta \epsilon_2) - \epsilon_3 (\Delta \lambda_1 + \Delta \lambda_2)}{\Delta \epsilon_1 \Delta \lambda_2 - \Delta \epsilon_2 \Delta \lambda_1}, \\
\beta_1 &= \frac{-\Delta \lambda_2}{\Delta \epsilon_1 \Delta \lambda_2 - \Delta \epsilon_2 \Delta \lambda_1}, \quad \beta_2 = \frac{(\Delta \lambda_2 + \Delta \lambda_1)}{\Delta \epsilon_1 \Delta \lambda_2 - \Delta \epsilon_2 \Delta \lambda_1}, \\
\gamma_1 &= \frac{\Delta \epsilon_2}{\Delta \epsilon_1 \Delta \lambda_2 - \Delta \epsilon_2 \Delta \lambda_1}, \quad \gamma_2 = \frac{-(\Delta \epsilon_2 + \Delta \epsilon_1)}{\Delta \epsilon_1 \Delta \lambda_2 - \Delta \epsilon_2 \Delta \lambda_1}.
\end{aligned} \tag{14}$$

Substituting (11)-(14) in (10) leads to the following representation of the optimal value function:

$$\phi(\epsilon, \lambda) = b_0 + b_1 \epsilon + b_2 \lambda + b_3 \epsilon \lambda + b_4 \epsilon^2 + b_5 \lambda^2, \tag{15}$$

where

$$\begin{aligned}
b_0 &= a_0 + a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_1 \alpha_2 + a_4 \alpha_1^2 + a_5 \alpha_2^2, \\
b_1 &= a_1 \beta_1 + a_2 \beta_2 + a_3 (\alpha_1 \beta_2 + \alpha_2 \beta_1) + 2a_4 \alpha_1 \beta_1 + 2a_5 \alpha_2 \beta_2, \\
b_2 &= a_1 \gamma_1 + a_2 \gamma_2 + a_3 (\alpha_1 \gamma_2 + \alpha_2 \gamma_1) + 2a_4 \alpha_1 \gamma_1 + 2a_5 \alpha_2 \gamma_2, \\
b_3 &= a_3 (\gamma_1 \beta_2 + \gamma_2 \beta_1) + 2a_4 \gamma_1 \beta_1 + 2a_5 \gamma_2 \beta_2, \\
b_4 &= a_3 \beta_1 \beta_2 + a_4 \beta_1^2 + a_5 \beta_2^2, \\
b_5 &= a_3 \gamma_1 \gamma_2 + a_4 \gamma_1^2 + a_5 \gamma_2^2.
\end{aligned}$$

Clearly (15) is a quadratic function of ϵ and λ . Because (ϵ_1, λ_1) , (ϵ_2, λ_2) and (ϵ_3, λ_3) are three arbitrary points in the non-trivial invariancy region, the claim of the theorem follows directly from (15). The proof is complete. \square

Corollary 2.3 *The boundary of a non-trivial invariancy region consists of a finite number of line segments, and on each such line segment the optimal value function is a univariate quadratic function.*

Proof: We know that the optimal value function on the open set representing the non-trivial invariancy region is quadratic and represented by (15). By Lemma 2.1 the closure of the non-trivial invariancy region is a convex polyhedron. Thus, the non-trivial invariancy region boundary consists of a finite number of half-lines and/or line segments. On each line segment an optimal partition is defined by construction. The bi-parametric CQO problem can be considered as a uni-parametric problem on each line segment by computing appropriate linear relation between two parameters. Using continuity of the optimal value function we get that in the limit for the border line segment, we also obtain a single quadratic function. So, the optimal value function on each border line segment of the non-trivial invariancy region boundary is a single univariate quadratic function. \square

The range of parameter variation is an interval of the real line when $\lambda = \epsilon$. In this case one can identify the range of parameters via solving the following two auxiliary LO

problems [8]:

$$\begin{aligned} \lambda_\ell = \min_{\lambda, x, y, s} \{ & \lambda : Ax - \lambda \Delta b = b, \ x_B \geq 0, \ x_{\mathcal{N} \cup \mathcal{T}} = 0, \\ & A^T y + s - Qx - \lambda \Delta c = c, \ s_{\mathcal{N}} \geq 0, \ s_{\mathcal{B} \cup \mathcal{T}} = 0 \}, \end{aligned} \quad (16)$$

and

$$\begin{aligned} \lambda_u = \max_{\lambda, x, y, s} \{ & \lambda : Ax - \lambda \Delta b = b, \ x_B \geq 0, \ x_{\mathcal{N} \cup \mathcal{T}} = 0, \\ & A^T y + s - Qx - \lambda \Delta c = c, \ s_{\mathcal{N}} \geq 0, \ s_{\mathcal{B} \cup \mathcal{T}} = 0 \}. \end{aligned} \quad (17)$$

where $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ is the optimal partition for $\epsilon = \lambda = 0$.

Remark 2.4 *These auxiliary problems are simpler when either Δb or Δc is a zero vector. One may find the details in [4].*

Remark 2.5 *If solving of problems (16) and (17) lead to the singleton $\{0\}$, then the invariancy region $\mathcal{IR}(\Delta b, \Delta c)$ is a one dimensional set or the singleton $\{0\}$. In this case, the $\mathcal{IR}(\Delta b, \Delta c)$ is a trivial region. Otherwise, the region is non-trivial invariancy region.*

The invariancy region that contains the origin $(\epsilon, \lambda) = (0, 0)$ is referred to as the actual invariancy region.

