A Continuous-Review Inventory Model with Disruptions at Both Supplier and Retailer

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Abstract

We consider a continuous-review inventory problem for a retailer who faces random disruptions both internally and externally (from its supplier). We formulate the expected inventory cost at this retailer and analyze the properties of the cost function. In particular, we show that the cost function is quasi-convex and therefore can be efficiently optimized to numerically find the optimal order size from the retailer to the supplier. Computational experiments provide additional insight into the problem. In addition, we introduce an effective approximation of the cost function. Our approximation can be solved in closed form, which is useful when the model is embedded into more complicated supply chain design or management models.

Key Words: Inventory Model; Supply Disruptions; Safety Stock

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1 Introduction

In this paper, we study a continuous-review inventory problem with a single supplier and a single retailer. As in the classical EOQ model, we assume that the demand is constant and deterministic and that the retailer uses a zero-inventory-ordering (ZIO) policy to manage its inventory. Unlike the EOQ model, however, our model considers random disruptions at both the supplier and the retailer. The main difference between these two types of disruptions is that a disruption to the retailer destroys all on-hand inventory at the retailer, while disruptions at the supplier simply prevent the retailer from receiving replenishment orders.

Our model is related to the EOQ models with disruptions formulated by Parlar and Berkin (1991) and Berk and Arreola-Risa (1994) but differs in two important ways. The first is that the previous papers consider only external disruptions, not disruptions to the retailer itself. The second difference between our model and most existing disruption models is our assumption that a disruption to the retailer destroys the inventory at the retailer. This assumption is motivated by recent disasters such as Hurricane Katrina in 2005, which destroyed large inventories of lumber, coffee, and other products stored along the Gulf Coast (Reuters 2005).

We formulate the expected working inventory cost (which includes the ordering, holding and shortage costs) as a function of the retailer's order quantity for the problem described above. We prove that the cost function is quasi-convex and can therefore be efficiently optimized using standard numerical algorithms. In addition, we numerically

1. evaluate the shape of the cost function at the retailer as customer demand varies;
2. compare the optimal order quantity with the classical EOQ solution to demonstrate the importance of considering supplier and retailer disruptions when making inventory decisions;
3. study the effects of disruptions at the retailer and the supplier on the working inventory cost, as well as on the demand fill rate, at the retailer;
4. demonstrate that disruptions at the retailer can have a more significant impact on the retailer's cost and fill rate than disruptions at the supplier, even though most research on inventory models (even research that considers disruptions) ignores disruptions at the retailer.

Moreover, since the cost function provided in this paper is complex and cannot be minimized in closed form, we provide tight, more tractable approximations for the optimal order quantity and cost, as well as corresponding error bounds. We show that the approximate optimal cost is a concave and increasing function of the customer demand; this property allows it to be embedded into other more complicated supply chain design models in place of the original cost function for added tractability (see Section 8 for further discussion). Based on the approximate optimal order quantity, we also discuss the safety stock level at the retailer to protect against supplier disruptions.

The rest of this paper is organized as follows. In Section 2, we review the related literature. We formulate the expected working inventory cost at the retailer in Section 3 and analyze the properties of the cost function in Section 4. In Section 5, we conduct numerical experiments to obtain insight into the problem. We propose a tight approximation for the optimal cost at the retailer and analyze its properties in Section 6. In Section 7, we interpret the order quantity in terms of safety stock held at the retailer to protect against supplier disruptions. The paper is concluded in Section 8. All proofs for the results presented in this paper can be found in Appendix A.

2 Literature Review

The study by Parlar and Berkin (1991) is among the earliest works that incorporate supply disruptions into classical inventory models. They analyze the supply uncertainty problem for an EOQ model in which the supply is available only during an interval of random length, and then unavailable for another interval of random length. Using the renewal reward theorem, they formulate the expected cost as a function of the order quantity. Their work is based on two main assumptions. First, at any time, the decision maker knows the availability status of the supplier; second, the retailer follows a ZIO policy. Their cost function is shown to be incorrect in two respects by Berk and Arreola-Risa (1994), who propose a corrected cost function. This function cannot be minimized in closed form, and has not been proven to be convex (though it is unimodal). Our model differs from these models by considering both internal and external disruptions (i.e., disruptions to the retailer and the supplier) and by considering the case in which internal disruptions cause on-hand inventory to be destroyed.

Snyder (2008) develops a simple but tight approximation for the model introduced by Berk and Arreola-Risa (1994). He shows that his approximate cost function not only is convex but also yields a closed-form solution and behaves similarly to the classical EOQ cost function in several important ways. Heimann and Waage (2005) relax the ZIO assumption made by Berk and Arreola-Risa (1994), and then derive a closed-form solution using an approximation similar to Snyder's. The models of both Snyder (2008) and Heimann and Waage (2005) are used by Ross, Rong, and Snyder (2008) for a model with time-varying demands and disruption rates. Our approximation (Section 6) is similar in spirit but uses a higher-order polynomial approximation than the earlier approximations do.

Parlar and Perry (1995) relax two assumptions made by Parlar and Berkin (1991). First, they assume that the decision maker may not be aware of the ON-OFF status of the supply and may ascertain the state only at a cost. Second, they relax the ZIO assumption and treat the reorder point as a decision variable. In addition, they consider both deterministic and random yields at the supplier.

In addition to randomness in supply, Gupta (1996) assumes that customer demands are generated according to a Poisson process and that shortages result in lost sales. He considers a non-zero (but deterministic) lead time. A more general model is studied by Parlar (1997), who allows the lead time to be stochastic.

The works cited above assume there is only one supplier. Tomlin (2006) presents a dual-sourcing model in which orders may be placed with either a cheap but unreliable supplier or an expensive but reliable supplier. He considers a very general supplier-recovery process. Sheffi (2001) also discusses a dual-sourcing problem in the context of a a small illustrative example; no analytical formulation is provided. Chopra, Reinhardt, and Mohan (2007) and Schmitt and Snyder (2007) consider two-supplier models under both supply disruptions and yield uncertainty in the context of single-period and infinite-horizon models, respectively.

More generally, Dada, Petruzzi and Schwarz (2007) consider a newsvendor problem in which the retailer is served by multiple suppliers, all of which may be unreliable. They develop a modeling framework in which the newsvendor can diversify the risk of insufficient supply by spreading orders among the available suppliers, who differ in terms of cost and supply distribution functions. They establish properties of the optimal solution and obtain corresponding insights into the tradeoff between cost and reliability.

Most of the papers cited above propose numerical algorithms to find optimal solutions, rather than deriving closed-form solutions. As we have addressed, closed-form solutions can provide insights that numerical solutions cannot and are useful in the case when the model is to be embedded into more complicated models. Also, all of the above works ignore disruptions and recoveries at the retailer. This paper contributes to the literature in these two respects.

3 Model Formulation

Figure 1 illustrates a typical inventory pattern at the retailer when disruptions at both the retailer and the supplier are considered. As in Snyder (2008) and Berk and Arreola-Risa (1994), we assume that the retailer faces constant and positive customer demand with rate D (units/year)[1], and that the supplier lead-time is zero.

Both the retailer and the supplier may experience available ("ON") and disrupted ("OFF") states. When the retailer's inventory level hits 0, it attempts to place an order with the supplier, e.g., at time A in Figure 1. If the supplier is OFF (e.g., at time B), the retailer must wait until the supplier has recovered (time C) to place an order. If the retailer is disrupted (time D), all inventory is lost, and it cannot place a new order until the disruption has ended (time E). It is also possible that a retailer experiences a disruption (time F) and the supplier experiences a disruption (time G) before the retailer disruption ends (time H), in which case the retailer must wait until the supplier disruption ends (time I) before placing an order.

We define an *inventory cycle* at the retailer to be the duration between two consecutive successful shipments from the supplier, and use T to denote the inventory cycle length (a random variable). We use w to denote the expected waiting time until the supplier is available, assuming that the supplier is not available when the retailer is ON and wishes to place an order.

Note that the retailer will not have inventory on hand to serve its customers either when it is disrupted or when it is waiting for a supplier disruption to end. In this event, we consider both the lost-sales case and the time-independent backorder case (i.e., unmet demands are backordered, and the shortage cost is charged per unit, not per unit time). Figure 1 is drawn using the former assumption, in which case it is optimal to place an order of the same size, Q, regardless of the system state just before the order is placed. We explain later in this section that the models for these two cases are identical, except that in the lost-sales case, Q represents the order quantity, while in the time-independent backorder case it represents an order-up-to level, i.e., the retailer orders Q plus any backlogged demand. In both cases, the shortage cost

[1]We conduct simulation studies in Appendix B to show that our model is robust to violations of this assumption when customer demands instead follow a Poisson process with rate D.

is charged per unit of unmet demand.

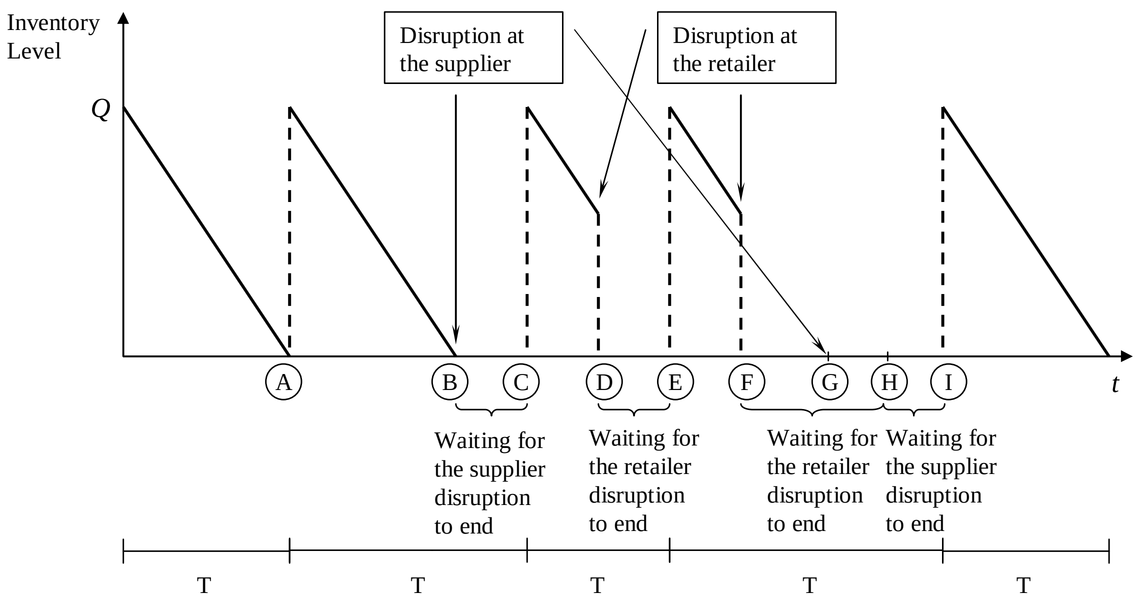

Figure 1: Inventory policy at the retailer with random cycle length T.

Following most of the research on inventory management under random supply disruptions (e.g., Parlar and Perry 1995, Gupta 1996), we assume that the durations of the ON and OFF cycles at the supplier are independently and identically distributed (i.i.d.) exponential random variables with rates λ and ψ, respectively. Therefore, the expected durations of the ON and OFF cycles at the supplier are $\frac{1}{\lambda}$ and $\frac{1}{\psi}$, respectively. We refer to λ and ψ as the disruption and recovery rates at the supplier, respectively. Similarly, we assume that the durations of the ON and OFF cycles at the retailer follow i.i.d. exponential distributions with disruption rate α and recovery rate β. All of the random durations are assumed to be mutually independent.

We believe that the exponential distribution is reasonable in this context since exponential distributions are often used to model the time between independent events that happen at a constant average rate, and otherwise are "often good approximations to the actual distributions" (Ross 2006, Chapter 5.1). In addition, the exponential distribution possesses various mathematical properties, such as the memoryless property, that facilitate our analysis. Similar models could be constructed using alternate distributions; we expect that such models would be significantly more difficult to analyze but would yield similar insights.

