Inventory Sharing under Decentralized Preventive Transshipments

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Abstract

We consider preventive transshipments between two stores in a decentralized system with two demand subperiods. Replenishment orders are made before the first subperiod, and the stores may make transshipments to one another between the subperiods. We prove that the transshipment decision has a dominant strategy, called a control-band conserving transfer policy, under which each store chooses a quantity to transship in or out that will keep its second-subperiod starting inventory level within a range called a control band. We prove that the optimal replenishment policy is a modified base-stock policy in which the order-up-to level depends on the initial inventory and capacity level at the other store. Finally, we prove that there does not exist a transfer price that coordinates the decentralized supply chain. Our research also explains many of the differences between preventive and emergency transshipments, including differences in the optimal transfer policies and the existence or nonexistence of transfer prices that coordinate the system.

Keywords: inventory sharing; preventive transshipment; dominant strategy

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1. Introduction

Risk pooling is an effective means for improving the performance of a supply chain. Since physical pooling of inventory is often impractical, preventive transshipments serve as an important tool for realizing partial risk pooling by reducing inventory imbalances among different locations and helping firms cope with future demand uncertainty. Risk pooling in this sense need not be only within a single organization but may be extended across organizations. In this study, we use a game-theoretic approach to study the optimal behavior of two stores that may transship to one another between replenishment cycles, but that make ordering and transshipment decisions independently in a decentralized control environment.

We consider two independent stores selling the same product and facing separate demands. The time horizon includes two subperiods with demand realizations in each. The stores order the product from a supplier at the beginning of the first subperiod. After first-subperiod demands are realized, it may be beneficial for both stores to transship some products from one store to another in order to achieve a better starting-inventory level for the second subperiod.

Tagaras [34] defines two types of transshipments under periodic review. **Emergency** transshipments occur at the end of the period, after demand information is fully realized. **Preventive** transshipments occur between two consecutive subperiods, when only partial demand information has been realized. If demand in the second sub-period is zero, the preventive transshipment problem reduces to a single-period emergency transshipment problem since there is no further demand information in the second sub-period.

Preventive transshipments are advantageous if the transshipment leadtime is positive and/or unsatisfied demand is lost since in these cases, transshipped items cannot be used to satisfy demands that have already occurred. Sometimes preventive transshipments are a strategic choice in order to reduce stockout costs and to grant companies more flexibility to handle the uncertainty in the system. Our paper is concerned primarily with preventive transshipments.

Coordination under decentralized systems is essential for strategic alliances among companies. Although recent years have seen a rising trend of merger-and-acquisition (M&A) activities in the retail industry [5] that may help retailers achieve economies of scale, many firms choose instead to form strategic alliances to achieve some of the benefits of M&A while maintaining their own identities. Our study can also be used to compare these two alternatives.

The decentralized preventive transshipment setting can also be used to model other business environments. For example, an original equipment manufacturer (OEM) in the electronics industry may use its excess capacity to produce for its competitor [28]. Transshipments then represent re-allocating the two firms' allotted manufacturing capacity. Moreover, even within an organization, functional managers may act in their own interest [23]. Then transshipments can help to relieve the losses due to managers' unaligned incentives. Suppose that a company manufactures a variety of products and each is operated by a different product manager. Common components can be transshipped among product lines to balance the production needs of different products. Similarly, in the service industry, different groups within a company may recruit employees individually, but they may re-assign staff members later to balance project workloads.

In some cases, preventive transshipments can be a vital tool in rectifying critical shortages. In October 2004, the U.S. faces a serious shortage of flu vaccine shots. However, the manufacturing process for flu vaccines takes months, so that even expedited production would have arrived too late. Some U.S. citizens travelled to Canada to buy flu shots, but many went without [11]. One of the solutions proposed is to have reciprocal regulatory agency approvals so that Canada and Europe can transship vaccines to the U.S. in case of future shortages [30]. Moreover, the transshipments should be made in advance, before the flu season, since it takes time for flu shots to take effect. This reinforces the necessity of proactive preventive transshipments rather than reactive emergency transshipments.

1.1 Structure of Optimal Transfer and Replenishment Policies

We analyze the optimal ordering and transshipment behavior for each store using backward induction. We first prove that each store has a dominant transshipment strategy, called a control-band conserving (CBC) transfer policy under which the store attempts to keep its second-subperiod starting inventory level within a "control band." The policy has two parameters, a transship-up-to level and a transship-down-to level. If the store begins the second subperiod with an inventory level below [above] the control band, it proposes to transship in [out] to bring its inventory level to the transship-up-to [transship-down-to] level, and if its inventory level is already within the control band, it does not transship in or out.

Next, we prove that there is a unique pair of order quantities that is a Nash equilibrium (NE) for both stores. Then we extend our analysis to account for non-zero starting inventory and finite capacity and prove that each store's optimal first-subperiod replenishment policy, accounting for its own and its partner's optimal transshipment policies, is a modified base-stock policy under which the optimal base-stock level depends on the initial inventory and storage capacity of the other store.

1.2 Non-existence of Coordinating Linear Transfer Price

We prove that there is no linear transfer price that can coordinate the supply chain under the decentralized system by showing that the CBC transfer policy cannot be optimal for the centralized system. (Linear transfer prices are standard in the literature on noncooperative transshipment games, e.g. Rudi et al. [31] and Hu et al. [19].) That is, transfer payments alone are not sufficient to make the decentralized transshipment system attain the same profit as the centralized one. This is a significant difference between emergency and preventive transshipments, as Hu et al. [19] give necessary and sufficient conditions for

Table 1: Contribution to Literature on Two-Store Transshipment Games

Transshipment Type	Optimal Transshipment Policy	Order Qty NE (0 Starting Inventory)	Optimal Order Policy	Coordinating Transfer Price?
Preventive	CBC transfer policy	Unique NE (Special case: Güllü et al. [16]; we prove in general)	Modified base-stock	Does not exist
Emergency	Complete pooling Rudi et al. [31]	Unique NE Rudi et al. [31]	Modified base-stock	Exists under certain conditions (e.g., identical retailers) Hu et al. [19]

the existence of a coordinating transfer price under emergency transshipments.

Moreover, for the purpose of comparison, we assume that the two stores are considering M&A or a strategic alliance. We define four supply chain configurations and compare the performance of each. In the *merged system* (MS), one of the two stores is closed and the other serves both demand streams. In the *centralized system* (CS), order and transshipment decisions are made centrally to maximize the total profit of both stores. In the *decentralized system* (DS), each store makes order and transshipment decisions independently in order to maximize its own profit. In the *separate system* (SS), no transshipment is allowed.

We can think of the MS and CS systems as two different forms that may occur as a result of M&A activities. In the SS system, the two stores do not cooperate in any way. The DS system may result from a strategic alliance; it is the primary focus of the analysis of this paper since the other three systems are well studied.

We evaluate the stores' losses and gains under these four configurations, and prove that the largest profit, ordered from the best to the worst, is generated from the MS, CS, DS, and SS systems. In our computational study, we compare the total profit in the four systems and provide managerial insights.

1.3 Contributions

Table 1 summarizes the results that have been established in the literature on two-store transshipment games under periodic review and our contribution to this literature. (We review the literature more thoroughly in Section 2.) Results in cells without citations are proved in this paper. In particular:

1. We prove the form of the dominant preventive transshipment strategy and provide an explanation of the difference between the form of the policy under emergency and preventive transshipments.

2. We prove that there is a unique order-quantity NE for general preventive transshipments.

3. We prove that the optimal order policy is a modified base-stock policy. This policy is also optimal for emergency transshipments since they are a special case of preventive transshipments.

4. We prove that there is no coordinating transfer price under decentralized preventive transshipments.

5. We explain property differences between emergency and preventive transshipments.

The remainder of this paper is structured as follows. We review the related literature in Section 2. In Section 3, we describe the model framework. We analyze the structure of the optimal transfer and replenishment policies under the decentralized preventive transshipment game in Sections 4 and 5. In Section 6, we compare the profitability of the four different supply chain configurations and show that there is no coordinating transfer price. We present a numerical study in Section 7, and in Section 8, we conclude our findings and suggest possible avenues for future research.

2. Literature Review

We first review the literature on centralized transshipments and then on decentralized transshipments.

Herer and Rashit [17], Herer et al. [18], Hu et al. [20], Karmarkar [22], Robinson [29], Tagaras [34, 35], Tagaras and Cohen [36], and Wee and Dada [37], among others, study emergency transshipments. There is a considerably smaller body of literature on preventive transshipments. Das [10] studies a two-subperiod model with a convex, differentiable preventive transshipment cost function. Assuming that the derivative of the transshipment cost function is sufficiently large or that the transshipment cost function is subadditive, he proves that the optimal transfer policy is what he calls a "base-stock conserving" (BSC) transfer policy. In a BSC transfer policy, each store sets a base-stock level for the second subperiod. After the demand in the first subperiod is realized, if one store's inventory level is less [greater] than its base-stock level, it wants to transship in [out] to bring its inventory level as close to its base-stock level as possible. The BSC transfer policy is a special case of the control-band conserving (CBC) transfer policy we propose in Section 4 in which the transship-up-to and transship-down-to levels are equal.

Jonsson and Silver [21] assume that replenishment orders occur every H periods and that transshipments are allowed at the end of the $(H-1)$st period. Replenishments and transshipments may both have positive lead times. The paper studies the impact of preventive transshipments on service level. Comez et al. [8] divide the replenishment cycle into N short periods and assume that at most one unit of demand may occur in each short period, and at only one of the two retailers. Transshipment lead times may be positive. Each retailer sends a transshipment request to the other when it runs out of inventory in a replenishment cycle. The paper provides an analysis of how much inventory each retailer should withhold based on of the number of remaining short periods until the next replenishment cycle.