3 Detecting the Boundary of an Invariancy Region

In this section, we describe the tools to identify a non-trivial invariancy region. Recall that for $\epsilon = \lambda$, the bi-parametric CQO problem reduces to uni-parametric CQO problem. This trivial observation suggests choosing a method to convert the bi-parametric CQO problem into uni-parametric CQO problems. We start with finding some points on the boundary of the invariancy region. To accomplish this, we select the lines passing through the origin as

$$\lambda = t\epsilon. \quad (18)$$

For now, we assume that the slope t is positive. Substituting (18) into the problem $QP(\Delta b, \Delta c, \epsilon, \lambda)$ converts it to the following uni-parametric CQO problem:

$$\min \left\{ (c + \epsilon \overline{\Delta c})^T x + \frac{1}{2} x^T Q x \mid Ax = b + \epsilon \Delta b, \ x \geq 0 \right\}, \quad (19)$$

where $\overline{\Delta c} = t \Delta c$. This way we can solve two associated auxiliary LO problems (16) and (17) to identify the range of variation for parameter ϵ when equation (18) holds. These two auxiliary LO problems are:

$$\begin{aligned} \lambda_\ell = \min_{\lambda, x, y, s} \{ & \lambda : Ax - \lambda \Delta b = b, \ x_B \geq 0, \ x_{\mathcal{N} \cup \mathcal{T}} = 0, \\ & A^T y + s - Qx - \lambda \overline{\Delta c} = c, \ s_{\mathcal{N}} \geq 0, \ s_{\mathcal{B} \cup \mathcal{T}} = 0 \}, \end{aligned} \quad (20)$$

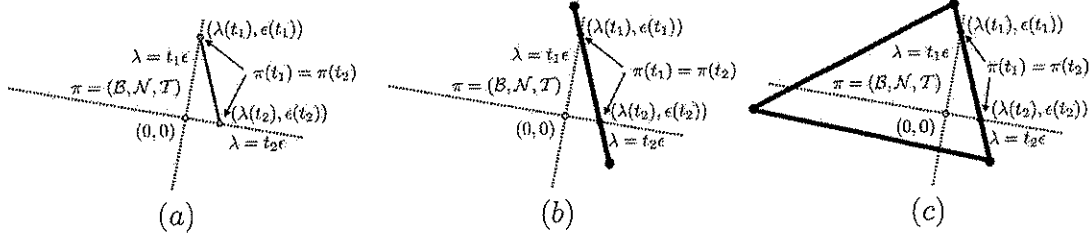


Figure 1: Case 1 Illustration

and

$$\begin{aligned} \lambda_u = \max_{\lambda, x, y, s} \{ & \lambda : Ax - \lambda \Delta b = b, \ x_B \geq 0, \ x_{\mathcal{N} \cup \mathcal{T}} = 0, \\ & A^T y + s - Qx - \lambda \overline{\Delta c} = c, \ s_{\mathcal{N}} \geq 0, \ s_{\mathcal{B} \cup \mathcal{T}} = 0 \}, \end{aligned} \quad (21)$$

where $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$ is the optimal partition for $\epsilon = \lambda = 0$.

Let us consider the case when the problem (20) is bounded for two given distinct nonzero values t_1 and t_2 (if the problem is unbounded, we know that the invariancy region is unbounded) and their objective values are $\epsilon(t_1)$ and $\epsilon(t_2)$, respectively. Thus, two points $(\epsilon(t_1), \lambda(t_1))$ and $(\epsilon(t_2), \lambda(t_2))$ of the invariancy region boundary are known.

Let $\pi(t_1) = (\mathcal{B}(t_1), \mathcal{N}(t_1), \mathcal{T}(t_1))$ and $\pi(t_2) = (\mathcal{B}(t_2), \mathcal{N}(t_2), \mathcal{T}(t_2))$ be optimal partitions at $(\epsilon(t_1), \lambda(t_1))$ and $(\epsilon(t_2), \lambda(t_2))$, respectively. There are two possibilities for these optimal partitions.

□ **Case 1:** $\pi(t_1) = \pi(t_2)$. The following lemma states that in this case we are able to identify a part of the boundary of the invariancy region that is a line on the “ $\epsilon - \lambda$ ” plane.

Lemma 3.1 *Let (ϵ_1, λ_1) and (ϵ_2, λ_2) be two distinct points on the “ $\epsilon - \lambda$ ” plane, where the optimal partitions at these points are identical. Then, the optimal partition for any point on the line segment connecting these two points is invariant, and equal to the optimal partition at the two given points.*

Proof: Let $\alpha\epsilon + \beta\lambda = 1$ be the line passing via two points (ϵ_1, λ_1) and (ϵ_2, λ_2) . We take both α and β to be non-zero, otherwise we are back to the uni-parametric case described in [8]. So, we have:

$$\lambda = \frac{1}{\beta} - \frac{\alpha}{\beta}\epsilon. \quad (22)$$

Substitution of (22) into the CQO problem $QP(\Delta b, \Delta c, \epsilon, \lambda)$ reduces it to a uni-parametric CQO problem as follows:

$$\min = \left\{ (\bar{c} + \epsilon \overline{\Delta c})^T x + \frac{1}{2} x^T Q x \mid Ax = b + \epsilon \Delta b, \ x \geq 0 \right\}, \quad (23)$$

where $\bar{c} = c + \frac{1}{\beta} \Delta c$ and $\overline{\Delta c} = \frac{\alpha}{\beta} \Delta c$. It is proven that the range of parameter variation in this case is an interval of the real line and for any ϵ in this range the optimal partition is invariant [8]. The proof is complete. □

Using Lemma 3.1 we can conclude that the boundary of the invariancy region contains a segment of the line (22). To identify the end points of this line segment, we need to find the invariancy interval for problem (23). The following theorem provides two auxiliary CQO problems to achieve it. The proof is straightforward and is omitted.