Below, we formulate $E[T]$, the expected inventory cycle length, and the expected cost per cycle at the retailer, and then apply the renewal reward theorem to derive the average cost per year at the retailer.

3.1 Expected Inventory Cycle Length $E[T]$

To formulate $E[T]$, we first formulate two key quantities: (1) the probability that the supplier is in the OFF state t time units after the start of an inventory cycle (we denote this probability $\phi(t)$), and (2) w, the expected waiting time if the supplier is OFF when an order attempt is made.

3.1.1 Formulation of $\phi(t)$

We construct a two-state continuous-time Markov chain for the ON and OFF states of the supplier. Under the assumptions made above, the Markov chain remains in the ON state for an exponentially distributed length of time with rate λ before going to the OFF state, where it spends an exponentially distributed length of time with rate ψ before returning to the ON state. Since the supplier is, by definition, ON at the beginning of each inventory cycle, the probability that the supplier is OFF $t(\geq 0)$ time units after the start of an inventory cycle is given by

$$\phi(t) = \frac{\lambda}{\lambda + \psi} \left(1 - e^{-(\lambda+\psi)t}\right)$$

(Ross 1996, p. 243).

3.1.2 Formulation of w

We construct a four-state continuous-time Markov chain to model the ON and OFF states at the retailer and supplier. The state space is $S = \{11, 10, 01, 00\}$, where the two digits represent the state of the supplier and retailer, respectively (see Figure 2). The time the Markov chain spends in each state before transitioning into a different state is exponentially distributed with the transition rates indicated in the figure.

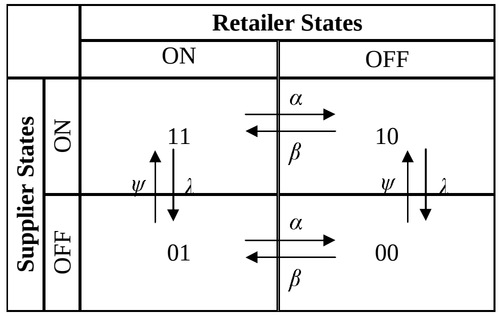

Figure 2: A four-state continuous-time Markov chain for the ON and OFF states at the retailer and the supplier.

We use $E[L_{i,j}]$ to denote the expected time until the system reaches state $j \in S$, given that the system is currently in state $i \in S$. Since w represents the expected time until both locations are ON, given that the supplier is OFF when an order attempt is made, $w \equiv E[L_{01,11}]$. It follows from Ross (1996, Section 5.5) that $E[L_{i,j}]$ equals the expected time the chain takes to leave state i plus any additional time after it leaves i before it enters j:

$$E[L_{i,j}] = \frac{1}{\upsilon_i} + \sum_{k \in S, k \neq i} P_{i,k} E[L_{k,j}],$$

where $P_{i,j}$ is the transition probability from state i to j, and υ_j is the rate at which the chain leaves state i.

From Ross (1996, Section 5.2), we have $\upsilon_{01} = \alpha + \psi$, $P_{01,11} = \frac{\psi}{\alpha+\psi}$, and $P_{01,00} = \frac{\alpha}{\alpha+\psi}$. Thus,

$$\begin{aligned} E[L_{01,11}] &= \frac{1}{\alpha+\psi} + P_{01,11}E[L_{11,11}] + P_{01,00}E[L_{00,11}] \\ &= \frac{1}{\alpha+\psi} + \frac{\psi}{\alpha+\psi} \cdot 0 + \frac{\alpha}{\alpha+\psi} \cdot E[L_{00,11}]. \end{aligned} \tag{1}$$

Similarly,

$$E[L_{00,11}] = \frac{1}{\beta+\psi} + \frac{\beta}{\beta+\psi} \cdot E[L_{01,11}] + \frac{\psi}{\beta+\psi} \cdot E[L_{10,11}], \tag{2}$$

$$E[L_{10,11}] = \frac{1}{\beta+\lambda} + \frac{\beta}{\beta+\lambda} \cdot 0 + \frac{\lambda}{\beta+\lambda} \cdot E[L_{00,11}], \tag{3}$$

Combining equations (1)-(3), we derive

$$w = E[L_{01,11}] = \frac{1}{\psi}\left[1 + \frac{\alpha(\lambda+\psi)}{\beta(\alpha+\beta+\lambda+\psi)}\right].$$

3.1.3 Formulation of $E[T]$

We now formulate the expected inventory cycle length at the retailer, $E[T]$. We consider two cases.

- *Case 1.* If the retailer is not disrupted during the $\frac{Q}{D}$ time units after an inventory cycle begins, i.e., before its inventory level reaches zero (refer to the first and second cycles in Figure 1), the probability that the supplier is OFF at the moment the retailer's inventory level reaches 0 is $\phi(\frac{Q}{D})$, hence the expected inventory cycle length at the retailer is

$$\frac{Q}{D} + w\phi\left(\frac{Q}{D}\right) = \frac{Q}{D} + \frac{\lambda w}{\lambda+\psi}\left(1 - e^{-(\lambda+\psi)\frac{Q}{D}}\right).$$

- *Case 2.* If the retailer is disrupted during the $\frac{Q}{D}$ time units after a cycle begins (refer to the third and fourth cycles in Figure 1), then let Y denote the time from the beginning of the cycle to this disruption, and let Z denote the time to recover from the disruption. Then $Y \sim \exp(\frac{1}{\alpha})$ on $[0, \frac{Q}{D}]$ and $Z \sim \exp(\frac{1}{\beta})$. Therefore, the probability that the supplier is OFF when the retailer recovers from the disruption is $\phi(Y+Z)$, and hence the expected inventory cycle length at the retailer is

$$\begin{aligned} & Y + E_Z\left[Z + w\phi(Y+Z)\right] \\ = & Y + \int_0^\infty \left(Z + \frac{\lambda w}{\lambda+\psi}(1 - e^{-(\lambda+\psi)(Y+Z)})\right)\beta e^{-\beta Z} dZ \\ = & Y + \frac{1}{\beta} + \frac{\lambda w}{\lambda+\psi} - \frac{\beta\lambda w e^{-(\lambda+\psi)Y}}{(\lambda+\psi)(\beta+\lambda+\psi)}. \end{aligned}$$

  where $Y \sim \exp(\frac{1}{\alpha})$ on $[0, \frac{Q}{D}]$.

The probability that case 1 applies (i.e., that the retailer is not disrupted during the $\frac{Q}{D}$ time units after an inventory cycle begins) is $1 - \int_0^{\frac{Q}{D}} Y e^{-\alpha Y} dY = e^{-\alpha\frac{Q}{D}}$. Therefore, combining the two cases,

$$
\begin{aligned}
E[T] &= e^{-\alpha\frac{Q}{D}}\left[\frac{Q}{D}+\frac{\lambda w}{\lambda+\psi}(1-e^{-(\lambda+\psi)\frac{Q}{D}})\right] \\
&\quad +\int_0^{\frac{Q}{D}}\left[Y+\frac{1}{\beta}+\frac{\lambda w}{\lambda+\psi}-\frac{\beta\lambda w e^{-(\lambda+\psi)Y}}{(\lambda+\psi)(\beta+\lambda+\psi)}\right]\alpha e^{-\alpha Y}dY \\
&= e^{-\alpha\frac{Q}{D}}\left[\frac{Q}{D}+\frac{\lambda w}{\lambda+\psi}\left(1-e^{-(\lambda+\psi)\frac{Q}{D}}\right)\right]+\frac{1}{\alpha}-\left(\frac{Q}{D}+\frac{1}{\alpha}\right)e^{-\frac{\alpha Q}{D}}+\left(\frac{1}{\beta}+\frac{\lambda w}{\lambda+\psi}\right)\left(1-e^{-\frac{\alpha Q}{D}}\right) \\
&\quad -\frac{\alpha\beta\lambda w\left(1-e^{-(\alpha+\lambda+\psi)\frac{Q}{D}}\right)}{(\lambda+\psi)(\beta+\lambda+\psi)(\alpha+\lambda+\psi)} \\
&= \frac{\lambda w}{\lambda+\psi}\left(1-\frac{\alpha\beta}{(\beta+\lambda+\psi)(\alpha+\lambda+\psi)}\right)\left(1-e^{-(\alpha+\lambda+\psi)\frac{Q}{D}}\right)+\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\left(1-e^{-\frac{\alpha Q}{D}}\right) \\
&= \bar{A}\left(1-e^{-(\alpha+\lambda+\psi)\frac{Q}{D}}\right)+\bar{B}\left(1-e^{-\frac{\alpha Q}{D}}\right),
\end{aligned}
\tag{4}
$$

where we define

$$
\bar{A} = \frac{\lambda w}{\lambda+\psi}\left(1-\frac{\alpha\beta}{(\beta+\lambda+\psi)(\alpha+\lambda+\psi)}\right) = \frac{\lambda(\alpha+\beta)}{\beta\psi(\alpha+\lambda+\psi)} \tag{5}
$$

$$
\bar{B} = \frac{1}{\alpha}+\frac{1}{\beta} \tag{6}
$$

to simplify the notation.

3.2 Expected Working Inventory Cost Per Inventory Cycle

We use C to denote the working inventory cost per inventory cycle at the retailer. (We do not consider costs incurred by the supplier.) Since the working inventory cost consists of ordering, holding and shortage costs, $E[C]$ equals the sum of the expected values of these three costs, which we formulate in the following sections.

3.2.1 Ordering Cost per Cycle

The ordering cost that the retailer incurs for placing orders to the supplier has fixed and variable components; that is,

$$
E[\text{ordering cost/cycle}] = F + aQ, \tag{7}
$$

where F is the fixed cost per order and a is the variable cost per unit, which includes the purchase price, as well as any other costs such as shipping and handling.

This expression assumes that exactly Q units are ordered per cycle. In the time-independent backorder case, however, more than Q units are ordered during cycles that follow disruptions, since additional units must be ordered to clear the backlog. In this case, we account for the additional per-unit cost a for each backorder in the stockout cost, as we explain in Section 3.2.3.

3.2.2 Holding Cost per Cycle

To calculate the holding cost, we consider two cases:

- If the retailer is not disrupted before its inventory level reaches zero in an inventory cycle, then the holding cost in this inventory cycle is the same as the holding cost in the classical EOQ problem, which equals $hQ^2/2D$, where h is the yearly holding cost at the retailer per unit of product.

- If the retailer is disrupted before its inventory level reaches zero, then the holding cost in this inventory cycle is the product of h and the area below the inventory curve, which equals $h(Q+Q-DY)Y/2$, where Y is the time from the beginning of this inventory cycle to the disruption, and $Y \sim \exp(\frac{1}{\alpha})$ on $[0, \frac{Q}{D}]$.

We thus have

$$\begin{aligned}
E[\text{holding cost/cycle}] &= P(\text{retailer not disrupted before inventory reaches zero}) \cdot h\frac{Q^2}{2D} \\
&\quad + \int_0^{\frac{Q}{D}} h\frac{(2Q-YD)Y}{2}\alpha e^{-\alpha Y}dY \\
&= e^{-\frac{\alpha Q}{D}} h\frac{Q^2}{2D} + hQ\left[\frac{1}{\alpha} - \left(\frac{Q}{D}+\frac{1}{\alpha}\right)e^{-\frac{\alpha Q}{D}}\right] \\
&\quad - \frac{hD}{2}\left[\frac{2}{\alpha}\left(\frac{1}{\alpha} - \left(\frac{Q}{D}+\frac{1}{\alpha}\right)e^{-\frac{\alpha Q}{D}}\right) - \frac{Q^2}{D^2}e^{-\frac{\alpha Q}{D}}\right] \\
&= \frac{hQ}{\alpha} - \frac{hD}{\alpha^2}\left(1-e^{-\frac{\alpha Q}{D}}\right). \qquad (8)
\end{aligned}$$

3.2.3 Shortage Cost per Cycle

Due to the random disruptions at the supplier and the retailer, inventory is not always available to meet demands at the retailer. We use π to denote the per-unit shortage cost. This cost represents either the stockout cost per lost sale (in the lost-sales case) or the backorder cost per unit (in the time-independent backorder case). In the latter case, π consists of the corresponding shortage penalty *and* the unit ordering cost, a, to make up for the fact that the ordering cost formulated in Section 3.2.1 assumes that exactly Q units are ordered per inventory cycle.