Papers considering transshipments under continuous review include Axaster [3, 4], Archibald et al. [2], Kukreja et al. [24], and Lee [26]. Since our model assumes periodic review, we omit a more thorough description of these papers.

The papers cited above all assume a centralized control mechanism. Several recent studies have focused on decentralized transshipments. In a decentralized system, the store transshipping out typically pays the actual transshipment costs (e.g., transportation), whereas the store transshipping in pays the other store a transfer price. The stores' performance is evaluated individually. Rudi et al. [31] prove that in a single-period model with emergency transshipments, there is a unique Nash equilibrium (NE) of order quantities placed by the two stores at the beginning of the period. Hu et al. [19] incorporate unreliable suppliers into the setting proposed by Rudi et al. [31]. They show that there sometimes exists a transfer price under which the decentralized system achieves the same level of profit as the centralized system does, and they provide necessary and sufficient conditions for this to occur. (In contrast, we prove that a coordinating transfer price never exists for preventive transshipments).

Anupindi et al. [1], Granot and Sosic [15], and Sosic [33] study noncooperative games for order decisions and cooperative games for emergency transshipment decisions. Comez et al. [9] study the equilibrium behavior of two competing retailers when customers may switch between retailers if the first one they visit has a stockout and transshipment does not take place. Dong and Rudi [12] study the impact of wholesale prices in a one-wholesaler, N-identical-retailer setting with emergency transshipments. Zhang [38] provides an alternative proof for Dong and Rudi's main result. Zhao and Atkins [41] show that competition among retailers reduces the effectiveness of emergency transshipments.

Trading in a secondary market is another mechanism to realize something akin to transshipments. The transfer price is equal to the equilibrium price at which demand is equal to supply. Lee and Whang [25] study a two-period model with an infinite number of identical retailers. Trading is allowed at the end of the first period. Under this setting, the equilibrium price is independent of the realization of demand in the first period. Under the equilibrium price, each retailer's actual need for transshipments can always be satisfied, which is not the case with an exogenous transfer price for a small number of retailers, as in the models by Rudi et al. [31], Hu et al. [19] and ours. Chod and Rudi [7] study a model with trading under an equilibrium price employed by two retailers to hedge against an uncertain demand curve. Trading occurs after the investment decision (e.g. replenishment decision) and before the pricing decision. When the demand curve is isoelastic, they show that a decentralized system can perform as well as a centralized system.

The studies of decentralized transshipments mentioned above assume periodic review. Under continuous review with Poisson-distributed demand, Zhao et al. [40] propose an (S, K, Z) policy, where S, K, and Z represent, respectively, the order-up-to level, the demand-filling rationing level, and the requesting rationing level. They prove that the (S, K, Z) policy is optimal for each retailer when the rate at which it receives transshipment requests and the acceptance ratio of the transshipment requests it sends out are exogenous. Then they impose the (S, K, Z) policy on a 2-retailer decentralized system. Since the exact cost function is hard to analyze, they characterize the equilibrium behavior of two symmetric retailers based on an approximate cost function using the (S, K, Z) policy. (In contrast, we prove that the control-band conserving (CBC) transfer policy, defined in Section 4, also has two rationing levels and is a dominant transfer policy in a 2-store decentralized preventive transshipment system.) Earlier work by Grahovac and Chakravarty [14] and Zhao et al. [39] essentially study (S, K, K) and $(S, K, 0)$ policies, respectively, which are special cases of the (S, K, Z) policy.

The paper most closely related to ours is that of Güllü et al. [16]. They study a game in which two retailers can rebalance their orders from the supplier after partial demand information is observed and show that there exists a unique NE of order quantities. Their setting, which is abstracted from a cross-docking system, is equivalent to preventive transshipments with backorders, with the transfer price equal to the purchase cost, and with zero transshipment cost. This is a special case of our setting. Therefore, we provide a more extensive analysis of preventive transshipments under a more general range of assumptions.

3. Model Framework

Suppose there are two stores, indexed 1 and 2, owned by two different companies. Our analysis of this decentralized system considers a base case in which: (a) there is no capacity constraint for replenishment orders, and (b) unmet demands at the end of the first subperiod are lost. We relax (a) in 5.3 to derive the optimal replenishment policy, and we analyze the backorder case in the Appendix.

Let $\neg i \equiv 3 - i$, that is, $\neg i$ represents the index of the store other than store i, where $i = 1, 2$. Let π_i^j represent the profit of store i from subperiod j to subperiod 2 for $j = 1, 2$. Let Q_i and x_i represent the order quantity and proposed transshipment amount at store i.

The sequence of events for our base case is:

1. Store i places a replenishment order of size Q_i at the beginning of the first subperiod with unit cost c_i.

2. Store i observes its first-subperiod demand d_i^1 and earns a per-unit selling price p_i. It receives revenue $p_i \min(d_i^1, Q_i)$ in the first subperiod and observes its remaining inventory $I_i = \max(Q_i - d_i^1, 0)$. Unmet demand is lost.

3. Store i determines its proposed transshipment quantity x_i. If $x_i > 0$, then store i wants to transship out x_i units; if $x_i < 0$, then it wants to transship in $-x_i$ units. The actual transshipment amount from store i to store $\neg i$ is denoted $z_{i,\neg i}$, with $z_{i,\neg i} = -z_{\neg i,i}$. These quantities are determined based on x_i and $x_{\neg i}$. For example, if store 1 wants to sell x_1 units ($0 < x_1 \leq I_1$) and store 2 wants to buy $-x_2$ units ($x_2 < 0$), then store 1 transships out $z_{12} = \min(x_1, -x_2) > 0$ units, store 2 transships out $z_{21} = -z_{12} < 0$ units, and the inventory level of store i becomes $I_i - z_{i,\neg i}$. The other possible transshipment scenarios are similar.

   The store transshipping in pays a per-unit transfer price p^t to the store transshipping out. The store transshipping out pays a per-unit transshipment cost c^t, which is an actual cost incurred during transshipment (e.g., the shipping cost). Note that, whereas p^t is simply transferred between parties in the supply chain, c^t is paid externally and reduces the total supply chain profit. Both p^t and c^t are exogenous and fixed. (The exogenous value of p^t comes from an agreement between the two stores prior to the beginning of the game.)

4. Store i observes its second-subperiod demand d_i^2 and earns the same per-unit selling price, p_i. Unmet demand is lost. Excess inventory is salvaged with a unit salvage value of l_i. The revenue in the second subperiod is equal to $p_i \min(I_i - z_{i,\neg i}, d_i^2)$ and the salvage value is equal to $l_i \max(I_i - z_{i,\neg i} - d_i^2, 0)$.

To avoid trivial cases, we require $l_i < p^t - c^t$ and $p^t < p_i$ (transshipping out is preferable to salvaging but inferior to selling ($l_i < p^t - c^t < p_i$), and transshipping in can be done at a profit ($p^t < p_i$)) and $l_i < c_i < p_i$ (sales are made at a profit, and salvages are made at a loss). We summarize the cost and revenue parameters in Table 2.

The random demand at store i in subperiod j is denoted D_i^j, with realization d_i^j. The D_i^j are independent but not necessarily identically disttributed. F_i $[f_i]$ and G_i $[g_i]$ represent the cdf [pdf] of demand at store i in the first subperiod and the second subperiod, respectively. To simplify the analysis, we require $0 < f_i(d) < \infty$ and $0 < g_i(d) < \infty$ if $d > 0$ and $f_i(d) = g_i(d) = 0$ otherwise. Any demand distribution can be modified slightly to satisfy these conditions, for example by adding light tails where the distribution has zero probability mass.

Note that by requiring $0 < g_i(d)$, we are ruling out the case in which the second-subperiod demand is always 0 to prevent the preventive transshipment problem from reducing to a single-period emergency transshipment problem. Therefore, some of the results in this research do not strictly apply to the single-period emergency transshipment problem. For example, see Theorem 4 and the discussion that follows.

Throughout, we use $\overline{A}$ to denote the complement of a set A and $E[\cdot]$ to denote expectation. We use $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$.

Table 2: Cost and Revenue Parameters

c_i	purchase cost for store i
p_i	selling price for store i
l_i	salvage value for store i
p^t	transfer price
c^t	transshipment cost

Note: if the index i is dropped, then parameter represents the value for store 1.

4. Dominant Transfer Policy

We study the two-store preventive transshipment game over two subperiods using backward induction. That is, we study the transshipment decision first and then study the preceding order decision, given that the optimal transshipment strategy has been implemented. In this section, we show that the transshipment decision in the second subperiod has a dominant strategy, (that is, each store's optimal proposed transshipment quantity is independent of the other store's proposed transshipment quantity) and we prove the optimal form of the transfer policy.

Below, we focus our analysis on store 1 WLOG. We omit the index 1 for store 1 whenever no confusion is caused.

Store i proposes to transship x_i at the beginning of the second subperiod. If $x_1 > 0$, then the actual amount transshipped out from store 1 is $\max(0, \min(x_1, -x_2))$; if $x_1 < 0$, then the amount transshipped in to store 1 is $\max(0, \min(-x_1, x_2))$. Let I_i be store i's inventory level at the end of the first subperiod and let $I \equiv I_1$. Let $\pi^2(x_1, I, x_2)$ be the profit function of store 1 in the second subperiod given the proposed transshipment amounts x_1 and x_2 and the inventory I after the first subperiod at store 1. Referring to the sequence of events in Section 3, we get:

$$\pi^2(x_1, I, x_2) = \begin{cases} pI + (l-p)E[(I-D^2)^+], & x_1, x_2 \le 0 \text{ or } x_1, x_2 \ge 0 \\ \\ pI + (p-p^t)\min(-x_1, x_2) & x_1 < 0, x_2 > 0 \\ \quad + (l-p)E[(I - D^2 + \min(-x_1, x_2))^+], & \\ \\ pI + (p^t - c^t - p)\min(x_1, -x_2) & x_1 > 0, x_2 < 0 \\ \quad + (l-p)E[(I - D^2 - \min(x_1, -x_2))^+] & \end{cases} \tag{1}$$

Let

$$\alpha_i^h = \frac{p_i - (p^t - c^t)}{p_i - l_i} \tag{2}$$

$$\alpha_i^l = \frac{p_i - p^t}{p_i - l_i}. \tag{3}$$

Note that, by the assumptions made on the cost parameters in Section 3, these two values are strictly between 0 and 1. We refer to $G_i^{-1}(\alpha_i^h)$ as store i's *transship-down-to level* and to $G_i^{-1}(\alpha_i^l)$ as its *transship-up-to level*.