Theorem 3.2 *Let (ϵ_1, λ_1) and (ϵ_2, λ_2) be two distinct points on the “ $\epsilon - \lambda$ ” plane. Let $\bar{\pi} = (\bar{B}, \bar{N}, \bar{T})$ denote the optimal partition at these two points. Moreover, let $\alpha\epsilon + \beta\lambda = 1$ denote the line passing through these two points. By solving the following two auxiliary LO problems:*

$$\begin{aligned} \epsilon_\ell(\alpha, \beta) = \min_{\epsilon, x, y, s} \{ & \epsilon : Ax - \epsilon\Delta b = b, \ x_{\bar{B}} \geq 0, \ x_{\bar{N} \cup \bar{T}} = 0, \\ & A^T y + s - Qx - \epsilon\bar{\Delta}c = \bar{c}, \ s_{\bar{N}} \geq 0, \ s_{\bar{B} \cup \bar{T}} = 0 \}, \end{aligned} \quad (24)$$

and

$$\begin{aligned} \epsilon_u(\alpha, \beta) = \max_{\epsilon, x, y, s} \{ & \epsilon : Ax - \epsilon\Delta b = b, \ x_{\bar{B}} \geq 0, \ x_{\bar{N} \cup \bar{T}} = 0, \\ & A^T y + s - Qx - \epsilon\bar{\Delta}c = \bar{c}, \ s_{\bar{N}} \geq 0, \ s_{\bar{B} \cup \bar{T}} = 0 \}. \end{aligned} \quad (25)$$

where $\bar{c} = c + \frac{1}{\beta}\Delta c$ and $\bar{\Delta}c = \frac{\alpha}{\beta}\Delta c$, one can identify the two vertices of the invariancy region as $(\epsilon_\ell(\alpha, \beta), \lambda_\ell(\alpha, \beta))$ and $(\epsilon_u(\alpha, \beta), \lambda_u(\alpha, \beta))$, where $\lambda_\ell(\alpha, \beta) = \frac{1}{\beta} - \frac{\alpha}{\beta}\epsilon_\ell(\alpha, \beta)$ and $\lambda_u(\alpha, \beta) = \frac{1}{\beta} - \frac{\alpha}{\beta}\epsilon_u(\alpha, \beta)$.

Remark 3.3 *Observe that one of these auxiliary LO problems can be unbounded. In this case, the actual invariancy region is unbounded. If both problems (24) and (25) are unbounded then there is only one invariancy region on the “ $\epsilon - \lambda$ ” plane.*

Case 1 is illustrated at Figure 1. Figure 1(a) shows two points with identical optimal partition and Figure 1(b) depicts the corresponding identified transition line segment. Figure 1(c) illustrates the non-trivial invariancy region with its boundary.

□ **Case 2:** $\pi(t_1) \neq \pi(t_2)$. Let $(\bar{\epsilon}, \bar{\lambda})$ be an arbitrary point on the line segment between the two points (ϵ_1, λ_1) and (ϵ_2, λ_2) . Moreover, let $\bar{\pi} = (\bar{B}, \bar{N}, \bar{T})$ denote the associated optimal partition at $(\bar{\epsilon}, \bar{\lambda})$. We distinguish three cases for these optimal partitions.

Subcase 2.1: $\bar{\pi} \neq \pi$, $\bar{\pi} \neq \pi(t_1)$ and $\bar{\pi} \neq \pi(t_2)$. In this case, all three points (ϵ_1, λ_1) , (ϵ_2, λ_2) and $(\bar{\epsilon}, \bar{\lambda})$ are on the boundary of the invariancy region. Moreover, the points (ϵ_1, λ_1) and (ϵ_2, λ_2) are the transition points on the boundary of the invariancy region. Figure 2 illustrates this case. The statement follows directly from Corollary 2.3.

Subcase 2.2: $\bar{\pi} \neq \pi$ and either $\bar{\pi} = \pi(t_1)$ or $\bar{\pi} = \pi(t_2)$ holds. Without loss of generality, let $\bar{\pi} \neq \pi(t_1)$, but $\bar{\pi} = \pi(t_2)$. In this case, both points (ϵ_2, λ_2) and $(\bar{\epsilon}, \bar{\lambda})$ are on a single boundary line of the invariancy region and consequently, Theorem 3.2 can be used to identify two vertices of the invariancy region. We claim that in this case, (ϵ_1, λ_1) is one of the endpoints on this part of the invariancy region. Because, if (ϵ_1, λ_1) is not one of the endpoints of this line segment, then the endpoint should

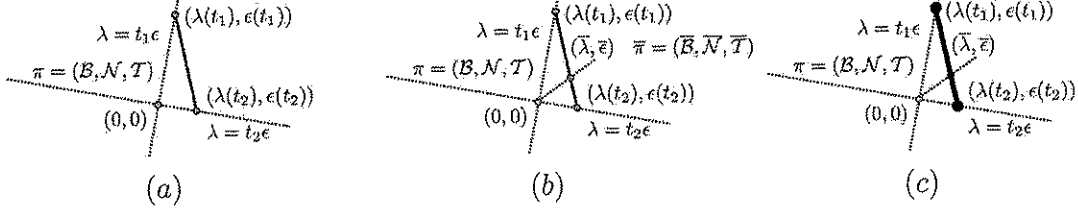


Figure 2: Case 2.1 Illustration

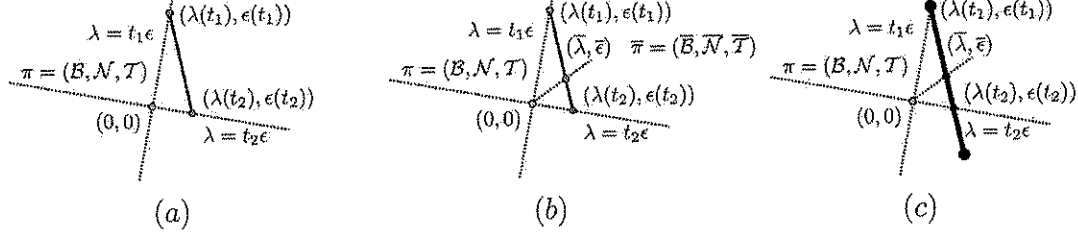


Figure 3: Case 2.2 Illustration

be somewhere between (ϵ_1, λ_1) and $(\bar{\epsilon}, \bar{\lambda})$. It means that (ϵ_1, λ_1) is not on the line segment between (ϵ_2, λ_2) and $(\bar{\epsilon}, \bar{\lambda})$. It is obvious that (ϵ_1, λ_1) and $(\bar{\epsilon}, \bar{\lambda})$ are not in the neighboring line segment, because two adjacent line segments could not have the same slope. Thus, these two points lay on different line segments. This situation contradicts the convexity of the invariency region. This subcase is illustrated on Figure 3.