To calculate the expected shortage cost per inventory cycle at the retailer, we first formulate the expected time in each cycle during which the retailer has inventory on hand. As in Section 3.2.2, if the retailer is not disrupted before its inventory level reaches zero, then it has inventory on hand for a duration of Q/D time units; otherwise, it has inventory on hand for a duration of Y time units, where $Y \sim \exp(\frac{1}{\alpha})$, $0 \leq Y \leq \frac{Q}{D}$. Therefore, the expected time that the retailer has inventory within an inventory cycle is

$$\frac{Q}{D}e^{-\alpha\frac{Q}{D}} + \int_0^{\frac{Q}{D}} Y\alpha e^{-\alpha Y}\,dY$$

and the expected time that the retailer does not have inventory is

$$E[T] - \left(\frac{Q}{D}e^{-\alpha\frac{Q}{D}} + \int_0^{\frac{Q}{D}} Y\alpha e^{-\alpha Y}\,dY\right).$$

Thus,

$$\begin{aligned} E[\text{shortage cost/cycle}] &= \pi D\left[E[T] - \frac{Q}{D}e^{-\alpha\frac{Q}{D}} - \int_0^{\frac{Q}{D}} Y\alpha e^{-\alpha Y}\,dY\right] \\ &= \pi D\left[E[T] - \frac{1}{\alpha}\left(1 - e^{-\frac{\alpha Q}{D}}\right)\right]. \end{aligned} \tag{9}$$

Combining (7)–(9), we can derive the expected total cost per cycle:

$$E[C] = F + aQ + \frac{hQ}{\alpha} - \frac{hD}{\alpha^2}\left(1 - e^{-\frac{\alpha Q}{D}}\right) + \pi D\left[E[T] - \frac{1}{\alpha}\left(1 - e^{-\frac{\alpha Q}{D}}\right)\right]$$

3.3 Expected Working Inventory Cost per Year

Using the well known renewal reward theorem (Ross 1996, p. 133), the long-run average annual working inventory cost is:

$$\mathcal{I}(Q) \equiv \frac{E[C]}{E[T]} = \pi D + \frac{F + (a + \frac{h}{\alpha})Q - \left(1 - e^{-\alpha \frac{Q}{D}}\right)\left(\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}\right)}{E[T]}, \tag{10}$$

where $E[T]$ is given in (4). We express this cost as a function of Q.

We use Q^* to denote the Q that minimizes $\mathcal{I}(Q)$. In addition, we use $\mathcal{I}^*(D)$ to denote the optimal cost treated as a function of the demand. In Sections 4 and 5, we study properties of $\mathcal{I}(Q)$ and $\mathcal{I}^*(D)$, suggest efficient solution algorithms to compute Q^*, and conduct computational experiments to obtain managerial insights.

4 Properties of $\mathcal{I}(Q)$

4.1 Special Cases

We first study the function $\mathcal{I}(Q)$ for three special cases: 1) when the supplier is never disrupted, 2) when the retailer is never disrupted, and 3) when neither the retailer nor the supplier is ever disrupted. (All proofs are contained in Appendix A.)

Property 1 *When the supplier is never disrupted ($\lambda = 0$), the function $\mathcal{I}(Q)$ is convex with respect to Q; and $\mathcal{I}^*(D)$ is a concave function of D.*

Property 2 *When the retailer is never disrupted ($\alpha = 0$), the function $\mathcal{I}(Q)$ reduces to the cost function proposed by Berk and Arreola-Risa (1994).*

Property 3 *When neither the supplier nor the retailer is ever disrupted ($\alpha = 0$ and $\lambda = 0$), the cost function $\mathcal{I}(Q)$ reduces to the classical EOQ function.*

The proof of Property 3 follows the same logic as that of Property 2 and is omitted.

4.2 Shape of $\mathcal{I}(Q)$

We now introduce three results that describe the shape of $\mathcal{I}(Q)$ as a function of Q.

Lemma 1 *(a)* $\lim_{Q\to 0} \frac{\partial}{\partial Q}\mathcal{I}(Q) < 0$*;*

(b) $\lim_{Q\to\infty} \frac{\partial}{\partial Q}\mathcal{I}(Q) = (a + \frac{h}{\alpha})/(\bar{A} + \bar{B}) > 0$*, where* $\bar{A}$ *and* $\bar{B}$ *are as defined in (5)–(6).*

In other words, $\mathcal{I}(Q)$ is decreasing for small values of Q and increasing for large ones. Corollary 1 naturally follows from Lemma 1; its proof is omitted.

Corollary 1 Q^* *is always finite and positive.*

Property 4 $\mathcal{I}(Q)$ *is quasi-convex in* $Q \geq 0$*.*

It follows from Property 4 that any standard method for solving single-dimensional unconstrained quasi-convex optimization problems, such as the bisection or golden section search, can be applied to compute Q^*.

Property 5 proposes a lower bound for $\mathcal{I}^*(D)$, which we denote by $LB(D)$.

Property 5 *Let*

$$LB(D) = \begin{cases} \pi D + \frac{F+(a-\pi)\frac{D}{\alpha}}{\bar{A}+\bar{B}}, & \text{if } D \geq \alpha F/(\pi - a) \\ \pi D + \frac{F+(a-\pi)\frac{D}{\alpha}}{\bar{A}\frac{\alpha+\lambda+\psi}{\alpha}+\bar{B}}, & \text{if } D < \alpha F/(\pi - a) \end{cases} \tag{11}$$

Then $\mathcal{I}^*(D) \geq LB(D)$*.*

5 Numerical Experiments

In this section, we report our computational experiments on the behaviors of the optimal cost and optimal solution.

5.1 Shape of $\mathcal{I}^*(D)$ as a Function of D

Our goal in this subsection is to demonstrate that $\mathcal{I}^*(D)$ is an increasing and concave function of D. Using (18) in Appendix A, we can obtain the relationship between Q^* and D by letting $\frac{\partial}{\partial Q}\mathcal{I}(Q) = 0$, and hence formulate $\frac{\partial}{\partial D}Q^*$ and $\frac{\partial^2}{\partial D^2}Q^*$ explicitly. We can then formulate $\frac{\partial}{\partial D}\mathcal{I}^*(D)$ and $\frac{\partial^2}{\partial D^2}\mathcal{I}^*(D)$. However their expressions (which are omitted here) are too complex to prove monotonicity and concavity analytically. (Such proofs are possible for the approximate model presented in Section 6, or for special cases such as the one in Property 1.)

Instead, we conduct computational experiments to check the values of the derivatives at many points (by changing the value of D) to see whether $\mathcal{I}^*(D)$ is a concave, increasing function of D at these points.

We first generated 5,000 random instances in which all parameters were uniformly drawn from the intervals shown in Table 1. For each data set, we checked the values of $\frac{\partial}{\partial D}\mathcal{I}^*(D)$ and $\frac{\partial^2}{\partial D^2}\mathcal{I}^*(D)$ at 500 points randomly generated in [1, 10000]. Our experimental results show that $\frac{\partial}{\partial D}\mathcal{I}^*(D)$ is always positive and $\frac{\partial^2}{\partial D^2}\mathcal{I}^*(D)$ is always negative at the points we tested, confirming monotonicity and concavity numerically for these instances.

Table 1: Parameter intervals for random instances

Parameter	Interval	Parameter	Interval
F	[5, 20]	α	[0.01, 10]
a	[1, 5]	β	[α, 365]
π	[$2a$, $10a$]	λ	[0.01, 10]
h	[0.01, 0.5]	ψ	[λ, 365]

5.2 Q^* vs. the Classical EOQ Solution

If we use Q_E to denote the classical EOQ solution, then $Q^* - Q_E$ can be interpreted as the optimal safety stock level held at the retailer to protect against supplier disruptions. Our numerical experiments show that $Q^* - Q_E$ tends to be large when the supplier is often unavailable and tends to be small (or even negative, i.e. $Q^* < Q_E$) when the retailer is often disrupted. In addition, we note that the recovery rate at the retailer does not seem to have an impact on the value of Q^*. We do not report our experimental results here since these observations conform to our intuition that the order size is influenced by the availability of the supplier and the disruption frequency at the retailer: When the supplier is often unavailable, the retailer holds additional inventory (safety stock) to maintain a certain level of service; whereas when the retailer itself is often disrupted, it has to keep less inventory to avoid the inventory loss caused by disruptions. We study the approximate safety stock level at the retailer analytically in Section 7.

We now demonstrate, using the following computational experiments, the savings that results from using Q^* instead of Q_E if disruptions are present. We consider the lost-sales case in this section, but our qualitative observations are still valid for the time-independent backorder case except for the different definition of order quantity in the time-independent backorder case, as discussed at the beginning of Section 3.

In the experiments reported below, we fix $F = 6$, $a = 2$, $D = 1000$, $h = 0.2$, $\pi = 10$ and $\beta = 24$. (We also conducted 20 additional sets of experiments in which these parameters were randomly generated from the intervals shown in Table 1. Similar observations were obtained as those reported in this section, but these are omitted due to space considerations.) We vary α, λ and ψ. (We compare Q^* and the EOQ solution as β varies below.) For each set of parameters, we calculate Q^* using the bisection search, and calculate Q_E using the classical EOQ formula. We then calculate $(\mathcal{I}(Q_E) - \mathcal{I}(Q^*))/\mathcal{I}(Q_E)$ to determine the advantage from using Q^* instead of Q_E as the order size from the retailer to the supplier. Table 2 reports the computational results.

Remark. It follows from the definitions of Q^* and Q_E that $\mathcal{I}(Q^*) \leq \mathcal{I}(Q_E)$. The main purpose of this experiment is to study the magnitude of this difference as the disruption parameters change, and explore the conditions under which the cost savings from using Q^* instead of Q_E are most significant.

Table 2 demonstrates that

- the advantage from using Q^* instead of Q_E first decreases, then increases as the retailer's disruptions become more frequent (α increases), except when the supplier is often available (λ is small and/or ψ is large), in which case the advantage from using Q^* increases as the disruption rate at the retailer increases;

Table 2: Savings from using Q^* instead of the classical EOQ solution Q_E when $\beta = 24$ and α, λ and ψ vary

α	λ	ψ	$\frac{\mathcal{I}(Q_E)-\mathcal{I}(Q^*)}{\mathcal{I}(Q_E)}$ (%)	α	λ	ψ	$\frac{\mathcal{I}(Q_E)-\mathcal{I}(Q^*)}{\mathcal{I}(Q_E)}$ (%)
0.01	5	12	16.58	0.01	1	12	3.93
0.05	5	12	15.25	0.05	1	12	3.13
0.1	5	12	13.61	0.1	1	12	2.36
0.5	5	12	5.86	0.5	1	12	0.07
1	5	12	1.85	1	1	12	0.62
5	5	12	5.94	5	1	12	15.44
10	5	12	17.25	10	1	12	26.00
0.01	0.1	12	0.09	0.01	0.01	12	<0.01
0.05	0.1	12	<0.01	0.05	0.01	12	0.04
0.1	0.1	12	0.01	0.1	0.01	12	0.17
0.5	0.1	12	1.60	0.5	0.01	12	2.19
1	0.1	12	4.39	1	0.01	12	5.11
5	0.1	12	19.56	5	0.01	12	20.75
10	0.1	12	29.99	10	0.01	12	30.37
5	1	6	12.10	0.01	1	6	12.80
5	1	12	15.44	0.01	1	12	3.93
5	1	24	18.09	0.01	1	24	0.63
5	1	48	19.70	0.01	1	48	0.05
5	1	96	20.45	0.01	1	96	<0.01

- when the retailer's disruption rate (α) is small, the advantage from using Q^* increases as the supplier's disruption rate (λ) increases, while the opposite is true if the retailer's disruption rate is large;

- when the retailer's disruption rate (α) is small, the advantage from using Q^* decreases as the supplier's recovery rate (ψ) increases, while the opposite is true if the retailer's disruption rate is large.