Theorem 1. *Store i has the following dominant strategy:*

1. *When $I_i > G_i^{-1}(\alpha_i^h)$, propose to transship out $I_i - G_i^{-1}(\alpha_i^h)$;*
2. *When $I_i < G_i^{-1}(\alpha_i^l)$, propose to transship in $G_i^{-1}(\alpha_i^l) - I_i$;*
3. *Do nothing if $I_i \in [G_i^{-1}(\alpha_i^l), G_i^{-1}(\alpha_i^h)]$.*

(The proof of this theorem and all other results are provided in the Appendix.)

We call this policy a *control-band conserving* (CBC) transfer policy, which combines elements of a base-stock conserving transfer policy (see, e.g., Das [10]) and a control-band ordering policy (see, e.g., Zipkin [42]). A CBC policy has two conserving levels, the transship-down-to level and the transship-up-to level. Store i's proposed transshipment quantity depends only on store i's own starting inventory in the second sub-period and is not affected by the inventory and proposed transshipment quantity at the other store. When the first-subperiod ending inventory at store i is above the transship-down-to level, store i wants to reduce its inventory to as close to the transship-down-to level as possible by transshipping out. When the inventory is below the transship-up-to level, store i wants to increase its inventory to as close to the transship-up-to level as possible by transshipping in. Store i does nothing when the inventory lies between the band defined by the transship-up-to level and the transship-down-to level.

This policy is similar to a control-band policy [42, page 428] except that it refers to transshipment rather than replenishment/returns. The base-stock conserving (BSC) transfer policy by Das [10] is a special case of the CBC policy under which the transship-up-to level is equal to the transship-down-to level.

Figure 1 depicts the movement of both stores' on-hand inventories during the transshipment process following the dominant strategy in Theorem 1. If the on-hand inventories at the end of the first subperiod fall in area $\overline{A_1 \cup A_2}$, then either both retailers wish to transship out [in] or at least one store wishes to do nothing. Therefore, no actual transshipment occurs. If the on-hand inventories fall in area $A_1 \cup A_2$, then the inventory moves along the line $I_1 + I_2 = K$ toward the center, where K is the sum of the inventories of the two retailers after the first subperiod, and stops when it hits the boundary of $A_1 \cup A_2$.

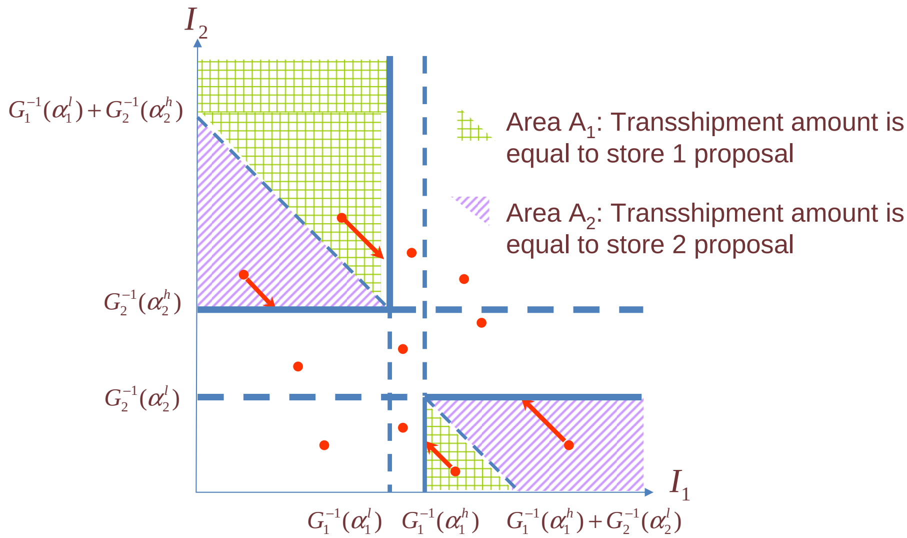


Figure 1: Relationship between Optimal Transshipments and First-Subperiod Ending Inventory

Theorem 1 is important for the implementation of the strategy, since the dominant strategy ensures that each store's transshipment proposal is independent of its opponent's order quantity and the realization of the demand in the first subperiod.

As the transfer price p^t increases, $(G_1^{-1}(\alpha_1^l), G_2^{-1}(\alpha_2^l))$ and $(G_1^{-1}(\alpha_1^h), G_2^{-1}(\alpha_2^h))$ move toward $(0,0)$. This changes the probability that a transshipment occurs. (The probability could increase or decrease, depending on the cost parameters.) As c^t increases, $(G_1^{-1}(\alpha_1^l), G_2^{-1}(\alpha_2^l))$ does not change, but $(G_1^{-1}(\alpha_1^h), G_2^{-1}(\alpha_2^h))$ moves away from $(0,0)$. This reduces the probability of transshipment. When $c^t = 0$, $(G_1^{-1}(\alpha_1^l), G_2^{-1}(\alpha_2^l))$ coincides with $(G_1^{-1}(\alpha_1^h), G_2^{-1}(\alpha_2^h))$. This is the exact form of the base stock conserving (BSC) transfer policy [10], in which the transship-up-to and transship-down-to levels are equal. Therefore, we have the following corollary.

Corollary 1. *When the transshipment cost is negligible, that is $c^t = 0$, the control-band conserving (CBC) transfer policy degenerates to the base stock conserving (BSC) transfer policy.*

Das proves that a BSC policy is optimal under centralized control when the transshipment cost function is convex and differentiable and its derivative is sufficiently large or the cost function is subadditive. We have now proved that, in addition, a BSC policy is optimal under decentralized control with zero transshipment cost.

Moreover, Rudi et al. [31] show that, under decentralized emergency transshipments, complete pooling (transferring as much inventory as is on-hand at one store to the other store, but no more than its demand shortfall) is optimal. This is because the stores do not need to conserve any inventory for their own future demand. But in the preventive transshipment setting, on-hand inventory can be retained to serve future demand. This causes different optimal transshipment policies under the two settings. However, emergency transshipments are a special case of preventive transshipments in which the demand in the second subperiod is equal to 0. In this case, $G_i(d) \approx 1$ for $d > 0$, and so $G_i^{-1}(\alpha_i^h) \approx G_i^{-1}(\alpha_i^l) \approx 0$. This is identical to the complete-pooling policy when the first-subperiod ending inventory (I) can be negative; that is, we allow backorders at the end of the first subperiod. Therefore, we have:

Corollary 2. *When $D_i^2 \equiv 0$ for $i = 1, 2$, the control-band conserving (CBC) transfer policy degenerates to the complete-pooling policy.*

According to the dominant strategy, we can obtain the optimal profit based on x_2 and I:

$$\pi^2(I, x_2) = \begin{cases} pI + (l-p)E[(I - D^2)^+], & I \leq G^{-1}(\alpha^h), x_2 \leq 0 \text{ or} \\ & I \geq G^{-1}(\alpha^l), x_2 \geq 0 \\ \\ pI + (p - p^t)\min(G^{-1}(\alpha^l) - I, x_2) & I < G^{-1}(\alpha^l), x_2 > 0 \\ +(l-p)E[(I - D^2 + \min(G^{-1}(\alpha^l) - I, x_2))^+], & \\ \\ pI + (p^t - c^t - p)\min(I - G^{-1}(\alpha^h), -x_2) & I > G^{-1}(\alpha^h), x_2 < 0 \\ +(l-p)E[(I - D^2 - \min(I - G^{-1}(\alpha^h), -x_2))^+], & \end{cases} \quad (4)$$

We can easily obtain Lemma 1:

Lemma 1.

1. *$\pi^2(I, x_2)$ is a continuous function in the vector (I, x_2).*

2. *$\pi^2(I, x_2)$ is an increasing function in I when x_2 is fixed.*

3. *$\pi^2(I, x_2)$ is an increasing function in x_2 when I is fixed to a value less than $G^{-1}(\alpha^l)$. $\pi^2(I, x_2)$ is a decreasing function in x_2 when I is fixed to a value greater than $G^{-1}(\alpha^h)$. $\pi^2(I, x_2)$ is a constant in x_2 when I is between $[G^{-1}(\alpha^l), G^{-1}(\alpha^h)]$.*

4. *Partial derivatives of $\pi^2(I, x_2)$ exist almost everywhere. Absolute values of the partial derivatives are bounded. In the area where partial derivatives do not exist, the subgradient is bounded too.*

Lemma 1.2 indicates that $\pi^2(I, x_2)$ increases in I since the purchase cost is calculated in the first subperiod. Lemma 1.3 shows that when one store wants to transship in [out], its profit increases [decreases] in the proposed transshipment amount of the other store. Since $\pi^2(I, x_2)$ involves min and max functions, it is possible that the partial derivative may not exist in a certain region such as at $x_2 = 0$. Lemma 1.4 indicates that the measure of such a region is zero. When we consider the ordering decision in the next subsection, we need $E[\pi^2(I, x_2)]$ to be differentiable with respect to I and x_2. Lemma 1.4 ensures this since the demand pdf is finite.