Subcase 2.3: $\bar{\pi} = \pi$. In this case, it is immediately understood that the point $(\bar{\epsilon}, \bar{\lambda})$ belongs to the invariency region $\mathcal{IR}(\Delta b, \Delta c)$. In this case, we solve the problems (20) and (21) for $\bar{\Delta c} = t_3 \Delta c$, where $t_3 = \frac{\bar{\lambda}}{\bar{\epsilon}}$.

First we prove that problem (20) for $\bar{\Delta c} = t_3 \Delta c$ is not unbounded. To the contrary, let (20) is unbounded. One can consider that $t_1 = \kappa t_3$, where $\kappa > 0$. In this case, it leads to the conclusion that problem (24) is unbounded which contradicts the assumption. Thus, let ϵ_3 be the optimal objective function value of problem (20). Consequently λ_3 is known. Let $\pi(t_3) = (\mathcal{B}(t_3), \mathcal{N}(t_3), \mathcal{T}(t_3))$ be the optimal

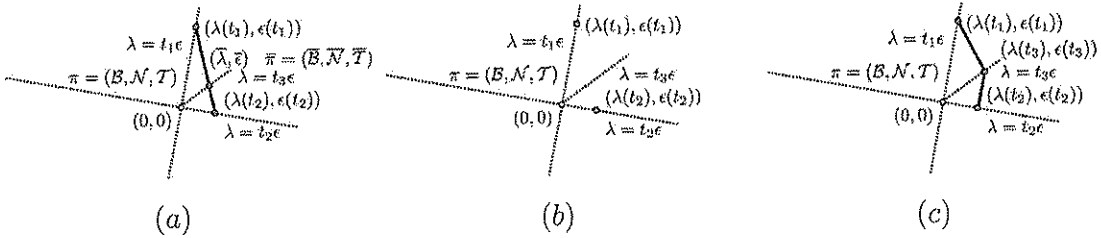


Figure 4: Case 2.3 Illustration

partition at (ϵ_3, λ_3) . Henceforth, the situation between optimal partitions $\pi(t_1)$ and $\pi(t_3)$ and optimal partitions $\pi(t_2)$ and $\pi(t_3)$ would fall in one of the two cases: Case 1 or 2. Now, we consider pairs of points $(\epsilon_1, \lambda_1) - (\epsilon_3, \lambda_3)$ and $(\epsilon_3, \lambda_3) - (\epsilon_2, \lambda_2)$ again to determine if Case 1 or 2 applies to each pair. We repeat this procedure until the invariancy region boundary between the points (ϵ_1, λ_1) and (ϵ_2, λ_2) is completely traced. Figure 4 illustrates the procedure. In this way, we can continue to identify all the transition points (and transition lines) of the invariancy region. Since the number of optimal partitions is finite, and all auxiliary LO problems can be solved in polynomial time, thus identifying the borders of the actual invariancy region is done in polynomial time in the number of optimal partitions.

Now, we can summarize the procedure of identifying all transition points (vertices) and transition lines (edges) in an invariancy region. Lets assume that we know an initial inner point of the invariancy region and one of the edges (Figure 5(a) and (b) shows how to find an inner point of the region). We are going to "shoot" by solving problem (24) or (25) counter-clockwise from the initial point to identify each edge (see Figure 5(c-f)). As we already know one of the edges, we exclude all the angles α_{exp} between the initial point and the two vertices v_1 and v_2 of the known edge from the candidate angles to shoot. So, we shoot in the angle $v_0 - v_2$ plus in the small angles β and 2β and identify the optimal partition in the two points we get. Here we use Case 1 or Case 2 described above to find the invariancy region boundary between the vertex v_2 and the point we get when shooting in the angle 2β . If the optimal partition is the same for the points in the directions β and 2β , we compute the vertices of this new edge e_2 and verify if one of those correspond to a vertex of the previously known edge e_1 . If it is not the case, then bisection is used to identify the missing edges between e_1 and e_2 . We continue in this manner until all edges of the invariancy region are identified.

4 Transition from an Invariancy Region to the Adjacent Invariancy Regions

The first step of the algorithm is to determine the bounding box for the values of ϵ . Due to the fact that ϵ is the parameter appearing in the constraints, the problem $QP(\Delta b, \Delta c, \epsilon, \lambda)$ may become infeasible for large or small ϵ values. Determining the bounding box is done as in many computational geometry algorithms [5, 17]. To find the range of ϵ where the parametric problem $QP(\Delta b, \Delta c, \epsilon, \lambda)$ is feasible, we solve the following problem starting from the initial point (λ_0, ϵ_0) :

$$\min \left\{ (c + \Delta c \lambda_0)^T x + \frac{1}{2} x^T Q x \mid Ax = b + \epsilon \Delta b, x \geq 0 \right\}. \quad (26)$$

Solving problem (26) gives the values of ϵ_{\min} and ϵ_{\max} that (see Figure 6(a)) are the lower and the upper feasibility bounds for the bi-parametric problem $QP(\Delta b, \Delta c, \epsilon, \lambda)$. Observe that we may have either $\epsilon_{\min} = -\infty$ or $\epsilon_{\max} = +\infty$.