These three observations suggest that the advantage from using Q^* instead of Q_E is significant when either the retailer is disrupted frequently and the supplier is often available, or the retailer is seldom disrupted but the supplier is often unavailable. This is because Q^* differs from Q_E in two main cases: it decreases when the retailer's disruption rate is large (to avoid potential loss of inventory) and increases when the supplier is often unavailable (to protect against supply unavailability). When neither of these conditions is true, Q^* does not deviate much from Q_E, and when both are true, the two effects tend to cancel each other out, and therefore Q^* is again close to Q_E. But if one of the conditions is true, then Q^* is significantly different from Q_E and the benefit of using Q^* is more pronounced.

We now fix $\psi = 12$ and adjust the values of α, β and λ to compare Q^* with the EOQ solution. Please refer to Table 3.

Table 3: Savings from using Q^* instead of the classical EOQ solution when $\psi = 12$ and α, β and λ vary

α	β	λ	$\frac{\mathcal{I}(Q_E)-\mathcal{I}(Q^*)}{\mathcal{I}(Q_E)}$ (%)
5	6	1	7.45
5	12	1	11.37
5	24	1	15.44
5	48	1	18.80
5	96	1	21.10
5	6	0.01	9.83
5	12	0.01	15.14
5	24	0.01	20.75
5	48	0.01	25.46
5	96	0.01	28.73
0.01	6	1	3.91
0.01	12	1	3.92
0.01	24	1	3.93
0.01	48	1	3.93
0.01	96	1	3.94

We notice from Table 3 that the advantage from using Q^* rather than Q_E is significant when the recovery rate at the retailer is large, especially when the retailer is easily disrupted or the supplier is often available. This is because Q^* tends to be small when the retailer faces a large disruption rate or the supplier has high availability. Therefore, when the retailer is disrupted, it will lose less inventory if it used Q^* than it would if it used Q_E as the order size. Thus, ordering and holding costs can be reduced by using Q^*. On the other hand, when the recovery rate at the retailer is large, shortage costs tend to be smaller, which makes the ordering and holding costs a larger portion of the total cost. Thus, when the retailer has large disruption and recovery rates and the supplier has high availability, the advantage from using Q^* as the order size is more pronounced.

5.3 Effect of Disruptions on Unit Cost and Fill Rate

In this subsection, we study the effect of supply disruptions on the average working inventory cost of serving a single unit of customer demand at the retailer (we call this the *unit cost*), as well as on the customer demand fill rate. Table 4 displays the results for a particular set of instances; based on our other experiments, not reported here, these results appear to be representative of a broader set of instances.

Table 4 is obtained via the following steps when $F = 6$, $a = 2$, $h = 0.2$, $\pi = 10$ and $\psi = 12$, and the values of α, β and λ vary:

1. For fixed input parameters, and for the values of α, β and λ provided in the first column of the table, we apply bisection search to compute Q^* when D equals 10, 100 and 1000.

2. For each set of values of α, β and λ, and for each $D = 10$, 100, and 1000, we substitute Q^* computed in step 1 into $\mathcal{I}(Q)$ to obtain $\mathcal{I}(Q^*)$, and divide it by the corresponding D value to obtain the unit cost, i.e. the cost of serving one unit of customer demand at the retailer.

3. Since we assume that the retailer faces constant customer demand, the customer demand fill rate equals the average percentage of an inventory cycle during which the retailer has inventory on hand. Referring to the discussion in Section 3.2.3, we have

$$\text{fill rate} = \frac{1}{E[T]}\left(\frac{Q}{D}e^{-\frac{\alpha Q}{D}} + \int_0^{\frac{Q}{D}} Ye^{-\alpha Y}dY\right) = \frac{1 - e^{-\frac{\alpha Q}{D}}}{\alpha E[T]}.$$

   Therefore, for each D value, we substitute Q^* computed in step 1 into this formula to obtain the fill rate.

We make the following observations based on Table 4:

- The retailer's availability has a more significant impact on the unit cost and fill rate than the supplier's availability.

- The supplier's availability has an apparent influence on the unit cost and fill rate only when the retailer is often disrupted.

- The unit cost at the retailer decreases as the demand, D, increases.

We can see from the first observation that good decisions made "downstream" in the supply chain (at the retailer, in our case) can reduce the impact of uncertainty "upstream" (at the

Table 4: The effect of supply disruptions on the unit cost and customer demand fill rate at the retailer when $F = 6$, $a = 2$, $h = 0.2$ and $\pi = 10$

			$D = 10$		$D = 100$		$D = 1000$	
α	β	λ	Unit Cost	Fill Rate (%)	Unit Cost	Fill Rate (%)	Unit Cost	Fill Rate (%)
0.01	24	5	2.59	99.03	2.34	98.34	2.29	98.14
0.1	24	5	2.87	98.20	2.49	97.28	2.43	97.01
1	24	5	4.54	91.75	3.41	89.85	3.25	89.31
10	24	5	10.91	59.57	6.61	57.61	6.00	55.96
0.01	24	0.01	2.52	99.96	2.17	99.95	2.05	99.93
0.1	24	0.01	2.76	99.58	2.26	99.57	2.10	99.55
1	24	0.01	4.29	95.99	2.85	95.97	2.48	95.95
10	24	0.01	10.47	70.56	5.75	70.55	4.73	70.54
0.01	24	0	2.52	99.96	2.17	99.95	2.05	99.93
0.1	24	0	2.76	99.58	2.25	99.57	2.10	99.55
1	24	0	4.29	95.99	2.85	95.97	2.48	95.95
10	24	0	10.47	70.56	5.75	70.55	4.72	70.54
5	24	0	5.86	82.76	3.63	82.76	3.40	82.76
5	24	0.01	5.86	82.73	3.63	82.72	3.40	82.70
5	24	0.1	5.87	82.50	4.50	82.37	3.73	82.23
5	24	1	5.96	80.36	4.72	79.20	4.07	78.04
5	24	10	8.44	66.52	5.76	65.45	5.44	65.23
0.01	24	0	2.52	99.96	2.17	99.95	2.05	99.93
0.01	24	0.01	2.52	99.96	2.17	99.95	2.05	99.93
0.01	24	0.1	2.52	99.93	2.17	99.87	2.07	99.74
0.01	24	1	2.54	99.69	2.22	99.32	2.16	99.07
0.01	24	10	2.63	98.61	2.40	97.85	2.36	97.68
0.01	12	5	2.60	98.99	2.34	98.30	2.30	98.10
0.1	12	5	2.90	97.79	2.52	96.87	2.46	96.61
1	12	5	4.75	88.15	4.66	86.43	3.51	85.98
10	12	5	11.43	46.52	7.38	45.56	6.91	44.48
0.01	12	0.01	2.52	99.91	2.17	99.91	2.06	99.89
0.1	12	0.01	2.79	99.17	2.29	99.16	2.14	99.14
1	12	0.01	4.51	92.30	3.13	92.28	2.77	92.26
10	12	0.01	10.83	54.53	6.72	54.52	5.92	54.51
0.01	12	0	2.52	99.91	2.17	99.91	2.06	99.89
0.1	12	0	2.79	99.17	2.29	99.16	2.13	99.14
1	12	0	4.51	92.30	3.13	92.28	2.77	92.26
10	12	0	10.83	54.53	6.72	54.52	5.92	54.51
5	12	0	8.40	70.59	5.28	70.59	4.61	70.59
5	12	0.01	8.40	70.57	5.28	70.56	4.61	70.55
5	12	0.1	8.40	70.39	5.30	70.28	4.65	70.18
5	12	1	8.44	68.69	5.50	67.80	4.94	66.91
5	12	10	8.67	54.15	6.38	51.65	6.11	50.61
0.01	12	0	2.52	99.91	2.17	99.91	2.06	99.89
0.01	12	0.01	2.52	99.91	2.17	99.91	2.06	99.89
0.01	12	0.1	2.53	99.89	2.18	99.83	2.08	99.70
0.01	12	1	2.54	99.65	2.23	99.28	2.16	99.03
0.01	12	10	2.63	98.57	2.40	97.81	2.37	97.64

supplier) on the performance of the supply chain. However, downstream uncertainty has a more significant impact on the supply chain's performance, and is more difficult to mitigate via ordering policies, than upstream uncertainty, since there is no inventory buffer to protect against downstream disruptions.

The second observation can be intuitively explained as follows: If the retailer is seldom disrupted but the supplier is often disrupted, the retailer can increase its order quantity to reduce its order frequency and the percentage of time that it does not have any stock on hand. By doing so, the retailer reduces its shortage cost, which is usually much larger (per unit) than the inventory holding cost, and hence the unit cost at the retailer does not increase much. However, if the retailer and the supplier are both disrupted often, the retailer may not be able to reduce its shortage cost by holding extra inventory since such inventory is often destroyed during disruptions at the retailer. Therefore, in this case the retailer has little ability to buffer against disruptions at the supplier, and hence the retailer's unit cost may be significantly influenced by such disruptions.

Observation 3 is consistent with our numerical experiments reported in Section 5.1, which show that $\mathcal{I}^*(D)$ is a concave and increasing function of D.

6 A Tight Approximation for $\mathcal{I}^*(D)$

Although $\mathcal{I}(Q)$ possesses convenient properties such as Property 4, we cannot derive a closed-form solution for Q^* because of the complexity of $\mathcal{I}(Q)$ and of its derivative. A closed-form solution is useful since we can directly embed it into more complex models (see Section 8). In this section, we derive a closed-form approximate formula for Q^*, based on which we approximate $\mathcal{I}^*(D)$.

Combining (4) and (10), we have

$$\mathcal{I}(Q) = \pi D + \frac{F + \left(a + \frac{h}{\alpha}\right) Q - \left(1 - e^{-\alpha\frac{Q}{D}}\right)\left(\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}\right)}{\bar{A}(1 - e^{-(\alpha+\lambda+\psi)\frac{Q}{D}}) + \bar{B}\left(1 - e^{-\frac{\alpha Q}{D}}\right)}. \tag{12}$$

Our approximation consists of two steps. First, in most realistic instances, the recovery rate, ψ, is relatively large. Therefore, $e^{-(\alpha+\lambda+\psi)\frac{Q}{D}}$ is close to zero. For example, if retailer and supplier ON cycles both last for 1 year, on average ($\alpha = \lambda = 1$), supplier OFF cycles last for 1 month, on average ($\psi = 12$), and orders are placed quarterly ($Q/D = 0.25$), then

$e^{-(\alpha+\lambda+\psi)\frac{Q}{D}} = 0.03$. We therefore omit this term in (12) and approximate $\mathcal{I}(Q)$ using

$$\pi D + \frac{F + \left(a + \frac{h}{\alpha}\right) Q - \left(1 - e^{-\alpha\frac{Q}{D}}\right)\left(\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}\right)}{\bar{A} + \bar{B}\left(1 - e^{-\frac{\alpha Q}{D}}\right)}$$

$$= \pi D - \frac{\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}}{\bar{B}} + \frac{F + \left(a + \frac{h}{\alpha}\right) Q + \frac{\bar{A}\left(\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}\right)}{\bar{B}}}{\bar{A} + \bar{B} - \bar{B}e^{-\frac{\alpha Q}{D}}} \quad (13)$$

Property 6 *(13) is a convex function of Q for $Q \geq 0$ and obtains its minimum at some $Q > 0$. Moreover, the minimum value of (13), expressed as a function of D, is a concave function of D for $D > 0$.*