5. Optimal Replenishment Policy

In this section, we first assume that there is no initial inventory or capacity constraints at either store. Given that the optimal transshipment strategy has been implemented, we study the optimal order quantity for store i assuming that the transshipment amount for store $\neg i$ is exogenous. We then transform the results to the case in which the transshipment decision at store $\neg i$ is endogenous, based on its order quantity and first-subperiod realized demand. We show that the profit function π_i^1 is continuous and concave with respect to store i's own order quantity to establish the existence of a pure NE for the order quantities. Then we show that the best response function of the order quantity of store i decreases in the order quantity of store $\neg i$ and that its slope has absolute value less than 1. Therefore, the uniqueness of the pure NE is guaranteed by the contraction mapping theorem.

In 5.3, we relax the assumptions that the stores have unlimited storage capacity. We show that the optimal replenishment policy for each store is a modified base-stock policy in which the optimal base-stock level depends on storage capacity of the other store.

5.1 Exogenous Store-2 Transshipment Decision

We now determine the optimal order quantity for store 1. In this subsection, we assume that store 2 proposes a transshipment amount directly, regardless of its order quantity and realization of demand in the first subperiod, i.e. store 1 takes store 2's proposed transshipment amount as exogenous. (We relax this assumption in Section 5.2.) Then the total profit of store 1 is

$$\tilde{\pi}^1(Q, x_2) \quad = \quad (p-c)Q - pE[(Q-D^1)^+] + E[\pi^2((Q-D^1)^+, x_2)]. \tag{5}$$

The notation $\tilde{\pi}$ is meant to suggest that this is not the final profit function since it depends on x_2. (In Section 5.2 we will determine the profit as a function of Q_1 and Q_2 only.)

Lemma 2. *1. $\tilde{\pi}^1(Q, x_2)$ is continuous and partial derivatives exist everywhere.*

2. When $x_2 \leq 0$,

$$\begin{aligned} \frac{\partial \tilde{\pi}^1(Q, x_2)}{\partial Q} \quad = \quad & (p-c) + (p^t - c^t - p)\Pr(Q - G^{-1}(\alpha^h) + x_2 < D^1 \leq Q - G^{-1}(\alpha^h)) \\ & +(l-p)\left[\Pr(Q - G^{-1}(\alpha^h) < D^1 \leq Q - D^2)\right. \\ & \left. + \Pr(D^1 \leq Q - G^{-1}(\alpha^h) + x_2, D^1 + D^2 \leq Q + x_2)\right]. \end{aligned}$$

3. *When* $0 < x_2 \leq G^{-1}(\alpha^l)$,

$$\begin{aligned}\frac{\partial \tilde{\pi}^1(Q, x_2)}{\partial Q} &= (p-c) + (p^t - p)\Pr(Q - G^{-1}(\alpha^l) < D^1 \leq Q - G^{-1}(\alpha^l) + x_2) \\ &\quad + (l-p)\left[\Pr(Q - G^{-1}(\alpha^l) + x_2 < D^1 \leq Q, D^1 + D^2 \leq Q + x_2) \right. \\ &\quad \left. + \Pr(D^1 \leq Q - G^{-1}(\alpha^l), D^1 + D^2 \leq Q)\right].\end{aligned}$$

4. *When* $x_2 > G^{-1}(\alpha^l)$,

$$\begin{aligned}\frac{\partial \tilde{\pi}^1(Q, x_2)}{\partial Q} &= (p-c) + (p^t - p)\Pr(Q - G^{-1}(\alpha^l) < D^1 \leq Q) \\ &\quad + (l-p)\Pr(D^1 \leq Q - G^{-1}(\alpha^l), D^1 + D^2 \leq Q).\end{aligned}$$

5. $\frac{\partial \tilde{\pi}^1(Q, x_2)}{\partial Q}$ *is continuous and differentiable and its partial derivatives are bounded.*

6. $\tilde{\pi}^1(Q, x_2)$ *is a strictly concave function with respect to* Q. *In particular,* $\frac{\partial^2 \tilde{\pi}^1(Q, x_2)}{\partial Q^2} < 0$.

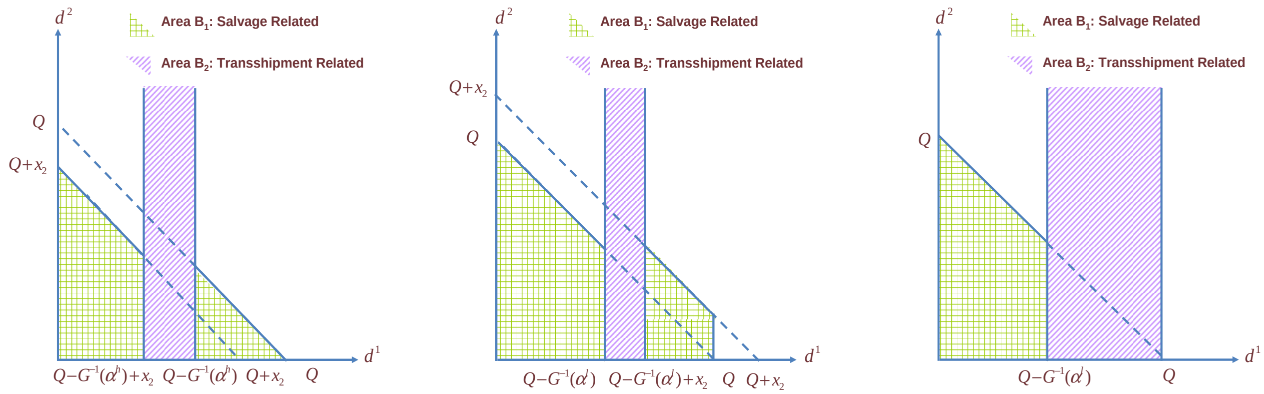
7. *When* $x_2 \leq G^{-1}(\alpha^l)$, *we have* $\frac{\partial^2 \tilde{\pi}^1(Q, x_2)}{\partial Q^2} < \frac{\partial^2 \tilde{\pi}^1(Q, x_2)}{\partial Q \partial x_2} < 0$. *When* $x_2 > G^{-1}(\alpha^l)$, *we have* $\frac{\partial^2 \tilde{\pi}^1(Q, x_2)}{\partial Q \partial x_2} = 0$.

In Figure 2, we show how to interpret $\frac{\partial \tilde{\pi}^1(Q, x_2)}{\partial Q}$ geometrically. The figure has three parts, corresponding to the three cases in Lemma 2.2, 2.3, and 2.4. Figure 2.a depicts the case $x_2 \leq 0$, which is addressed by Lemma 2.2. If (d^1, d^2) is located in the area B_1, that is, $d^1 + d^2 \leq Q + x_2$ and $d^1 \leq Q - G^{-1}(\alpha^h) + x_2$, or $d^1 + d^2 \leq Q$ and $d^1 \geq Q - G^{-1}(\alpha^h)$, any incremental order quantity δQ can be neither transshipped nor sold; it can only be salvaged. In both cases, store 1's current order quantity is sufficient to meet its own total demand. In the former case, it is also sufficient to meet store 2's transshipment request, while in the latter case, it does not wish to transship out. Therefore, in both cases, the additional order quantity δQ cannot be used either to transship or to satisfy store-1 demand. If (d^1, d^2) is in B_2, i.e., $Q - G^{-1}(\alpha^h) + x_2 \leq d^1 \leq Q - G^{-1}(\alpha^h)$, then the current order quantity is insufficient to meet store 2's transshipment request, so the incremental order quantity δQ is used to transship. In all other areas, the incremental order quantity δQ is used to satisfy customer demand.

Therefore, when $x_2 \leq 0$, the marginal cost of an increase of δQ is the purchase cost c, while the marginal revenue is increased sales to the customer with probability under area $\overline{B_1 \cup B_2}$, increased salvage value with probability under area B_1, and increased transshipment revenue with probability under area B_2. Similarly, Figure 2.b and 2.c depict the cases for Lemma 2.3 and 2.4.

5.2 Store 2 Makes Order Quantity Decision

We now extend the previous result to the case in which store 2 makes transshipment decisions endogenously based on its order quantity and realized demand in the first subperiod. The proposed transship-



(a: $x_2 \leq 0$) (b: $0 < x_2 \leq G^{-1}(\alpha^l)$) (c: $x_2 > G^{-1}(\alpha^l)$)

Figure 2: Interpretation of $\partial\tilde{\pi}^1(Q, x_2)/\partial Q$ for Lost-Sales Case

ment amount at store 2 is provided in Theorem 1; let $x_2(Q_2, d_2^1)$ denote this amount. Then

$$x_2(Q_2, d_2^1) = \begin{cases} (Q_2 - d_2^1)^+ - G_2^{-1}(\alpha_2^h), & \text{if } (Q_2 - d_2^1)^+ > G_2^{-1}(\alpha_2^h) \\ (Q_2 - d_2^1)^+ - G_2^{-1}(\alpha_2^l), & \text{if } (Q_2 - d_2^1)^+ < G_2^{-1}(\alpha_2^l) \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

The expected profit for store 1 is therefore

$$\pi^1(Q_1, Q_2) \quad = \quad E_{D_2^1}[\tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))] \tag{7}$$

Lemma 3. *1. $\pi^1(Q_1, Q_2)$ is continuous and differentiable.*

2. $\pi^1(Q_1, Q_2)$ is a strictly concave function with respect to Q_1.

3. $\frac{\partial^2\pi^1(Q_1,Q_2)}{\partial Q_1^2} < \frac{\partial^2\pi^1(Q_1,Q_2)}{\partial Q_1 \partial Q_2} \leq 0$. Therefore, store 1's best response function to the order quantity of store 2 (Q_2) is a decreasing function and the absolute value of its slope is less than 1.