After identifying the feasibility bounds in the " $\epsilon - \lambda$ " plane, we choose $\epsilon_{\min} \neq \infty$ or $\epsilon_{\max} \neq -\infty$. Let $\epsilon = \epsilon_{\min}$ and the optimal partition at the point $(\lambda_0, \epsilon_{\min})$ is $\pi_{\min} =$

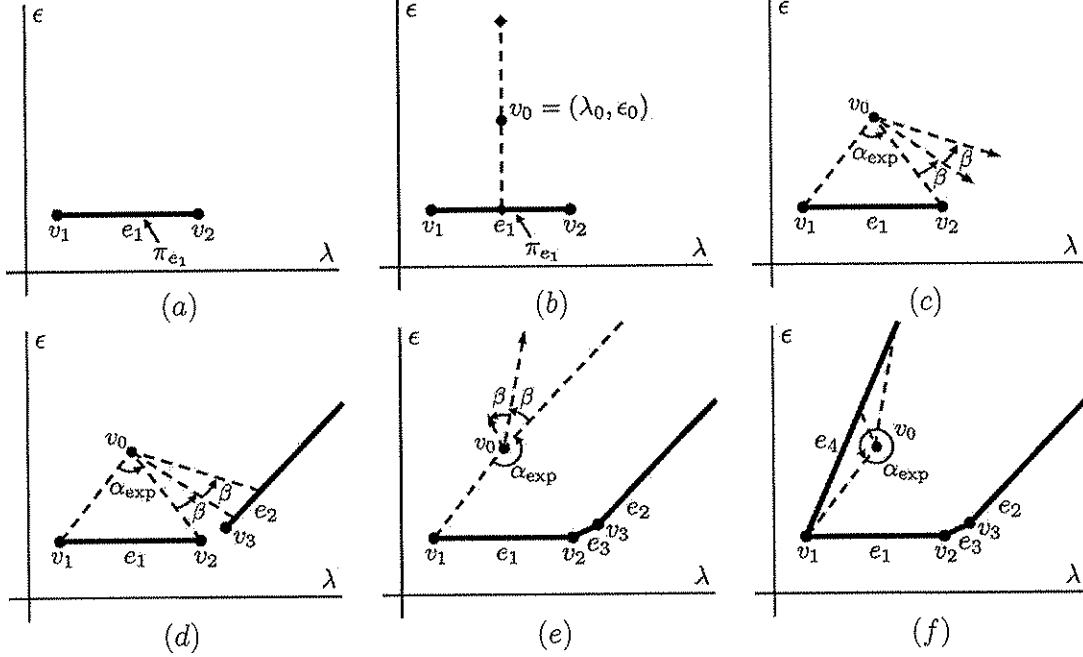


Figure 5: Invariancy Region Exploration Algorithm

$(\mathcal{B}_{\min}, \mathcal{N}_{\min}, \mathcal{T}_{\min})$. Then we can solve problems (16) and (17) with the optimal partition $\pi = \pi_{\min}$ and $\lambda \Delta c$ replaced by $\epsilon_{\min} \Delta c$ to identify the edge on the line $\epsilon = \epsilon_{\min}$ (see Figure 6(b)). If the point $(\lambda_0, \epsilon_{\min})$ is a singleton, we find the invariancy interval to the right from it. Now, we have an edge of one of the invariancy regions and we can get an initial inner point of that invariancy region selecting a point on the edge and utilizing Algorithm 1 from [8]. Using that initial inner point, we can identify the first non-trivial invariancy region including all of its edges and vertices as described in Section 3 (see Figure 6(c)).

To enumerate all invariancy regions in the bounding box, we use concepts and tools [5, 17] from computational geometry. The algorithm that we are going to present possess some similarities with polygon subdivision of the space and planar graphs. Our algorithm is essentially the subdivision of the bounding box into convex polyhedrons that can be unbounded. First, we introduce the notation and data structures used in computational geometry to describe these type of problems. Second, we show how to modify those data structures to create the complete algorithm for invariancy region enumeration.

The geometric objects involved in the given problem are vertices, edges and cells (faces), see Figure 7. Cells correspond to the non-trivial invariancy regions. Edges and vertices are trivial invariancy regions, each edge connects two vertices. It is important to notice that cells can be unbounded if the corresponding invariancy region is unbounded. That is why we need to extend the representation of the vertex to allow incorporating the information that the vertex can represent the virtual endpoint of the unbounded edge if the corresponding cell is unbounded. For instance, edge e_1 on Figure 7 is unbounded, so in addition to its first endpoint v_1 , we add another virtual endpoint being any point

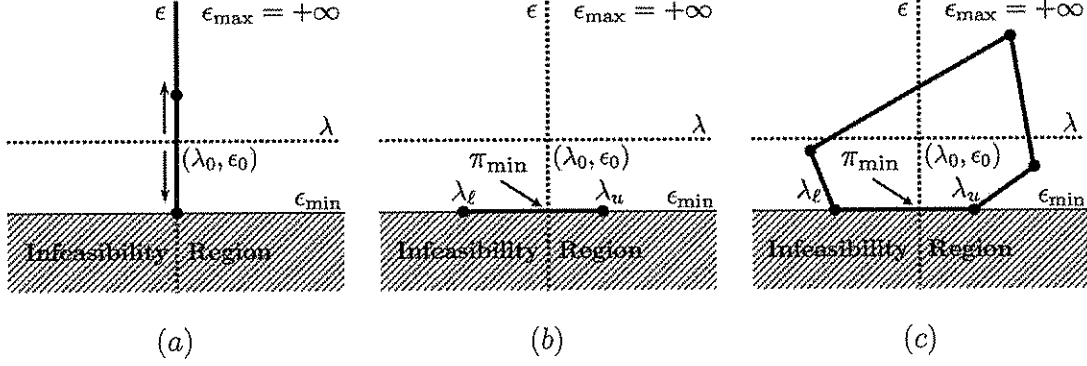


Figure 6: Algorithm Initialization

on the edge except v_1 . Consequently, each vertex need to be represented not only by its coordinates (x, y) , but also by the third coordinate z that indicates if it is a virtual vertex and the corresponding edge is unbounded. Another note to make is that the optimal partition may not be unique for each vertex or edge. First, at every virtual vertex, the optimal partition is the same as on the corresponding edge. Second, we may have situations when the optimal partition is the same on the incident edges and vertices if those are on the same line (edges e_2 and e_7 and vertex v_3 has the same optimal partition on Figure 7).