It follows from Property 6 that the Q that minimizes (13) is the only positive solution to the following first-order condition of (13):

$$\frac{a + \frac{h}{\alpha}}{\bar{A} + \bar{B} - \bar{B}e^{-\frac{\alpha Q}{D}}} - \frac{\left[F + \left(a + \frac{h}{\alpha}\right) Q + \frac{\bar{A}\left(\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}\right)}{\bar{B}}\right] \frac{\bar{B}\alpha}{D} e^{-\frac{\alpha Q}{D}}}{\left(\bar{A} + \bar{B} - \bar{B}e^{-\frac{\alpha Q}{D}}\right)^2} = 0,$$

which is equivalent to

$$\left(a + \frac{h}{\alpha}\right)(\bar{A} + \bar{B}) - \left[\frac{\alpha F}{D} + \left(a + \frac{h}{\alpha}\right) + \frac{\bar{A}\left(\frac{h}{\alpha} + \pi\right)}{\bar{B}} + \left(a + \frac{h}{\alpha}\right)\frac{\alpha Q}{D}\right]\bar{B}e^{-\frac{\alpha Q}{D}} = 0$$

Since we cannot obtain a closed-form solution to this equation, we replace $e^{-\frac{\alpha Q}{D}}$ with its second-order Taylor-series expansion, $\left(1 + \frac{\alpha Q}{D} + \frac{\alpha^2 Q^2}{2D^2}\right)^{-1}$. The resulting equation,

$$\left(a + \frac{h}{\alpha}\right)(\bar{A} + \bar{B})\left(1 + \frac{\alpha Q}{D} + \frac{\alpha^2 Q^2}{2D^2}\right) - \left[\frac{\alpha F}{D} + \left(a + \frac{h}{\alpha}\right) + \frac{\bar{A}\left(\frac{h}{\alpha} + \pi\right)}{\bar{B}} + \left(a + \frac{h}{\alpha}\right)\frac{\alpha Q}{D}\right]\bar{B}$$

$$= \left(a + \frac{h}{\alpha}\right)(\bar{A} + \bar{B})\frac{\alpha^2 Q^2}{2D^2} + \left(a + \frac{h}{\alpha}\right)\bar{A}\frac{\alpha Q}{D} - \left[\frac{\alpha F\bar{B}}{D} + \bar{A}(\pi - a)\right] = 0, \quad (14)$$

can be solved in closed form; its solution is

$$\hat{Q} \equiv D \cdot \frac{-\bar{A} + \sqrt{\bar{A}^2 + \frac{2\alpha(\bar{A}+\bar{B})\left(\frac{\alpha F\bar{B}}{D} + \bar{A}(\pi - a)\right)}{(\alpha a + h)}}}{(\bar{A} + \bar{B})\alpha} \quad (15)$$

We denote this expression $\hat{Q}$ and use it to approximate Q^*. ((14) also has another real root, but it is negative and infeasible as an order size.)

Remark. We use a second-order polynomial function to approximate $e^{-\frac{\alpha Q}{D}}$, rather than a third- or higher-order polynomial, which would be more precise, because the purpose of the approximation is to derive a closed-form approximate solution for Q^*. Third- and fourth-order polynomial functions do have closed-form formulas, but they are too complicated to permit further analysis, and higher-order polynomials generally do not have closed-form formulas.

Property 7 *When the retailer is never disrupted ($\alpha = 0$), $\hat{Q}$ reduces to the approximate EOQD solution derived by Snyder (2008).*

Property 8 *When neither the retailer nor the supplier is ever disrupted ($\lambda = \alpha = 0$), $\hat{Q}$ reduces to the classical EOQ solution.*

These properties are intuitive, since Snyder's model is a special case of ours in which $\alpha = a = 0$, and the EOQ model is a special case of ours in which $\lambda = \alpha = 0$.

Having approximated Q^* by $\hat{Q}$, we now approximate $\mathcal{I}^*(D)$. By applying the approximations introduced above, $\mathcal{I}(Q)$ can be approximated by:

$$\hat{\mathcal{I}}(Q) \equiv \pi D - \frac{\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}}{\bar{B}} + \frac{F + \left(a + \frac{h}{\alpha}\right) Q + \frac{\bar{A}\left(\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}\right)}{\bar{B}}}{\bar{A} + \bar{B} - \frac{\bar{B}}{1 + \frac{\alpha Q}{D} + \frac{\alpha^2 Q^2}{2D^2}}}$$

Since $\hat{Q}$ satisfies (14), we have

$$\left(a + \frac{h}{\alpha}\right)(\bar{A} + \bar{B})\left(1 + \frac{\alpha \hat{Q}}{D} + \frac{\alpha^2 \hat{Q}^2}{2D^2}\right) - \left[\frac{\alpha F}{D} + \left(a + \frac{h}{\alpha}\right) + \frac{\bar{A}\left(\frac{h}{\alpha} + \pi\right)}{\bar{B}} + \left(a + \frac{h}{\alpha}\right)\frac{\alpha \hat{Q}}{D}\right]\bar{B} = 0,$$

and hence

$$\left(a + \frac{h}{\alpha}\right)\left[(\bar{A} + \bar{B})\left(1 + \frac{\alpha \hat{Q}}{D} + \frac{\alpha^2 \hat{Q}^2}{2D^2}\right) - \bar{B}\right] - \left[F + \frac{\bar{A}\left(\frac{hD}{\alpha^2} + \frac{\alpha\pi}{D}\right)}{\bar{B}} + \left(a + \frac{h}{\alpha}\right)\hat{Q}\right]\frac{\alpha \bar{B}}{D} = 0.$$

Therefore,

$$\left(a + \frac{h}{\alpha}\right)\left[(\bar{A} + \bar{B}) - \frac{\bar{B}}{1 + \frac{\alpha \hat{Q}}{D} + \frac{\alpha^2 \hat{Q}^2}{2D^2}}\right] = \left[F + \frac{\bar{A}\left(\frac{hD}{\alpha^2} + \frac{\alpha\pi}{D}\right)}{\bar{B}} + \left(a + \frac{h}{\alpha}\right)\hat{Q}\right]\frac{\alpha \bar{B}}{D\left(1 + \frac{\alpha \hat{Q}}{D} + \frac{\alpha^2 \hat{Q}^2}{2D^2}\right)},$$

based on which we derive

$$\begin{aligned}\hat{\mathcal{I}}(\hat{Q}) &= \pi D - \frac{\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}}{\bar{B}} + \frac{\left(a + \frac{h}{\alpha}\right) D \left(1 + \frac{\alpha \hat{Q}}{D} + \frac{\alpha^2 \hat{Q}^2}{2D^2}\right)}{\alpha \bar{B}} \\ &= \pi D - \frac{\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}}{\bar{B}} + \frac{F + \left(\frac{aD}{\alpha} + \frac{hD}{\alpha^2}\right) + \frac{\bar{A}\left(\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}\right)}{\bar{B}} + \left(a + \frac{h}{\alpha}\right)\hat{Q}}{\bar{A} + \bar{B}}\end{aligned}$$

$$= \pi D + \frac{F + \frac{(a-\pi)D}{\alpha} + \left(a + \frac{h}{\alpha}\right)\hat{Q}}{\bar{A} + \bar{B}} \tag{16}$$

(The second equality follows from (14).)

We use $\hat{\mathcal{I}}^*(D)$ to denote this expression when it is treated as a function of D. We now examine the accuracy from using $\hat{\mathcal{I}}^*(D)$ to approximate $\mathcal{I}^*(D)$. In particular, we wish to evaluate the approximation error $r \equiv |\hat{\mathcal{I}}^*(D) - \mathcal{I}^*(D)|/\hat{\mathcal{I}}^*(D)$. (To compute r, we calculate Q^* using the bisection search, and compare $\hat{\mathcal{I}}^*(D)$ and $\mathcal{I}^*(D)$ numerically.) In addition, we define the following error bound:

$$eb = \max\left\{\frac{\mathcal{I}(\hat{Q})}{\hat{\mathcal{I}}^*(D)}, \frac{\hat{\mathcal{I}}^*(D)}{LB(D)}\right\} - 1,$$

where $LB(D)$ is defined in (11).

Property 9 *If we use $\hat{\mathcal{I}}^*(D)$ to approximate $\mathcal{I}^*(D)$, then $r \leq eb$.*

We tested the values of eb and r numerically on 100,000 random instances with input parameters drawn uniformly from the ranges given in Table 1, and with customer demand D drawn uniformly from [1,10000]. We summarize our results as follows:

- $r \leq 1\%$ for 61,299 instances (61.30%)
- $r \leq 5\%$ for 95,796 instances (95.80%)
- $r \leq 10\%$ for 99,064 instances (99.06%)
- the average value of r is 1.04%
- $eb \leq 10\%$ for 54,983 instances (54.98%)
- $eb \leq 20\%$ for 88,424 instances (88.42%)
- $eb \leq 30\%$ for 97,606 instances (97.61%)
- the average value of eb is 16.39%

This demonstrates that $\hat{\mathcal{I}}^*(D)$ is a quite tight approximation of $\mathcal{I}^*(D)$.

The following result is useful for embedding the approximate model into larger models, as described in Section 8:

Property 10 *$\hat{\mathcal{I}}^*(D)$ is an increasing and concave function of D.*

Although we have assumed $D > 0$, Property 10 still applies when $D \geq 0$ if we define $\hat{\mathcal{I}}^*(D) = 0$ when $D = 0$.

7 Safety Stock at the Retailer

We suggested in Section 5.2 that $Q^* - Q_E$ can be interpreted as the optimal safety stock level held at the retailer to protect against supplier disruptions. Since Q^* cannot be derived in closed form, we instead analyze $\hat{Q} - Q_E$, which is the approximate optimal safety stock level held at the retailer.

Property 11 *(a) $\hat{Q}$ is independent of β.*

(b) $\hat{Q}$ is monotone with respect to λ and ψ.

(c) If $\psi \geq \frac{h}{a}$, then $\hat{Q}$ is a decreasing function of α.

The proof of Property 11(c) relies on the additional condition $\psi \geq \frac{h}{a}$. This condition is not particularly restrictive since typically the recovery rate $\psi \geq 1$ (i.e., the durations of supplier OFF cycles are typically no more than one year) and $h \leq a$. Therefore, we assume that this condition is true in this paper.

Corollary 2 follows from Property 11(b). It uses the following condition:

$$\sqrt{D} > \frac{\sqrt{2F}(\sqrt{\alpha a + h + 2\alpha(\pi - a)} + \sqrt{\alpha a + h})}{2(\pi - a)} \tag{17}$$

Corollary 2 *When (17) holds, $\hat{Q}$ is an increasing function of λ and a decreasing function of ψ. When (17) does not hold, $\hat{Q}$ is a decreasing function of λ and an increasing function of ψ.*

Parts (a) and (c) of Property 11 and the first part of Corollary 2 are intuitive and conform with our observations in Section 5.2. The second part of Corollary 2 can be explained as follows: We assume a ZIO policy at the retailer, and the retailer will receive shipments from the supplier as long as both the retailer and the supplier are available. When the demand rate is small compared to the retailer's disruption rate, it is very costly to serve the small customer demand due to the high risk of inventory loss. In this case, if the supplier is often unavailable, by reducing the order quantity, the retailer can significantly increase the duration during which it does not have inventory on hand and does not have to serve customers, so that it can reduce the waste caused by disruptions. Though the shortage cost increases as the order size decreases, it will not increase much since the demand rate is small.

We now consider two cases to analyze the impact of disruptions at the retailer and the supplier on the safety stock level at the retailer.

Case (1): α is large or D is small so that (17) does not hold

It follows from Corollary 2 that $\hat{Q}$ is an increasing function of the supplier availability. Thus, $\hat{Q}$ reaches its maximum when the supplier is always available (i.e. $\lambda = 0$ or $\psi = +\infty$); by (15), this maximum is given by $\sqrt{\frac{2FD}{\alpha a+h}}$. Therefore,

$$\hat{Q} \leq \sqrt{\frac{2FD}{\alpha a + h}} \leq \sqrt{\frac{2FD}{h}} = Q_E.$$

In other words, when the retailer is often disrupted or the demand rate is small, so that (17) is violated, no safety stock should be kept at the retailer.