Then we have:

Theorem 2. *There exists a unique Nash equilibrium of order quantities.*

Notice that we assume the initial inventory is zero at both stores. If the initial inventory is non-zero, results similar to Theorem 2 can be shown. This sets the stage for multi-period models in future studies.

5.3 Capacity Constraints

If the capacity is V_i^0 for store i, then store i needs to solve

$$\max_{Q_i \leq V_i^0} \pi_i^1(Q_i, Q_{\neg i}). \tag{8}$$

Let Q_i^{0*} be the NE order quantity for store i without capacity constraint.

Proposition 1. *Let $B_i(Q)$ be the best response function of the order quantity for store i when store $\neg i$ sets its order quantity equal to Q, ignoring the constraint $B_i(Q) \leq V_i^0$. Then for $i = 1, 2$,*

1. *When $V_i^0 \leq Q_i^{0*}$ and $V_{\neg i}^0 > Q_{\neg i}^{0*}$, then $Q_i^* = V_i^0$ and $Q_{\neg i}^* = \min(V_{\neg i}^0, B_{\neg i}(V_i^0))$.*
2. *When $V_i^0 > Q_i^{0*}$ and $V_{\neg i}^0 \leq Q_{\neg i}^{0*}$, then $Q_i^* = \min(V_i^0, B_i(V_{\neg i}^0))$ and $Q_{\neg i}^* = V_{\neg i}^0$.*
3. *When $V_i^0 \leq Q_i^{0*}$ and $V_{\neg i}^0 \leq Q_{\neg i}^{0*}$, then $(Q_i^*, Q_{\neg i}^*)=(V_i^0, V_{\neg i}^0)$.*
4. *When $V_i^0 > Q_i^{0*}$ and $V_{\neg i}^0 > Q_{\neg i}^{0*}$, then $(Q_i^*, Q_{\neg i}^*)=(Q_i^{0*}, Q_{\neg i}^{0*})$.*

Moreover, the NE of order quantities is unique in all cases.

This proposition shows that the stores should employ a modified base-stock policy under which the modified order-up-to level of one store is determined by the other store's capacity level. The following theorem follows immediately.

Theorem 3. *Store i's optimal order policy is a modified base-stock policy under which the unique optimal order-up-to level is $S_i^*(V_{\neg i}^0) = B_i(\min(Q_{\neg i}^{0*}, V_{\neg i}))$. Moreover, $S_i^*(V_{\neg i}^0)$ is decreasing in $V_{\neg i}$.*

Note that Lemma 3.3 also applies to the best response functions under emergency transshipments. Therefore, the same proof method can be employed to show that a modified base-stock policy is also optimal for a single-period emergency transshipment. Therefore, we have the following corollary.

Corollary 3. *In a single-period decentralized emergency transshipment problem, store i's optimal replenishment policy is a modified base-stock policy.*

6. Non-Existence of Coordinating Linear Transfer Price

In this section we prove that there is no coordinating linear transfer price. In order to show this property and evaluate the effectiveness of decentralized preventive transshipments, we introduce three other supply chain configurations.

The decentralized system (DS) is studied in Sections 4 and 5. We now study three other supply chain configurations that might result from possible alliances between the two stores. In the merged system (MS), there is only one store left to serve both stores' customers. We assume that the remaining store inherits the best properties of the two stores. In particular: it serves demand from both stores (there are no customer losses due to the closure), the unit purchase cost is $c^m = \min(c_1, c_2)$, the unit selling price is $p^m = \max(p_1, p_2)$, and the unit salvage value is $l^m = \max(l_1, l_2)$.

The centralized system (CS) is similar to DS except that both order and transshipment decisions are made in a centralized manner to maximize total supply chain profit. In the separate system (SS), transshipments are not allowed. That is, $x_i \equiv 0$.

Let π_i^d and π_i^s represent the profit of store i over the two subperiods for the DS and SS systems, respectively. Let π^m, π^c, π^d and π^s represent the total profit of the two stores over the two subperiods for MS, CS, DS and SS systems, respectively.

We provide formulations for the MS, CS and SS systems in the Appendix.

π^{m*} denotes the optimal profit for the MS system, and similarly for the other systems. Then we have the following proposition.

Proposition 2. *1. $\pi^{m*} \geq \pi^{c*} \geq \pi^{d*} \geq \pi^{s*}$, that is, $MS \geq CS \geq DS \geq SS$ in terms of total profit of the system.*

2. $\pi_i^{d*} \geq \pi_i^{s*}$, $i = 1, 2$, that is, both stores benefit from DS compared with SS.

Proposition 2 indicates that MS is the best configuration. However, we have ignored the fact that closing one store usually results in a loss of customers. Taking this into consideration, MS may not be better than the other systems. Nevertheless, our MS system is still useful to study since it provides a benchmark for the best results that the system can possibly achieve. The proposition also indicates that participating in transshipment coordination always bring a benefit to both stores.

The next question is whether there exists a transfer price such that the whole supply chain is coordinated. Let Q_i^{c*} be the optimal order quantity of store i under the CS system and I_i be the inventory level at the beginning of the second subperiod at store i. Suppose that when $I_i = 0$ and $I_{\neg i} = Q_{\neg i}^{c*}$, it is optimal for store $\neg i$ to transfer a positive quantity to store i for $i = 1, 2$ under the CS system. In other words, there is a non-zero probability of the transshipments occurring between the two stores in both directions under the optimal CS system. Under this mild condition, we prove that a CBC transfer policy cannot be optimal in the CS system, and therefore that there is no coordinated transfer price.

Theorem 4. *When $I_i = 0$ and $I_{\neg i} = Q_{\neg i}^{c*}$, it is optimal for store $\neg i$ to transfer a positive quantity to store i for $i = 1, 2$ under the CS system. Then we have*

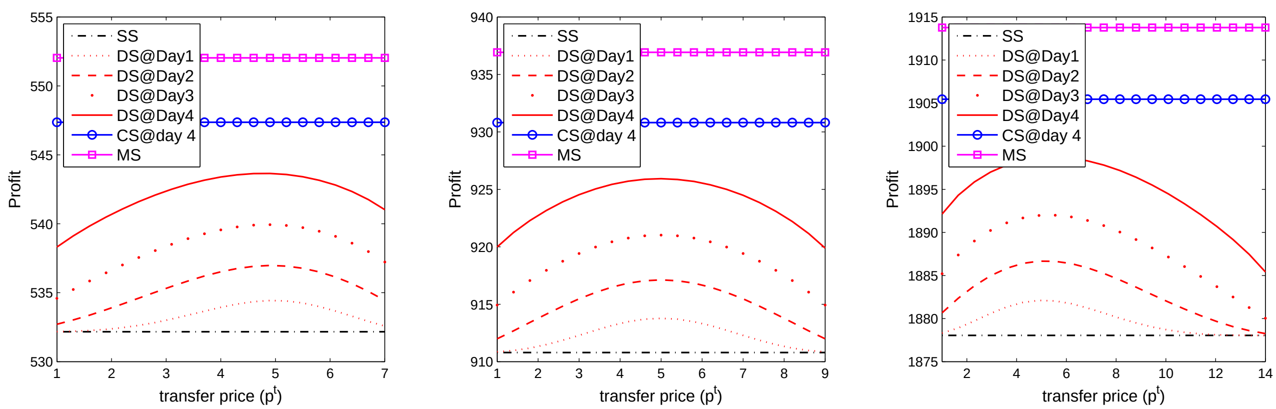
1. *A CBC transfer policy cannot be optimal in the CS system.*
2. *Let $\pi^{d*}(p^t)$ be the optimal profit in the DS system with transfer payment equal to p^t. Then there is no p^t such that $\pi^{c*} = \pi^{d*}(p^t)$.*

In contrast to our result, Rudi et al. [31] and Hu et al. [19] show that there may exist transfer prices that coordinate the system under emergency transshipments. Our result differs from theirs because a complete-pooling policy (defined in Section 4) is optimal under both the centralized and decentralized systems for emergency transshipments, making a coordinating transfer price possible. But under preventive transshipments, the optimal transshipment policies are different for the centralized and decentralized systems.

7. Numerical Study

In this section, we provide a numerical study to demonstrate several results from the previous sections, as well as to provide additional managerial insights.

We assume that there are two identical stores close to each other. Their demands per day from Monday to Friday are identically and independently normally distributed with mean (μ) 20 and standard deviation (σ) 5. The probability of negative demand can be ignored. Assume that there are four possible time points to have a preventive transshipment: the end of Monday, Tuesday, Wednesday or Thursday. We examine which day is the most beneficial to transship between the two stores. We set the salvage value $l = 0$, the purchase cost $c = 5$ and the transshipment cost $c^t = 0$. We consider three types of products, with selling price $p = 8, 10, 15$, that represent low-end, middle-end and high-end products, respectively. We want to compare the performance of the MS, CS, DS and SS systems. We calculate the optimal order quantities and profit under different systems. The optimal order quantities for CS and DS are solved numerically; the uniqueness of these solutions is guaranteed by Theorem 2 and Proposition 3.