The data structures that we use for storing the information about vertices, faces and cells are similar to the ones used in many computational geometry algorithms [5]. Traversing the cell is usually done counter-clockwise and we are going to follow that convention as well. The extension of the standard storage and representation model is that we allow convex polyhedrons (cells) to be unbounded and that we do not require the vertexes to be in general positions (three vertexes can be on one line). Moreover, we add some extra fields to the records representing each geometric object. The structures of the records corresponding to vertices, edges, cells and optimal partitions are:

```

vertex {
    vertex id
    coordinates (x,y,z)
    optimal partition code
}
edge {
    edge id
    vertex_1 id
    vertex_2 id
    incident cell_1
    incident cell_2
    optimal partition code
}
cell {
    cell id

```

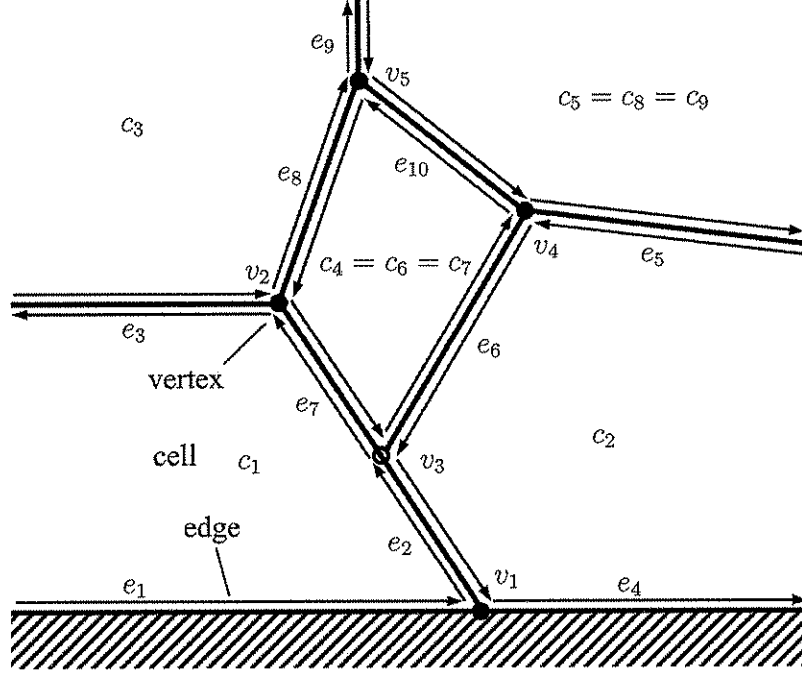


Figure 7: Computational Geometry Problem Representation

```

    list of edges
    optimal partition code
}
optimal partition {
    optimal partition code
    list of objects with this optimal partition
}

```

Optimal partitions are numerically encoded as an integer number for minimizing the storage as follows: partition $\mathcal{BB} \dots \mathcal{BB} = 0$, $\mathcal{BB} \dots \mathcal{BN} = 1$, $\mathcal{BB} \dots \mathcal{BT} = 2$, $\mathcal{BB} \dots \mathcal{NB} = 3$, etc. To do the conversion we apply the summation formula $\sum_{i=0}^{n-1} d_i 3^i$, where $d_i = 0$ if $i \in \mathcal{B}$, $d_i = 1$ if $i \in \mathcal{N}$ and $d_i = 2$ if $i \in \mathcal{T}$. This encoding (optimal partition code) allows not only saving storage, but also to establish the lexicographic order of the identified optimal partitions and to use binary search tree for verifying if the identified optimal partition was already encountered. We are using binary search trees for searching quickly among identified vertices, edges, cells and optimal partitions.

To enumerate all invariancy regions we use two queues that store indices of the cells that are already investigated and to be processed. At the start of the algorithm, the first cell enters the to-be-processed queue and the queue of completed cells is empty (c_1 is entering the to-be-processed queue on Figure 7). After that, we identify the cell c_1 including all faces and vertices starting from the known edge e_1 and moving counter-clockwise (please note that the virtual vertices corresponding to the unbounded edges are not shown on

Figure 7). Due to the fact that the optimal partition at the edge between the vertices v_1 and v_2 is the same, we are able to identify only the edge e_2 at the moment. Now, when we have identified all the edges incident to the cell c_1 we can add the potential cells corresponding to each of the edges to the to-be-processed queue, so $e_1 \rightarrow \emptyset$ (infeasible), $e_2 \rightarrow c_2$, $e_3 \rightarrow c_3$. So, we add c_2 and c_3 to the to-be-processed queue and move c_1 to the completed queue. Next, we start processing the cell c_2 as the first element of the to-be-processed queue. At this stage, we identify that the edge e_2 is shorter than the original one and we split it into two edges – e_2 and e_7 (note that the edges e_2 , e_7 and vertex v_3 have the same optimal partition). As the result of splitting edge e_2 into two edges, we need to add the cell c_4 that corresponds to the edge e_7 to the to-be-processed queue. We get $e_7 \rightarrow c_4$, $e_2 \rightarrow c_1$ (already processed), $e_4 \rightarrow \emptyset$ (infeasible), $e_5 \rightarrow c_5$, $e_6 \rightarrow c_6$ and add c_4 , c_5 , c_6 into the to-be-processed queue. Now, c_2 is moved to the completed queue. Next in the to-be processed queue is c_3 that gives us $e_3 \rightarrow c_1$ (already processed), $e_8 \rightarrow c_7$ and $e_9 \rightarrow c_8$. So, c_7 and c_8 are added to the to-be-processed queue and c_3 is moved to the completed queue. Next, we process c_4 and identify that $c_4 = c_6 = c_7$ based on checking the identified optimal partitions list and identified edges list. Here, $e_{10} \rightarrow c_9$. So, c_4 is moved to the completed queue and c_6 , c_7 are removed from the to-be-processed queue. Next in the to-be-processed queue is c_5 and we identify that $c_5 = c_8 = c_9$ and there are no more new edges. As the result, we move c_5 to the completed queue and remove c_8 and c_9 from the to-be-processed queue. The to-be-processed queue is empty now, so we have identified all the invariancy regions.