Case (2): α is small or D is large so that (17) holds

It follows from Corollary 2 that $\hat{Q}$ is a decreasing function of the supplier availability. In addition, we know from Property 11 that $\hat{Q}$ is always a decreasing function of α. Therefore, when the supplier has low availability and the retailer is seldom disrupted, $\hat{Q}$ tends to be large, and the safety stock level at the retailer is high; when the supplier has high availability and the retailer is often disrupted, $\hat{Q}$ is small and the safety stock level at the retailer is small or even zero. We use the following two examples to illustrate this conclusion:

Example 1: When the supplier is always available, as we discussed above, $\hat{Q} = \sqrt{\frac{2FD}{\alpha a+h}}$. If the retailer is sometimes disrupted, then $\alpha > 0$ and hence $\hat{Q} < Q_E$, which means no safety stock is held at the retailer.

Example 2: If the retailer is never disrupted ($\alpha = 0$) but the supplier is sometimes unavailable, since $\hat{Q}$ is a decreasing function of the availability of the supplier when (17) holds, $\hat{Q} > \sqrt{\frac{2FD}{\alpha a+h}} = \sqrt{\frac{2FD}{h}} = Q_E$. Therefore, the retailer should hold safety stock, and the safety stock level increases as the availability of the supplier decreases.

8 Conclusions

We consider a continuous-review inventory model for a problem with a single retailer and a single supplier, in which both the retailer and the supplier may be randomly disrupted. The results in this paper hold for both the time-independent backorder case and the lost-sales case. We study properties of the expected cost function and suggest numerical solution algorithms to obtain the optimal order size that the retailer should use when placing orders to the supplier. The computational experiments in this paper show that the cost savings from considering disruptions at the retailer and the supplier are significant, and that disruptions at the retailer have a much bigger impact on the working inventory cost and fill rate at the retailer than disruptions at the supplier do. We also propose a tight approximation for the cost function, since the exact function cannot be minimized in closed form. Based on this approximation, we further analyze the impact of supplier and retailer disruptions on the safety stock level at the retailer.

Our research in this paper can be extended in at least two respects. First, the model proposed in this paper can be embedded into integrated supply chain design models with both supplier and retailer disruptions and recoveries taken into consideration. For example, Qi, Shen, and Snyder (2007) embed the cost function from the present paper into a location-inventory model that considers disruptions at both the supplier and retailer echelons, just as Shen, Coullard, and Daskin (2003) embed the classical EOQ model into a location-inventory model without disruptions.

Figure 1 is based on the assumption that the retailer follows a ZIO inventory policy. When this assumption is relaxed, and the retailer can place orders to the supplier when its inventory level reaches $R \geq 0$, the resulting inventory policy is an (R, Q)-like policy and may perform better than the ZIO policy. In this case, we need to find both the optimal order quantity Q and the optimal reorder point R.

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Appendix A: Proofs

Lemma 2 is used in the proofs of Properties 1 and 6.

Lemma 2 *Suppose all parameters are positive and k and l are positive numbers such that $k \geq l$, $\frac{\alpha s}{Dt} > \frac{k-l}{l}$, and $\frac{k}{l} < e^{\frac{\alpha s}{Dt}}$. Then*

(a) $\frac{s+tX}{k-le^{-\frac{\alpha X}{D}}}$ *is a convex function of X in $X \geq 0$*

(b) If we use X^* to denote the value of $X \geq 0$ that minimizes the function $\frac{s+tX}{k-le^{-\frac{\alpha X}{D}}}$, then $X^* > 0$ and $\frac{s+tX^*}{k-le^{-\frac{\alpha X^*}{D}}}$ is a concave function of D for $D > 0$.

Proof. The first derivative of $\frac{s+tX}{k-le^{-\frac{\alpha X}{D}}}$ is

$$\frac{t}{k-le^{-\frac{\alpha X}{D}}} - \frac{(s+tX)\frac{\alpha l}{D}e^{-\frac{\alpha X}{D}}}{(k-le^{-\frac{\alpha X}{D}})^2},$$

and its second derivative is

$$-\frac{\frac{\alpha lt}{D}e^{-\frac{\alpha X}{D}}}{(k-le^{-\frac{\alpha X}{D}})^2} - \frac{\frac{\alpha lt}{D}e^{-\frac{\alpha X}{D}} - (s+tX)\frac{\alpha^2 l}{D^2}e^{-\frac{\alpha X}{D}}}{(k-le^{-\frac{\alpha X}{D}})^2} + \frac{2(s+tX)\frac{\alpha^2 l^2}{D^2}e^{-\frac{2\alpha X}{D}}}{(k-le^{-\frac{\alpha X}{D}})^3}.$$

Therefore, to prove part (a) of the lemma, it suffices to show that

$$f(X) \equiv \left(k-le^{-\frac{\alpha X}{D}}\right)\left[-2t+(s+tX)\frac{\alpha}{D}\right] + 2(s+tX)\frac{\alpha l}{D}e^{-\frac{\alpha X}{D}} > 0$$

for $X \geq 0$. Well,

$$\begin{aligned} f'(X) &= \frac{\alpha l}{D}e^{-\frac{\alpha X}{D}}\left[-2t+(s+tX)\frac{\alpha}{D}\right] + \left(k-le^{-\frac{\alpha X}{D}}\right)\frac{\alpha t}{D} + 2t\frac{\alpha l}{D}e^{-\frac{\alpha X}{D}} - 2(s+tX)\frac{\alpha^2 l}{D^2}e^{-\frac{\alpha X}{D}} \\ &= \left(k-le^{-\frac{\alpha X}{D}}\right)\frac{\alpha t}{D} - (s+tX)\frac{\alpha^2 l}{D^2}e^{-\frac{\alpha X}{D}}; \\ f''(X) &= \frac{\alpha^2 lt}{D^2}e^{-\frac{\alpha X}{D}} - \frac{\alpha^2 lt}{D^2}e^{-\frac{\alpha X}{D}} + (s+tX)\frac{\alpha^3 l}{D^3}e^{-\frac{\alpha X}{D}} = (s+tX)\frac{\alpha^3 l}{D^3}e^{-\frac{\alpha X}{D}}. \end{aligned}$$

Since

$$f'(-\infty) > 0, f'\left(-\frac{s}{t}\right) = \left(k-le^{\frac{\alpha s}{Dt}}\right)\frac{\alpha t}{D} < 0, f'(\infty) = \frac{\alpha kt}{D} > 0,$$

$f(X)$ is a concave function of X with one maximum point (we use X_1 to denote it) in $(-\infty, -\frac{s}{t}]$, and a convex function with one minimum point (we use X_2 to denote it) in $[-\frac{s}{t}, \infty)$. It follows from $f''(0) > 0$ that $X_1 < 0$ and $f(X) \geq f(X_2)$ for all $X \geq 0$.

When $X_2 \geq 0$, it follows from the first order condition that

$$f'(X_2) = \left(k-le^{-\frac{\alpha X_2}{D}}\right)\frac{\alpha t}{D} - (s+tX_2)\frac{\alpha^2 l}{D^2}e^{-\frac{\alpha X_2}{D}} = 0.$$

Therefore, for any $X \geq 0$,

$$\begin{aligned} f(X) \geq f(X_2) &= \left(k-le^{-\frac{\alpha X_2}{D}}\right)\left[-2t+(s+tX_2)\frac{\alpha}{D}\right] + 2t\left(k-le^{-\frac{\alpha X_2}{D}}\right) \\ &= \left(k-le^{-\frac{\alpha X_2}{D}}\right)(s+tX_2)\frac{\alpha}{D} > 0 \end{aligned}$$

due to the fact $k > l$.

When $X_2 < 0$, since $f(X)$ is a convex function with one minimum point at X_2 in $[-\frac{s}{t}, \infty)$, for any $X \geq 0$, we have

$$\begin{aligned} f(X) \geq f(0) &= (k-l)\left(-2t + s\frac{\alpha}{D}\right) + 2s\frac{\alpha l}{D} \\ &= (k-l)(-2t) + (k+l)s\frac{\alpha}{D} > 0 \end{aligned}$$

due to the fact that $\frac{\alpha s}{Dt} > \frac{k-l}{l} \geq \frac{2(k-l)}{k+l}$. This completes the proof of part (a).

We now prove the second part of the lemma. When $X = 0$, the first derivative of $\frac{s+tX}{k-le^{-\frac{\alpha X}{D}}}$ is

$$\frac{t}{k-l} - \frac{s\frac{\alpha l}{D}}{(k-l)^2} = \frac{lt}{(k-l)^2}\left(\frac{k-l}{l} - \frac{s\alpha}{tD}\right) < 0;$$

and when $X = +\infty$, the first derivative is $\frac{t}{k} > 0$. It follows from part (a) of the lemma that $X^* > 0$ and satisfies the first order condition. Therefore

$$\frac{t}{k-le^{-\frac{\alpha X^*}{D}}} - \frac{(s+tX^*)\frac{\alpha l}{D}e^{-\frac{\alpha X^*}{D}}}{\left(k-le^{-\frac{\alpha X^*}{D}}\right)^2} = 0.$$

We thus derive the relationship between X^* and D as

$$tkDe^{\frac{\alpha X^*}{D}} - (tD + \alpha s + \alpha t X^*)l = 0.$$

We use $F(X^*, D)$ to denote the left-hand side of the above equation.

Since

$$\begin{aligned} \frac{\partial F(X^*,D)}{\partial X^*} &= tk\alpha e^{\frac{\alpha X^*}{D}} - \alpha tl \\ \frac{\partial F(X^*,D)}{\partial D} &= tke^{\frac{\alpha X^*}{D}} - \frac{tk\alpha X^*}{D}e^{\frac{\alpha X^*}{D}} - tl, \end{aligned}$$

we have

$$\frac{\partial X^*}{\partial D} = -\frac{\frac{\partial F(X^*,D)}{\partial D}}{\frac{\partial F(X^*,D)}{\partial X^*}} = -\frac{tke^{\frac{\alpha X^*}{D}} - \frac{tk\alpha X^*}{D}e^{\frac{\alpha X^*}{D}} - tl}{tk\alpha e^{\frac{\alpha X^*}{D}} - \alpha tl} = -\frac{1}{\alpha} + \frac{\frac{X^*}{D}e^{\frac{\alpha X^*}{D}}}{e^{\frac{\alpha X^*}{D}} - l/k},$$

$$\frac{\partial \frac{X^*}{D}}{\partial D} = \frac{\frac{\partial X^*}{\partial D}}{D} - \frac{X^*}{D^2} = -\frac{1}{\alpha D} + \frac{\frac{X^*}{D^2}e^{\frac{\alpha X^*}{D}}}{e^{\frac{\alpha X^*}{D}} - l/k} - \frac{X^*}{D^2} = -\frac{1}{\alpha D}\left(1 - \frac{\alpha X^*}{D}\frac{l/k}{e^{\frac{\alpha X^*}{D}} - l/k}\right),$$

$$\text{and} \quad \frac{\partial e^{\frac{\alpha X^*}{D}}}{\partial D} = \alpha e^{\frac{\alpha X^*}{D}} \frac{\partial \frac{X^*}{D}}{\partial D}.$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 X^*}{\partial D^2} &= \frac{\frac{\partial \frac{X^*}{D}}{\partial D} e^{\frac{\alpha X^*}{D}} + \frac{X^*}{D} \frac{\partial e^{\frac{\alpha X^*}{D}}}{\partial D}}{e^{\frac{\alpha X^*}{D}} - l/k} - \frac{\frac{X^*}{D} e^{\frac{\alpha X^*}{D}} \frac{\partial e^{\frac{\alpha X^*}{D}}}{\partial D}}{\left(e^{\frac{\alpha X^*}{D}} - l/k\right)^2} \\
&= \frac{\frac{\partial \frac{X^*}{D}}{\partial D} e^{\frac{\alpha X^*}{D}} + \frac{X^*}{D} \alpha e^{\frac{\alpha X^*}{D}} \frac{\partial \frac{X^*}{D}}{\partial D}}{e^{\frac{\alpha X^*}{D}} - l/k} - \frac{\frac{X^*}{D} e^{\frac{\alpha X^*}{D}} \alpha e^{\frac{\alpha X^*}{D}} \frac{\partial \frac{X^*}{D}}{\partial D}}{\left(e^{\frac{\alpha X^*}{D}} - l/k\right)^2} \\
&= \frac{\frac{\partial \frac{X^*}{D}}{\partial D} e^{\frac{\alpha X^*}{D}}}{\left(e^{\frac{\alpha X^*}{D}} - l/k\right)^2} \left(e^{\frac{\alpha X^*}{D}} - l/k - \frac{\alpha X^*}{D} \cdot l/k\right) \\
&= -\frac{e^{\frac{\alpha X^*}{D}}}{\alpha D \left(e^{\frac{\alpha X^*}{D}} - l/k\right)^3} \left(e^{\frac{\alpha X^*}{D}} - l/k - \frac{\alpha X^*}{D} \cdot l/k\right)^2 \\
&< 0
\end{aligned}$$