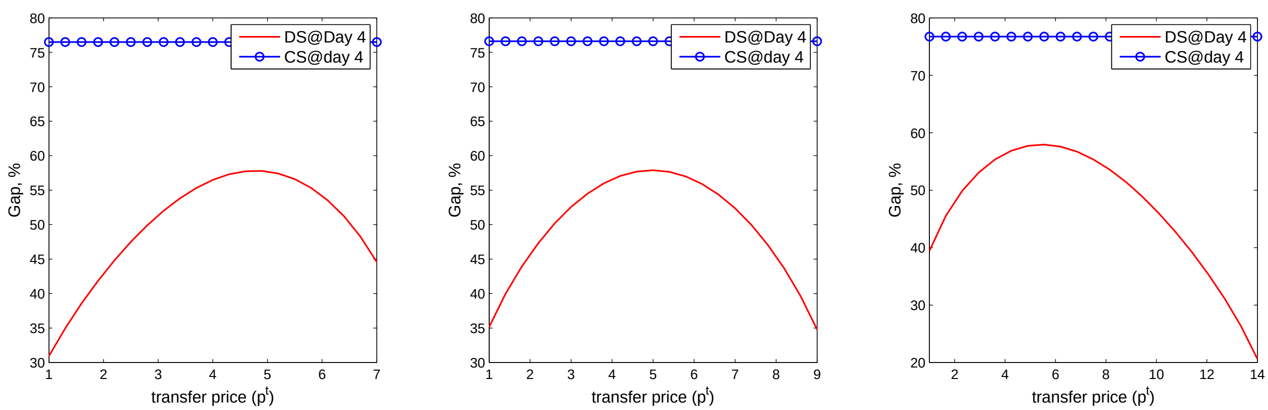


(a: $p = 8$) (b: $p = 10$) (c: $p = 15$)

Figure 3: Relationship between Total Profit and Transfer Price

In Figure 3, we plot the total profit as a function of p^t under various control mechanisms. Notice that only π^{d*} is affected by p^t, the unit transfer payment between the two stores, since the SS system completely ignores the transshipment, and the MS and CS systems is centralized, making the transfer payment irrelevant. From Figure 3, $\pi^{m*} \geq \pi^{c*} \geq \pi^{d*} \geq \pi^{s*}$ for all transfer prices and for all DS transshipment days. In the DS system, transshipment at day 4 outperforms the other three choices. This makes sense in light of a result from Jonsson and Silver [21] that, for an order cycle with H subperiods, the stockout probability is modest at the end of the $(H-1)$th period and quite low before the $(H-1)$th period when the daily demand mean is well above the daily demand standard deviation. Then the maximum benefit is attained when transshipments are made on day 4, when the stockout probability is modest and the inventory imbalance in the system is the second-greatest (day 5 is the most imbalanced, however it is worthless to transship at the end of the horizon when unsatisfied demand is lost immediately). Therefore, when we compare DS with the other control mechanisms, we assume that transshipments are made on day 4.

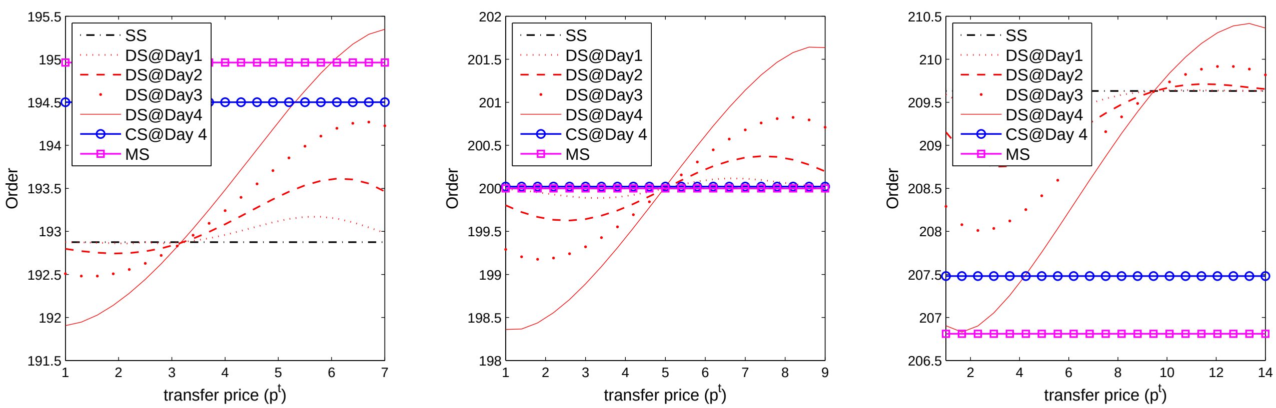
It is clear from Figure 3 that the other supply chain configurations all outperform SS. Although the relative improvement of all other control mechanisms compared with the SS system ranges from 1% to 4%, this still represents a substantial improvement since it refers to overall profit rather than only the holding, shortage cost and transshipment gains, which ignore $(p-c)E[D]$. After adjusting by $(p-c)E[D]$, the improvement ratio increases significantly.



(a: $p = 8$) (b: $p = 10$) (c: $p = 15$)

Figure 4: Performance Gap for Total Profit

Let $\gamma^c = (\pi^{c*} - \pi^{s*})/(\pi^{m*} - \pi^{s*}) \times 100\%$ and $\gamma^d = (\pi^{d*} - \pi^{s*})/(\pi^{m*} - \pi^{s*}) \times 100\%$. These ratios indicate how much of the gap between MS and SS can be filled by CS and DS at day 4, respectively. We plot γ^c and γ^d in Figure 4. The figure suggests that CS can close roughly 75% of the performance gap, and DS at day 4 can close roughly 50% of the gap, compared with MS. Compared to MS, CS and DS also have the additional benefit of not paying the extra effort of closing one store and potentially losing customers. As Section 6 indicates, there is no coordinating transfer price in general. Figure 4 shows that DS loses around $15\% - 20\%$ effectiveness compared with CS when the transfer price is set to maximize the total profit in the decentralized system. Moreover, the optimal transfer price is roughly equal to the purchase cost, regardless of the type of product. The intuition behind this is that the optimal inventory level for the second subperiod at each store (assuming equal purchase costs) would be equal to its corresponding newsboy order-up-to level if each store knew the realization of the demand in the first subperiod in advance. Since preventive transshipments are a tool to mitigate the demand uncertainty in the first subperiod, the transfer price being in the neighborhood of the purchase cost helps to keep the post-transshipment inventory level around the newsboy order-up-to level for the second subperiod. Moreover, small differences between the purchase price and the transfer price creates only a small increase in profit since the curve is relatively flat around its maximum.



(a: $p = 8$) (b: $p = 10$) (c: $p = 15$)

Figure 5: Relationship between Optimal Order Quantities and Transfer Price

Figure 5 illustrates the relationship between the sum of the orders at the two stores among various control mechanisms. When the transfer price is 5, where the overall profit is close to optimal for DS, the more profit the control mechanism has, the closer the sum of orders is to the mean demand (200). Usually, the mechanism with higher profit tends to utilize the risk pooling effect more, that is, the demand variance of the system is reduced. This brings the total optimal order quantity closer to the total demand mean. This matches the finding under emergency transshipments by Dong and Rudi [12] and Zhang [38] for normally distributed demand.

In order to verify the insights obtained from Figures 3-5, we performed a more thorough numerical study, whose results are summarized in Table 3. In this study, we generated random instances by drawing p and σ from $U(6, 20)$ and $U(0.5, 5)$, respectively. The values of l, c, c^t and μ are the same as those from the figures. Moreover, we assume that transshipments occur on day 4. In order to get p^{t*} for each (p, σ)

pair, we divide $[c + 0.2, p - 0.2]$ into 19 intervals with equal lengths. We examine the profit under the DS system for p^t at the 20 end points of these intervals as well as $p^t = 5$. We then pick the p^{t*} among all 21 p^t values that leads to the highest profit under the DS system.

From Table 3, we can see that 1) p^{t*} is close to the purchase cost $c = 5$; 2) γ^c is greater than 70% most of the time; 3) γ^d lies in (54, 58); 4) if $p < 10$ (i.e., the newsboy ratio is less than 0.5), the total order quantity is smallest for the SS system, then DS, then CS, and largest for DS, and all are less than the total mean demand ($\mu = 20 \times 2$ stores $\times$ 5 days = 200); and 5) if $p > 10$, (i.e., the newsboy ratio is greater than 0.5), the sequence of order quantity magnitudes is the reverse of that in 4) and all are greater than the total demand mean.

Next, we apply linear regression to estimate the impact of p and σ on p^{t*}, γ^c and γ^d. We estimate the coefficients in the following regression models: 1) $p^{t*} = a_1 + b_1 p + c_1 \sigma$; 2) $\gamma^c = a_2 + b_2 p + c_2 \sigma$; and 3) $\gamma^d = a_3 + b_3 p + c_3 \sigma$.

From Table 4, at a 95% significance level, the final regression models are 1) $p^{t*} = a_1 + b_1 p$; 2) $\gamma^c = a_2 + c_2 \sigma$; and 3) $\gamma^d = a_3 + b_3 p + c_3 \sigma$. High R^2's indicate that these regression models have good prediction power. We can see that p^{t*} is solely affected by p; c_1, the coefficient of σ, is statistically insignificant due to its high p-value in the significance test. Similarly, γ^c is solely affected by σ due to the high p-value of b_2, the coefficient of p. The linear regression models can be interpreted as follows. p^{t*} increases in p. Higher p's require higher p^t's to encourage transshipments. On the other hand, γ^c increases in σ. A smaller σ results in less imbalance between two stores. Such imbalance would reduce the benefit of transshipments. γ^d is affected by both p and σ. Notice that b_3 is negative. As p has very limited impact on the performance of the CS system, negative b_3 indicates that the gap between decentralized and centralized transshipments increases in p because stores would be less willing to transship. This is due to the fact that $b_1 < 1$, i.e., p^{t*} increases at a smaller pace than p.