Data: The CQO optimization problem and Δb , Δc

Result: Optimal partitions on all invariancy intervals, optimal value function

Initialization: compute bounding box in the “ $\epsilon - \lambda$ ” plane and compute inner point in one of the invariancy regions;

```

while not all invariancy regions are enumerated do
    run sub-algorithm to compute all edges and vertices of the current invariancy
    region;
    add all unexplored regions corresponding to each edge to the to-be-processed
    queue and move the current region to the queue of completed region indices;
    if to-be-processed queue of the unexplored regions is not empty then
        pull out the first region from the to-be-processed queue;
        compute an inner point of the new region;
    else
        return the data structure with all the invariancy regions, corresponding
        optimal partitions and optimal value function;
    end
end

```

Algorithm 1: Algorithm for Enumerating All Invariancy Regions

The proposed Algorithm 1 on page 15 runs in linear time in the output size (the constant $C \cdot n$ is 3). But, by the nature of the parametric problem, the number of vertices, edges and faces can be exponential in the input size. In our experiences worst case does not happen in practise very often though.

5 Illustrative Example

Here we present some illustrative numerical results for a simple example by using the algorithm outlined in Section 4. Computations can be performed by using any IPM solver for LO and CQO problems due to the fact that IPMs find a maximally complementary solution in the limit. We have used the McIPM [20] and MOSEK [1] solvers for our computations. Let us consider the following CQO problem with $x, c \in \mathbb{R}^5$, $b \in \mathbb{R}^3$, $Q \in \mathbb{R}^{5 \times 5}$ being a positive semidefinite symmetric matrix, $A \in \mathbb{R}^{3 \times 5}$ with $\text{rank}(A) = 3$. The problem data is as follows

$$Q = \begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, c = \begin{bmatrix} -16 \\ -20 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Delta c = \begin{bmatrix} 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}, A = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 11 \\ 8 \\ 20 \end{bmatrix}, \Delta b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

With this data the perturbed CQO instance is

$$\begin{aligned} \min \quad & (-16 + 7\lambda)x_1 + (-20 + 6\lambda)x_2 + 2x_1^2 + 2x_1x_2 + \frac{5}{2}x_2^2 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + x_3 = 11 + \epsilon \\ & 2x_1 + x_2 + x_4 = 8 + \epsilon \\ & 2x_1 + 5x_2 + x_5 = 20 + \epsilon \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned} \tag{27}$$

The result of our computations is presented in Figure 8 and Figure 9. Figure 8 shows the invariancy regions, the corresponding optimal partitions and the equations for the optimal value function. The optimal partitions for the invariancy intervals are shown in ovals, where each letter corresponds to the corresponding index being in one of the tri-partition sets \mathcal{B} , \mathcal{N} or \mathcal{T} . The partitions for the transition points are shown next to them. The solid dots correspond to the cases where the optimal partition in those transition points are different from the partitions on the neighboring invariancy intervals and invariancy regions. The circle at the point $\lambda = 10/3$, $\epsilon = -8$ corresponds to the case when the optimal partition for the whole line $\epsilon = -8$ is the same, but it differs for the other segments ending at the point. The graph of $\phi(\epsilon, \lambda)$ and the corresponding invariancy regions are presented on Figure 9. The piecewise quadratic optimal value function is drawn in different shades that correspond to the invariancy regions.

6 Conclusions and Future Work

We have extended the uni-parametric simultaneous perturbation results for CQO to the bi-parametric case. The algorithm outlined in the paper allows identifying all invariancy