The last inequality follows from the fact that $e^{\frac{\alpha X^*}{D}} > 1 > l/k$. We thus know that X^* is a concave function of D. Since

$$\frac{t}{k - le^{-\frac{\alpha X^*}{D}}} - \frac{(s + tX^*)\frac{\alpha l}{D} e^{-\frac{\alpha X^*}{D}}}{\left(k - le^{-\frac{\alpha X^*}{D}}\right)^2} = 0,$$

we have

$$\frac{s + tX^*}{k - le^{-\frac{\alpha X^*}{D}}} = \frac{D}{\alpha l} e^{\frac{\alpha X^*}{D}} = \frac{\frac{tD}{\alpha} + s + tX^*}{tk}.$$

We hence conclude that $\frac{s+tX^*}{k-le^{-\frac{\alpha X^*}{D}}}$ is a concave function of D too. ■

Proof of Property 1. When $\lambda \to 0$, $\bar{A} \to 0$, and hence

$$\begin{aligned}
\mathcal{I}(Q) &= \pi D + \frac{F + \left(a + \frac{h}{\alpha}\right) Q - \left(1 - e^{-\alpha \frac{Q}{D}}\right) \left(\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}\right)}{E[T]} \\
&= \pi D - \frac{\frac{hD}{\alpha^2} + \frac{\pi D}{\alpha}}{\bar{B}} + \frac{F + \left(a + \frac{h}{\alpha}\right) Q}{\bar{B}\left(1 - e^{-\alpha \frac{Q}{D}}\right)}
\end{aligned}$$

Because $\frac{F+\left(a+\frac{h}{\alpha}\right)Q}{1-e^{-\alpha\frac{Q}{D}}}$ satisfies all conditions of Lemma 2, $\mathcal{I}(Q)$ is a convex function of Q in $Q \geq 0$, and $\mathcal{I}^*(D)$ is a concave function of D in $D > 0$ when $\lambda \to 0$. ■

Proof of Property 2.

$$\begin{aligned}\lim_{\alpha\to 0} E[T] &= \lim_{\alpha\to 0}\left\{\frac{1}{\psi}\cdot\frac{\lambda}{\lambda+\psi}\left[1-\frac{\alpha\beta}{(\beta+\lambda+\psi)(\alpha+\lambda+\psi)}\right]\left(1-e^{-(\alpha+\lambda+\psi)\frac{Q}{D}}\right)+\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\left(1-e^{-\frac{\alpha Q}{D}}\right)\right\}\\ &= \frac{1}{\psi}\cdot\frac{\lambda}{\lambda+\psi}\left(1-e^{-(\lambda+\psi)\frac{Q}{D}}\right)+\frac{Q}{D}\end{aligned}$$

and,

$$\begin{aligned}\lim_{\alpha\to 0} E[C] &= \lim_{\alpha\to 0}\left\{F+aQ+\frac{hQ}{\alpha}-\frac{hD}{\alpha^2}\left(1-e^{-\frac{\alpha Q}{D}}\right)+\pi D\left[E[T]-\left(\frac{1}{\alpha}-\frac{1}{\alpha}e^{-\frac{\alpha Q}{D}}\right)\right]\right\}\\ &= F+aQ+\lim_{\alpha\to 0}\left\{\frac{h}{\alpha^2}\left[\alpha Q-D\left(1-e^{-\frac{\alpha Q}{D}}\right)\right]\right\}+\pi D\left[\frac{1}{\psi}\cdot\frac{\lambda}{\lambda+\psi}\left(1-e^{-(\lambda+\psi)\frac{Q}{D}}\right)+\frac{Q}{D}-\frac{Q}{D}\right]\\ &= F+aQ+\lim_{\alpha\to 0}\left\{\frac{hQ}{2\alpha}\left(1-e^{-\frac{\alpha Q}{D}}\right)\right\}+\pi D\left[\frac{1}{\psi}\cdot\frac{\lambda}{\lambda+\psi}\left(1-e^{-(\lambda+\psi)\frac{Q}{D}}\right)\right]\\ &= F+aQ+\frac{hQ^2}{2D}+\pi D\left[\frac{1}{\psi}\cdot\frac{\lambda}{\lambda+\psi}\left(1-e^{-(\lambda+\psi)\frac{Q}{D}}\right)\right]\end{aligned}$$

Therefore,

$$\mathcal{I}(Q)=\frac{F+aQ+\frac{hQ^2}{2D}+\frac{\lambda\pi D}{(\lambda+\psi)\psi}\left(1-e^{-(\lambda+\psi)\frac{Q}{D}}\right)}{\frac{Q}{D}+\frac{\lambda}{(\lambda+\psi)\psi}\left(1-e^{-(\lambda+\psi)\frac{Q}{D}}\right)}$$

as $\alpha\to 0$, which is exactly the same as the cost function proposed in Berk and Arreola-Risa (1994). ∎

Proof of Lemma 1. a) Based on (10), we have

$$\begin{aligned}\frac{\partial}{\partial Q}\mathcal{I}(Q) = &\frac{\left(a+\frac{h}{\alpha}\right)-\frac{\alpha}{D}e^{-\alpha\frac{Q}{D}}\left(\frac{hD}{\alpha^2}+\frac{\pi D}{\alpha}\right)}{E[T]}\\ &-\frac{\left[F+\left(a+\frac{h}{\alpha}\right)Q-\left(1-e^{-\alpha\frac{Q}{D}}\right)\left(\frac{hD}{\alpha^2}+\frac{\pi D}{\alpha}\right)\right]\frac{\partial}{\partial Q}E[T]}{E[T]^2}\end{aligned} \tag{18}$$

where

$$\frac{\partial}{\partial Q}E[T]=\bar{A}\frac{\alpha+\lambda+\psi}{D}e^{-(\alpha+\lambda+\psi)\frac{Q}{D}}+\bar{B}\frac{\alpha}{D}e^{-\frac{\alpha Q}{D}}.$$

When $Q\to 0$, $E[T]$ approaches zero, and $\frac{\partial}{\partial Q}E[T]=\bar{A}\frac{\alpha+\lambda+\psi}{D}+\bar{B}\frac{\alpha}{D}$. Therefore, when $Q\to 0$, $\frac{\partial}{\partial Q}\mathcal{I}(Q)=\frac{a-\frac{\alpha}{D}\frac{\pi D}{\alpha}}{E[T]}-\frac{F\frac{\partial}{\partial Q}E[T]}{E[T]^2}$ approaches infinity, and the first term is a first-order infinity while the second term is a second-order infinity. Thus the sign of $\frac{\partial}{\partial Q}\mathcal{I}(Q)$ is determined by the term with higher order of infinity, which is $-\frac{F\frac{\partial}{\partial Q}E[T]}{E[T]^2}$, and hence $\lim_{Q\to 0}\frac{\partial}{\partial Q}\mathcal{I}(Q)<0$.

b) When $Q\to\infty$, $e^{-\alpha\frac{Q}{D}}$ and $\frac{\partial}{\partial Q}E[T]$ both approach zero, and $E[T]=\bar{A}+\bar{B}$. It follows from (18) that $\frac{\partial}{\partial Q}\mathcal{I}(Q)=\frac{a+\frac{h}{\alpha}}{E[T]}=\frac{a+\frac{h}{\alpha}}{\bar{A}+\bar{B}}$, which is positive. ∎

Proof of Property 4. According to Lemma 1, to prove that $\mathcal{I}(Q)$ is quasi-convex, it suffices to show that there are at most two solutions to $\mathcal{I}(Q)=\epsilon$, where ϵ is an arbitrary real number. From (10), we know the equation $\mathcal{I}(Q)=\epsilon$ can be transformed into

$$(\pi D-\epsilon)\bar{A}\left[1-e^{-(\alpha+\lambda+\psi)\frac{Q}{D}}\right]+F+\left(a+\frac{h}{\alpha}\right)Q+\left(1-e^{-\alpha\frac{Q}{D}}\right)\left[(\pi D-\epsilon)\bar{B}-\frac{hD}{\alpha^2}-\frac{\pi D}{\alpha}\right]=0 \quad (19)$$

The second derivative of the left-hand side with respect to Q has the form $a_1e^{b_1Q}+a_2e^{b_2Q}$, where b_1 and b_2 are both negative. When the coefficients of the exponential functions in (19) are both zero, (19) reduces to a linear equation, which has exactly one solution. Otherwise, a_1 and a_2 will not both equal zero. Without loss of generality, we assume that $a_1\neq 0$. Since the equation $e^{(b_1-b_2)Q}=-a_2/a_1$ has at most one solution (because its left-hand side is an increasing function of Q, while its right-hand side is a constant), there is at most one solution to the equation $a_1e^{b_1Q}+a_2e^{b_2Q}=0$. Thus, based on Rolle's Theorem from calculus, there are at most three solutions to Equation (19), and hence to $\mathcal{I}(Q)=\epsilon$. In addition, it follows from Lemma 1 that it is impossible for $\mathcal{I}(Q)=\epsilon$ to have exactly three solutions when the number of its solutions is no more than three. We thus conclude that $\mathcal{I}(Q)=\epsilon$ has at most two solutions. ■

Proof of Property 5.

$$\begin{aligned}\mathcal{I}^*(D)&=\pi D+\frac{F+\left(a+\frac{h}{\alpha}\right)Q^*-\left(1-e^{-\alpha\frac{Q^*}{D}}\right)\left(\frac{hD}{\alpha^2}+\frac{\pi D}{\alpha}\right)}{\bar{A}\left[1-e^{-(\alpha+\lambda+\psi)\frac{Q^*}{D}}\right]+\bar{B}\left(1-e^{-\frac{\alpha Q^*}{D}}\right)}\\&\geq\pi D+\frac{F\left(1-e^{-\alpha\frac{Q^*}{D}}\right)+\left(a+\frac{h}{\alpha}\right)\frac{D}{\alpha}(1-e^{-\alpha\frac{Q^*}{D}})-\left(1-e^{-\alpha\frac{Q^*}{D}}\right)\left(\frac{hD}{\alpha^2}+\frac{\pi D}{\alpha}\right)}{\bar{A}\left[1-e^{-(\alpha+\lambda+\psi)\frac{Q^*}{D}}\right]+\bar{B}\left(1-e^{-\frac{\alpha Q^*}{D}}\right)}\\&=\pi D+\frac{F+(a-\pi)\frac{D}{\alpha}}{\bar{A}\frac{1-e^{-(\alpha+\lambda+\psi)\frac{Q^*}{D}}}{1-e^{-\frac{\alpha Q^*}{D}}}+\bar{B}}\end{aligned}$$

The inequality holds because $1-e^{-\frac{\alpha Q}{D}}\leq\frac{\alpha Q}{D}$ when $Q\geq 0$.