8. Conclusions and Future Directions

We consider a two-store system with preventive transshipments. We show that the optimal order decision for each store is a modified base-stock policy whose base-stock levels are determined by the initial inventory and capacity level at the other store. We also show that there is a dominant transfer policy, and that it takes the form of what we call a control-band conserving (CBC) transfer policy. Although the DS system does not perform as well as the CS system, it still significantly outperforms the SS system and is close to the most optimistic MS system if the transfer price is set close to the purchase cost. For normally distributed demand, the total order quantity in systems with transshipments is closer to the demand mean than those in the SS system, which reflects the risk pooling effect brought about by the transshipments.

	p	σ	p^{t*}	γ^c	γ^d	$Q_1^{s*}+Q_2^{s*}$	$Q_1^{d*}+Q_2^{d*}$	$Q_1^{c*}+Q_2^{c*}$	Q^{m*}
	6.58	3.01	4.84	74.96	57.44	190.53	192.23	192.66	193.30
	6.84	2.78	4.71	74.91	57.51	192.34	193.64	194.06	194.58
	6.88	3.09	4.73	75.13	57.55	191.65	193.09	193.53	194.10
	7.11	4.25	4.56	75.55	57.51	189.83	191.44	192.13	192.81
	7.17	2.15	4.6	74.43	57.33	195.02	195.82	196.14	196.48
	7.23	0.54	4.64	66.89	55.45	198.78	198.98	199.06	199.14
	9.39	0.6	5	68.86	55.34	199.78	199.82	199.83	199.84
	10.36	3.94	5	75.71	57.56	200.78	200.63	200.60	200.55
	11.26	3.75	5.09	75.64	57.51	202.36	201.94	201.83	201.67
	11.66	3.54	5.27	75.56	57.46	202.85	202.40	202.20	202.01
	12.23	2.48	5	74.95	57.19	202.56	202.07	201.98	201.81
	12.34	3.52	5	75.53	57.39	203.78	203.05	202.93	202.67
	12.54	3.58	5.05	75.55	57.41	204.11	203.34	203.18	202.91
	13.64	3.72	5.5	75.56	57.4	205.66	204.74	204.39	204.00
	13.96	0.96	5.62	71.56	55.71	201.56	201.32	201.21	201.11
	14.63	4.63	5.18	75.69	57.39	208.44	206.90	206.54	205.97
	14.73	1.16	5.22	72.4	56.02	202.15	201.76	201.67	201.52
	15.28	1.61	5.41	73.6	56.54	203.22	202.66	202.49	202.28
	15.56	1.95	5.5	74.17	56.79	204.04	203.35	203.13	202.86
	15.56	4.11	5.51	75.6	57.38	208.54	207.07	206.61	206.04
	15.71	4.59	5.56	75.66	57.41	209.69	208.05	207.51	206.85
	15.77	5	5.58	75.64	57.4	210.63	208.84	208.25	207.52
	15.89	1.5	5.62	73.31	56.39	203.24	202.70	202.51	202.29
	15.96	1.29	5.65	72.7	56.12	202.81	202.34	202.17	201.98
	16.12	4.69	5.7	75.65	57.38	210.40	208.68	208.06	207.35
	16.45	4.64	5.82	75.64	57.36	210.63	208.90	208.24	207.52
	17.12	3.54	6.05	75.34	57.16	208.66	207.29	206.71	206.12
	19.05	3.11	5.8	75.02	57.06	208.85	207.33	206.85	206.26
	19.63	0.96	5.97	70.63	54.98	202.83	202.35	202.19	202.00
	19.93	2.25	6.06	74.22	56.67	206.77	205.64	205.24	204.79
Mean			5.31	74.20	56.93				
Std			0.44	2.19	0.76				
Max			6.06	75.71	57.56				
Min			4.56	66.89	54.98				

Table 3: Results of Extended Numerical Study

	p^{t*}			γ^c			γ^d		
coefficient	a_1	b_1	c_1	a_2	b_2	c_2	a_3	b_3	c_3
value	3.9310	0.1043	-0.0008*	69.9113	0.0276*	1.3546	56.1794	-0.0495	0.4833
p-value	0.0000	0.0000	0.9692	0.0000	0.6259	0.0000	0.0000	0.0046	0.0000
R^2	0.88			0.72			0.80		

*Statistically insignificant

Table 4: Linear Regression Result

In this paper, we show that the inventory control policies are similar between preventive and emergency transshipments since complete pooling is a degenerate CBC policy, and a modified base-stock policy is optimal for both types of transshipments. The complexity of the control mechanisms is similar between emergency and preventive transshipments. In contrast, preventive transshipments provide more benefit and flexibility to the whole supply chain when the transshipment timing is allowed to be tuned.

However, we show that there is no coordinating linear transfer price under preventive transshipments. The gap between the CS and DS systems leaves room for M&A activities to improve the performance of the whole supply chain. For retailers who want to keep their independence while still seeking opportunities to improve their supply chain performance, it is necessary to explore other coordination mechanisms within strategic alliances so that they are not at a disadvantage compared to companies choosing M&A.

Transshipments can not only help firms cope with demand uncertainty but can also help mitigate supply uncertainty. Snyder and Shen [32] study the impact of inventory allocation in several supply chain configurations with supply disruptions. They show that supply uncertainty often results in properties that are opposite to those under demand uncertainty. It is worthwhile to investigate whether there is also a mirror-image effect between demand and supply uncertainty under preventive transshipments.

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9. Appendix

9.1 Formulation of MS, CS and SS

In the MS system, the two stores are merged into one store. There is no transshipment involved. We assumed that the cost [revenue] parameters take the minimum [maximum] value of the two stores, and that there are no customers lost due to the merger. Based on these optimistic assumptions, we can formulate the whole problem as a newsboy model.

$$\pi^m(Q^m) = p^m E[\min(Q^m, \sum_{i,j} D_i^j)] + l^m E[(Q^m - \sum_{i,j} D_i^j))^+] - c^m Q^m$$

The convolution $F_1 \circ F_2 \circ G_1 \circ G_2$ is the cdf of $\sum_{i,j} D_i^j$. We have

$$Q^{m*} = (F_1 \circ F_2 \circ G_1 \circ G_2)^{-1} \left(\frac{p^m - c^m}{p^m - l^m} \right) \quad (9)$$

$$\pi^{m*} = \pi^m(Q^{m*}) \quad (10)$$

For the SS system, there is also no transshipment involved since each store operates separately. We can model SS as two separate newsboy problems. The profit for store i follows

$$\pi_i^s(Q_i^s) = p_i E[\min(Q_i^s, \sum_j D_i^j)] + l_i E[(Q_i^s - \sum_j D_i^j))^+] - c_i Q_i^s$$

Then we have

$$Q_i^{s*} = (F_i \circ G_i)^{-1} \left(\frac{p_i - c_i}{p_i - l_i} \right) \quad (11)$$

$$\pi_i^{s*} = \pi_i^s(Q_i^{s*}) \quad (12)$$

$$\pi^{s*} = \pi_1^{s*} + \pi_2^{s*} \quad (13)$$

For the CS system, both transshipment and order decisions are centrally planned. If excess demand in the first subperiod is lost, we have the following formulation.

$$\begin{aligned} \pi^{c1}(Q_1^c, Q_2^c) &= E[p_1 \min(Q_1^c, D_1^1) + p_2 \min(Q_2^c, D_2^1)] - c_1 Q_1^c - c_2 Q_2^c \quad (14) \\ &\quad + E[\pi^{c2}([Q_1^c - D_1^1]^+, [Q_2^c - D_2^1]^+)] \\ (Q_1^{c*}, Q_2^{c*}) &= \arg\max \pi^{c1}(Q_1^c, Q_2^c) \quad (15) \\ \pi^{c*} &= \pi^{c1}(Q_1^{c*}, Q_2^{c*}) \quad (16) \end{aligned}$$

where

$$\begin{aligned} \pi^{c2}(I_1, I_2) &= \pi^{c2}(z^*(I_1, I_2), I_1, I_2) \quad (17) \\ z^*(I_1, I_2) &= \arg\max_{-I_2^+ \le z \le I_1^+} \pi^{c2}(z, I_1, I_2) \quad (18) \\ \pi^{c2}(z, I_1, I_2) &= E[p_1 \min(I_1 - z, D_1^2) + l_1(I_1 - z - D_1^2)^+] \\ &\quad + E[p_2 \min(I_2 + z, D_2^2) + l_2(I_2 + z - D_2^2)^+] - c^t |z| \quad (19) \end{aligned}$$

The first line in (14) represents the profit in the first subperiod. The second line represents the expected profit in the second subperiod over all possible (D_1^1, D_2^1) when both stores choose the optimal transshipment amount. By dropping the $^+$ in the second line of (14) and keeping the rest of (14) to (19) unchanged, we obtain the formulation for the backorder case.

Our formulation of the CS system is consistent with that of Das [10] except that Das assumes that the transshipment cost function is convex and differentiable everywhere. Das points out that $-I_2^+ \le z \le I_1^+$ is a nonconvex set in the backorder case. For the lost-sales case, although $\pi^{c2}(I_1, I_2)$ is concave for $(I_1, I_2) \in R_+^2$, its concavity is not preserved when we take the expectation $E[\pi^{c2}([Q_1^c - D_1^1]^+, [Q_2^c - D_2^1]^+)]$. Hence, he introduces additional assumptions on the transshipment cost function, such as requiring it to be subadditive or to have a sufficiently large derivative, to show that $\pi^{c1}(Q_1^c, Q_2^c)$ is concave in the backorder case.

Given these complications, we consider a special case for the decentralized system: We prove that if the two stores are identical and there is no transshipment cost, $\pi^{c1}(Q_1^c, Q_2^c)$ is concave in (Q_1^c, Q_2^c).

Proposition 3. *Suppose the two stores are identical and the transshipment cost is zero; that is,* $p = p_1 = p_2$, $c = c_1 = c_2$, $l = l_1 = l_2$, *the distribution of* D_i^j *is equal to that of* $D_{\neg i}^j$, *and* $c^t = 0$. *Then in the lost-sales case, we have*

1. $z^*(I_1, I_2) = \frac{I_1 - I_2}{2}$ *is a unique optimal transshipment quantity.*
2. $\pi^{c1}(Q_1^c, Q_2^c)$ *in* (14) *is strictly concave in* (Q_1^c, Q_2^c).