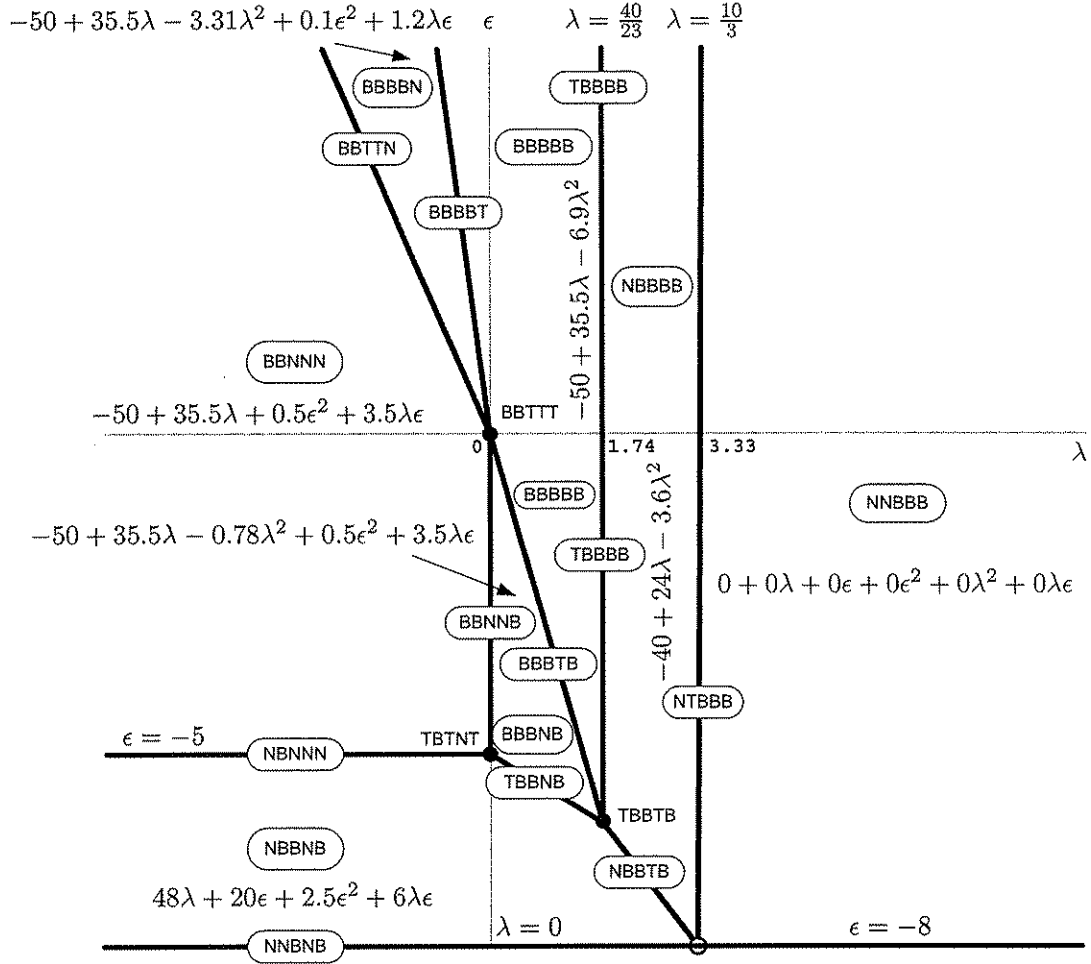


Figure 8: The Optimal Partitions and the Invariancy Regions

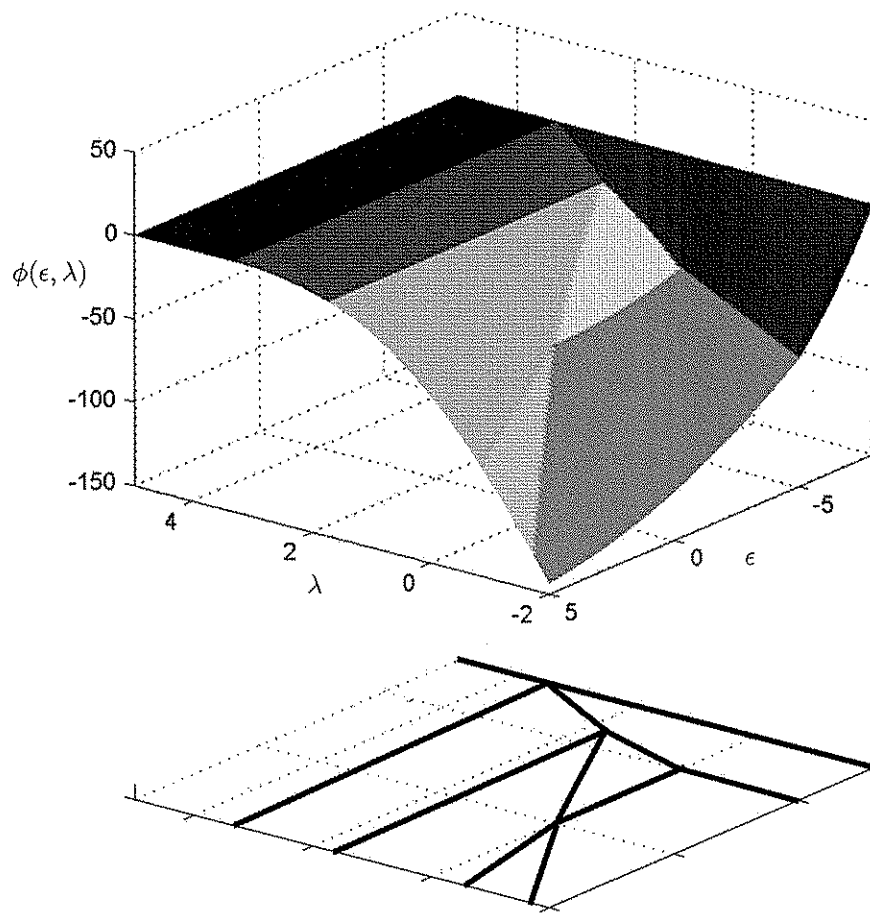


Figure 9: The Optimal Value Function (the image is rotated for better visibility)

regions where the optimal partition is invariant by solving a series of uni-parametric CQO problems. We can also compute the optimal value function and maximally complementary solutions on each invariancy region.

Even though all presented auxiliary optimization problems can be solved in polynomial time by IPMs and the number of different optimal partitions is finite, enumeration of all invariancy regions may not be achieved in polynomial time due to the fact that the number of different optimal partitions may increase exponentially with the cardinality of the index set. That is why the algorithm presented is linear in the output size, but not in the input size.

We would like to mention some differences of our algorithmic approach to parametric CQO optimization and the algorithm described in [18] which is implemented in [13]. First, in our study we consider simultaneous perturbation in the right-hand-side of the constraints and the linear term of the objective function with different parameters, while in [18] and related publications only perturbation in either the right-hand-side or the linear term of the objective is considered. Second, in [18] the authors define a critical region as the region of parameters where active constraints remain active. As the result, an important precondition for analysis in [18] is the requirement for either making non-degeneracy assumption or exploiting special tools for handling degeneracy, while, our algorithm does not require any non-degeneracy assumptions. Finally, the algorithm for parametric quadratic optimization described in [18] uses different parameter space exploration strategy than ours. Their recursive algorithm identifies a first critical (invariancy) region, and after that reverses the defining hyperplanes one by one in a systematic process to get a subdivision of the complement set. The regions in the subdivision are explored recursively. As the result, each critical (invariancy) region can be split among many regions and, consequently, all the parts has to be detected. Thus, each of the potentially exponential number of invariancy regions may be split among exponential number of regions, which makes their algorithm computationally expensive.

Future work inspired by this paper includes extending the methodology of solving bi-parametric programming problems to other classes of optimization problems and improving the implementation. To the best of our knowledge, this paper is the first attempt to study systematically bi-parametric convex quadratic optimization problems with different parameters in the coefficients of the objective function and right-hand-side of the constraints. Parametric optimization can be used not only for performing sensitivity analysis, but also for solving multi-objective and stochastic optimization problems [10]. Thus, extending our parametric analysis to other classes of Conic Linear Optimization problems is high in our priorities.

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