When $(\pi-a)D\geq\alpha F$,

$$\mathcal{I}^*(D)\geq\pi D+\frac{F+(a-\pi)\frac{D}{\alpha}}{\bar{A}+\bar{B}}$$

because

$$1-e^{-(\alpha+\lambda+\psi)\frac{Q^*}{D}}\geq 1-e^{-\frac{\alpha Q^*}{D}}.$$

When $(\pi-a)D<\alpha F$,

$$\mathcal{I}^*(D)\geq\pi D+\frac{F+(a-\pi)\frac{D}{\alpha}}{\bar{A}\frac{\alpha+\lambda+\psi}{\alpha}+\bar{B}}$$

because

$$\frac{1-e^{-(\alpha+\lambda+\psi)\frac{Q^*}{D}}}{\alpha+\lambda+\psi} \le \frac{1-e^{-\frac{\alpha Q^*}{D}}}{\alpha}$$

since $\frac{1-e^{-x}}{x}$ is a decreasing function of x when $x \ge 0$. ■

Proof of Property 6. Since $\bar{A}+\bar{B} \ge \bar{B}$,

$$\frac{\frac{\bar{A}\left(\frac{hD}{\alpha^2}+\frac{\pi D}{\alpha}\right)}{\bar{B}}+F}{a+\frac{h}{\alpha}} \cdot \frac{\alpha}{D} > \frac{\bar{A}\left(\frac{h}{\alpha}+\pi\right)}{\bar{B}\left(a+\frac{h}{\alpha}\right)} \ge \frac{\bar{A}}{\bar{B}} = \frac{(\bar{A}+\bar{B})-\bar{B}}{\bar{B}},$$

and

$$e^{\frac{\frac{\bar{A}\left(\frac{hD}{\alpha^2}+\frac{\pi D}{\alpha}\right)}{\bar{B}}+F}{a+\frac{h}{\alpha}} \cdot \frac{\alpha}{D}} > e^{\frac{\bar{A}}{\bar{B}}} \ge \frac{\bar{A}}{\bar{B}}+1,$$

this property directly follows from Lemma 2. ■

Proof of Property 7. When $\alpha=0$, $w=1/\psi$, $\bar{A}=\frac{\lambda}{\psi(\psi+\lambda)}$, and $\bar{B}\alpha=1$. From (15), we have

$$\begin{aligned}\hat{Q} &= -\frac{\lambda D}{\psi(\psi+\lambda)}+\sqrt{\left[\frac{\lambda D}{\psi(\psi+\lambda)}\right]^2+\frac{2\left[FD+\frac{\lambda D^2}{\psi(\psi+\lambda)}(\pi-a)\right]}{h}} \\ &= \frac{-\frac{\lambda hD}{\psi+\lambda}+\sqrt{\left(\frac{\lambda hD}{\psi+\lambda}\right)^2+2h\psi\left[\psi FD+\frac{\lambda D^2}{\psi+\lambda}(\pi-a)\right]}}{h\psi},\end{aligned} \tag{20}$$

which is the same as the approximate EOQD solution proposed by Snyder (2008) (Snyder assumes that $a=0$). ■

Proof of Property 8. When neither the retailer nor the supplier can be disrupted, $\alpha=\lambda=0$. It follows from (20) that $\hat{Q}=\frac{\sqrt{2h\psi(\psi FD)}}{h\psi}=\sqrt{\frac{2DF}{h}}$, which is the classical EOQ solution. ■

Proof of Property 9. When $\hat{\mathcal{I}}^*(D) \le \mathcal{I}^*(D)$,

$$r=\frac{\mathcal{I}^*(D)-\hat{\mathcal{I}}^*(D)}{\hat{\mathcal{I}}^*(D)}=\frac{\mathcal{I}^*(D)}{\hat{\mathcal{I}}^*(D)}-1 \le \frac{\mathcal{I}(\hat{Q})}{\hat{\mathcal{I}}^*(D)}-1,$$

and when $\hat{\mathcal{I}}^*(D) > \mathcal{I}^*(D)$,

$$r=\frac{\hat{\mathcal{I}}^*(D)-\mathcal{I}^*(D)}{\hat{\mathcal{I}}^*(D)} \le \frac{\hat{\mathcal{I}}^*(D)}{\mathcal{I}^*(D)}-1 \le \frac{\hat{\mathcal{I}}^*(D)}{LB(D)}-1.$$

Therefore, eb is a bound on the approximation error. ■

Proof of Property 10. According to (16), we only need to show that $\hat{Q}$ is concave and increasing with respect to D, which is true because of (15). ■

Proof of Property 11. It follows from (15) that

$$\hat{Q} = D \cdot \frac{-\bar{A}/\bar{B} + \sqrt{\bar{A}^2/\bar{B}^2 + \frac{2\alpha(\bar{A}/\bar{B}+1)\left[\frac{\alpha F}{D}+(\bar{A}/\bar{B})(\pi-a)\right]}{\alpha a+h}}}{(\bar{A}/\bar{B}+1)\alpha} \tag{21}$$

(a) Since $\bar{A} = \frac{\lambda(\alpha+\beta)}{\beta\psi(\alpha+\lambda+\psi)}$ and $\bar{B} = \frac{1}{\alpha} + \frac{1}{\beta}$, we have

$$\bar{A}/\bar{B} = \frac{\alpha\lambda}{\psi(\alpha+\lambda+\psi)}, \tag{22}$$

which is independent of β. Therefore, $\hat{Q}$ is independent of β by (21).

(b) It follows from (22) that $\bar{A}/\bar{B}$ is monotone with respect to λ and ψ. Thus, to prove (b), it suffices to show that (21) has a monotone relationship with $\bar{A}/\bar{B}$. Letting $X = \bar{A}/\bar{B}$ in (21), this is equivalent to showing that *either* for any $\epsilon > 0$, the equation

$$\frac{-X + \sqrt{X^2 + \frac{2\alpha(X+1)(\frac{\alpha F}{D}+X(\pi-a))}{\alpha a+h}}}{(X+1)\alpha} = \epsilon \tag{23}$$

has at most one nonnegative solution *or* the left-hand side is a constant (i.e., there exists $\epsilon > 0$ such that any nonnegative X is a solution to (23)).

(23) can be transformed into the equation

$$X^2 + \frac{2\alpha(X+1)\left[\frac{\alpha F}{D}+X(\pi-a)\right]}{\alpha a+h} = (X+\epsilon\alpha(X+1))^2,$$

which is equivalent to

$$\frac{2(\frac{\alpha F}{D}+X(\pi-a))}{\alpha a+h} = 2\epsilon X + \epsilon^2\alpha(X+1) \tag{24}$$

when X is nonnegative.

Since (24) is a linear equation of X, it, and hence (23), either has at most one nonnegative solution X or any nonnegative X is its solution, which completes the proof.

(c) By (22), following a similar transformation as in the proof of part (b), Equation (21) can be transformed into

$$\frac{2\left[\frac{F\psi(\alpha+\lambda+\psi)}{D}+\lambda(\pi-a)\right]}{\alpha a+h} = \frac{2\lambda\hat{Q}}{D} + \left(\frac{\hat{Q}}{D}\right)^2(\alpha+\psi)(\lambda+\psi),$$

which is equivalent to

$$\frac{2F\psi}{aD}+\frac{2\left[\frac{F\psi}{D}\left(\lambda+\psi-\frac{h}{a}\right)+\lambda(\pi-a)\right]}{\alpha a+h}=\frac{2\lambda\hat{Q}}{D}+\left(\frac{\hat{Q}}{D}\right)^2(\alpha+\psi)(\lambda+\psi) \tag{25}$$

Under the condition $\psi \geq \frac{h}{a}$, when α increases, the left-hand side of (25) is decreasing. Therefore, $\hat{Q}$ must be decreasing as α increases to make Equation (25) valid. ■

Proof of Corollary 2. As $\bar{A}/\bar{B} \to 0$, it follows from (21) that

$$\hat{Q}=D\cdot\frac{\sqrt{\frac{2\alpha(\frac{\alpha F}{D})}{\alpha a+h}}}{\alpha}=\sqrt{\frac{2FD}{\alpha a+h}}$$

As $\bar{A}/\bar{B} \to +\infty$, it follows from (21) that

$$\hat{Q}=D\cdot\frac{-\bar{A}/\bar{B}+\sqrt{\bar{A}^2/\bar{B}^2+\frac{2\alpha(\bar{A}/\bar{B})((\bar{A}/\bar{B})(\pi-a))}{\alpha a+h}}}{(\bar{A}/\bar{B})\alpha}=D\cdot\frac{-1+\sqrt{1+\frac{2\alpha(\pi-a)}{\alpha a+h}}}{\alpha}$$

If (17) holds, then

$$D>\frac{\frac{\sqrt{2FD}}{\sqrt{\alpha a+h}}\left(\sqrt{1+\frac{2\alpha(\pi-a)}{\alpha a+h}}+1\right)}{\frac{2(\pi-a)}{\alpha a+h}}$$

$$\Rightarrow \frac{-1+\sqrt{1+\frac{2\alpha(\pi-a)}{\alpha a+h}}}{\alpha}\cdot D>\sqrt{\frac{2FD}{\alpha a+h}}.$$

Therefore, the value of $\hat{Q}$ as $\bar{A}/\bar{B} \to +\infty$ is greater than that as $\bar{A}/\bar{B} \to 0$. Since $\hat{Q}$ has a monotone relationship with $\bar{A}/\bar{B}$ by Property 11(b), $\hat{Q}$ is an increasing function of $\bar{A}/\bar{B}$. Therefore, when (17) holds, $\hat{Q}$ is an increasing function of λ and a decreasing function of ψ. The second part of this corollary, when (17) is violated, can be proved similarly. ■

Appendix B: Constant Demands vs. Stochastic Demands

We assumed constant customer demand when we formulated the cost function in Section 3. We now conduct a simulation study to determine whether solutions derived from this cost function are "robust" regarding the customer demand types.

We built a simulation model using Arena 7.01 in which we relax the constant-demand assumption. Instead, we assume that demand follows a Poisson process with annual rate D. In the results reported below, we fix $F = 6$, $a = 2$, $h = 0.2$, $\pi = 10$ and $\psi = 12$, and vary λ, α and β. For each set of α, β and λ reported in Table 5, we

Table 5: Simulation studies on the constant customer demand assumption

α	β	λ	Average $(SIM^* - SIM)/SIM$	Average $(\widehat{SIM} - SIM)/SIM$
0.1	12	0.1	0.010436	0.009975
1	12	0.1	0.000000	0.009021
0.1	12	1	0.003682	0.006605
1	12	1	0.029310	0.047618
0.1	24	0.1	0.000000	0.000000
1	24	0.1	0.000701	0.002322
0.1	24	1	0.024692	0.062559
1	24	1	0.082106	0.074982
5	12	1	0.015640	0.093256
1	12	5	0.067794	0.063519
5	12	5	0.024135	0.011498
5	24	1	0.033106	0.062231
1	24	5	0.074540	0.046959
5	24	5	0.081012	0.036916

1. randomly draw values from the interval [1,10000] to serve as the annual demand rate D;
2. calculate the corresponding Q^* using bisection search;
3. calculate the corresponding $\hat{Q}$ using (15);
4. run 50 replications of the simulation model using Q^* and $\hat{Q}$ as the order sizes, respectively, and use SIM^* and $\widehat{SIM}$ to denote the corresponding resulting total costs;
5. seek the minimal total cost using the package Optquest for Arena by adjusting the order size, and denote SIM to be the resulting minimal total cost;
6. calculate $(SIM^* - SIM)/SIM$ and $(\widehat{SIM} - SIM)/SIM$ to see whether Q^* and $\hat{Q}$ perform well even under stochastic customer demand.

The simulation results are reported in Table 5. For each set of α, β and λ values, both of the "Average" columns reflect averages taken over the 50 replications. The average errors in Table 5 are all less than 10% and are mostly less than 5%, suggesting that the solutions obtained under the constant-demand assumption still perform well when this assumption is relaxed.