Proposition 3.1 says that the optimal transfer policy in the centralized system with two identical stores and no transshipment cost is for the two stores to evenly split their total on-hand inventory at the end of the first subperiod. This is different from the BSC policy introduced by Das since we imposed different assumptions on the transshipment cost function. Proposition 3.2 guarantees that the local maximizer is the global maximizer. This property helps us to carry out numerical comparisons between the CS and DS systems in Section 7.

The identical-stores requirement is only limited to Proposition 3.

9.2 Backorders

If we assume that unmet demands at the end of the first subperiod are backorderd instead of lost, Theorem 1 is not affected: a CBC transfer policy is still dominant in the backorder case. In this case, areas A_1 and A_2 in Figure 1 are not bounded by $I_1, I_2 \ge 0$. The total profit of store 1, as a function of store 1's order quantity and store 2's proposed transshipment quantity, is given by

$$\tilde{\pi}^{BO,1}(Q, x_2) = (p - c)Q - pE[(Q - D^1)^+] + E[\pi^2(Q - D^1, x_2)]$$

This expression is identical to (5), except that $(Q-D^1)^+$ is replaced with $Q-D^1$ in the last term since the second-subperiod starting inventory may be negative. The value of $\pi^2(I,x_2)$ is given in (4) for $I \geq 0$. For $I < 0$, we have:

$$\pi^2(I,x_2) = \begin{cases} 0, & \text{if } x_2 \leq 0 \\ (p-p^t)x_2, & \text{if } 0 < x_2 \leq -I \\ -(p-p^t)I + (p-p^t)\min(G^{-1}(\alpha^l), x_2+I) & \\ +(l-p)E[(\min(G^{-1}(\alpha^l), x_2+I) - D^2)^+], & \text{if } x_2 > -I \end{cases}$$

The following lemma is a mild modification of Lemma 2.

Lemma 4. *1. $\tilde{\pi}^{BO,1}(Q,x_2)$ is continuous and partial derivatives exist everywhere.*

2. When $x_2 \leq 0$,

$$\begin{aligned} \frac{\partial \tilde{\pi}^{BO,1}(Q,x_2)}{\partial Q} &= (p-c) \\ &+(p^t-c^t-p)\Pr(Q-G^{-1}(\alpha^h)+x_2 < D^1 \leq Q-G^{-1}(\alpha^h)) \\ &+(l-p)\left[\Pr(Q-G^{-1}(\alpha^h) < D^1 \leq Q-D^2)\right. \\ &\left.+\Pr(D^1 \leq Q-G^{-1}(\alpha^h)+x_2, D^1+D^2 \leq Q+x_2)\right] \end{aligned}$$

3. When $0 < x_2 \leq G^{-1}(\alpha)$,

$$\begin{aligned} \frac{\partial \tilde{\pi}^{BO,1}(Q,x_2)}{\partial Q} &= (p-c)+(p^t-p)\Pr(Q-G^{-1}(\alpha^l) < D^1 \leq Q-G^{-1}(\alpha^l)+x_2) \\ &+(l-p)\left[\Pr(Q-G^{-1}(\alpha^l)+x_2 < D^1 \leq Q, D^1+D^2 \leq Q+x_2)\right. \\ &+\Pr(Q < D^1 \leq Q+x_2, D^1+D^2 \leq Q+x_2) \\ &\left.+\Pr(D^1 \leq Q-G^{-1}(\alpha^l), D^1+D^2 \leq Q)\right] \end{aligned}$$

4. When $x_2 > G^{-1}(\alpha)$,

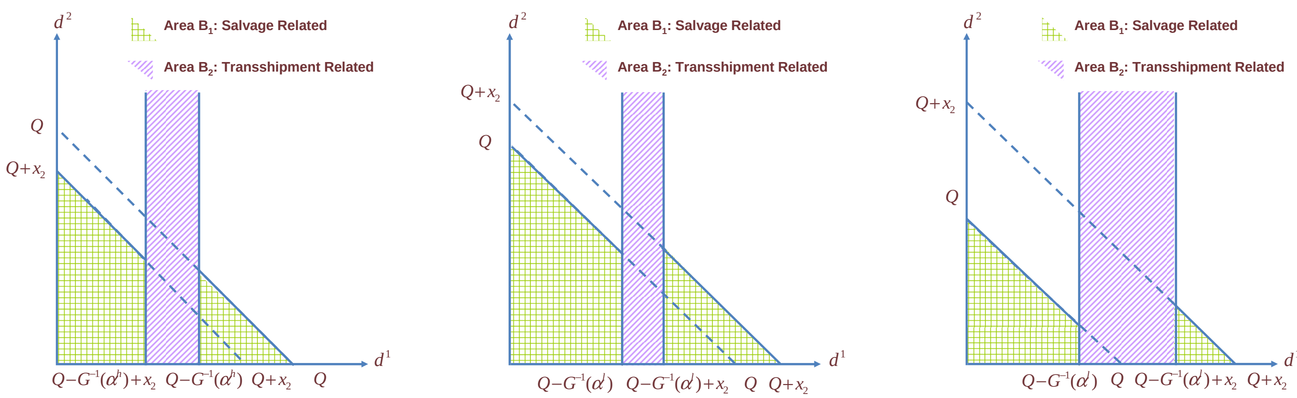
$$\begin{aligned} \frac{\partial \tilde{\pi}^{BO,1}(Q,x_2)}{\partial Q} &= (p-c)+(p^t-p)\left[\Pr(Q-G^{-1}(\alpha^l) < D^1 \leq Q)\right. \\ &\left.+\Pr(Q < D^1 \leq Q-G^{-1}(\alpha^l)+x_2)\right] \\ &+(l-p)\left[\Pr(D^1 \leq Q-G^{-1}(\alpha^l), D^1+D^2 \leq Q)\right. \\ &\left.+\Pr(Q-G^{-1}(\alpha^l)+x_2 < D^1 \leq Q+x_2, D^1+D^2 \leq Q+x_2)\right] \end{aligned}$$

5. $\frac{\partial \tilde{\pi}^{BO,1}(Q,x_2)}{\partial Q}$ is continuous and differentiable and its partial derivatives are bounded.

6. $\tilde{\pi}^{BO,1}(Q,x_2)$ is a strictly concave function with respect to Q. In particular, $\frac{\partial^2 \tilde{\pi}^{BO,1}(Q,x_2)}{\partial Q^2} < 0$.

7. $\frac{\partial^2 \tilde{\pi}^{BO,1}(Q,x_2)}{\partial Q^2} < \frac{\partial^2 \tilde{\pi}^{BO,1}(Q,x_2)}{\partial Q \partial x_2} < 0$.

The differences in parts 2-4 between Lemma 2 and Lemma 4 can be easily verified through a comparison of Figure 2 and Figure 6. When $x_2 \leq 0$, the backorder and lost-sales cases are identical. When $0 < x_2 \leq G^{-1}(\alpha^l)$, the only difference between the two cases is the situation in which $Q < D^1 \leq Q + x_2$ and $D^1 + D^2 \leq Q + x_2$. In the backorder case, when demand occurs in this area, a marginal unit does not generate sales but does generate salvage value, whereas the opposite is true in the lost-sales case. Therefore, the difference in the partial derivative between the backorders and lost-sales cases is $(l - p)\Pr(Q < D^1 \leq Q + x_2, D^1 + D^2 \leq Q + x_2)$. When $x_2 > G^{-1}(\alpha^l)$, there are two situations in which backorders and lost sales are different. One is $Q - G^{-1}(\alpha^l) + x_2 < D^1 \leq Q + x_2, D^1 + D^2 \leq Q + x_2$, in which case a marginal unit in the backorder case creates salvage instead of sales. The other is $Q < D^1 \leq Q - G^{-1}(\alpha^l) + x_2$, in which case a marginal unit in the backorder case saves a transfer payment but does not generate sales. The remainder of Lemma 4 is identical to Lemma 2, except part 7 since $\frac{\partial \tilde{\pi}^{BO,1}(Q,x_2)}{\partial Q}$ depends on x_2 in the backorder case.



(a: $x_2 \leq 0$) (b: $0 < x_2 \leq G^{-1}(\alpha^l)$) (c: $x_2 > G^{-1}(\alpha^l)$)

Figure 6: Interpretation of $\partial\tilde{\pi}^{BO,1}(Q, x_2)/\partial Q$ for Backorder Case

Store 2's optimal transshipment proposal is given by

$$x_2(Q_2, d_2^1) = \begin{cases} (Q_2 - d_2^1) - G_2^{-1}(\alpha_2^h), & \text{if } (Q_2 - d_2^1) > G_2^{-1}(\alpha_2^h) \\ (Q_2 - d_2^1) - G_2^{-1}(\alpha_2^l), & \text{if } (Q_2 - d_2^1) < G_2^{-1}(\alpha_2^l) \\ 0, & \text{otherwise} \end{cases}$$

This is idential (6) in Section 5.2 except that the second-subperiod starting inventory position may be negative.

The expected profit for store 1 with backorders is given by

$$\pi^{BO,1}(Q_1, Q_2) \quad = \quad E[\tilde{\pi}^{BO,1}(Q_1, x_2(Q_2, D_2^1))]. \tag{20}$$

We have the following lemma and theorem, which is analogous to Lemma 3 and Theorem 2.

Lemma 5. *1. $\pi^{BO,1}(Q_1, Q_2)$ is continuous and differentiable.*

2. $\pi^{BO,1}(Q_1, Q_2)$ is a strictly concave function with respect to Q_1

3. $\frac{\partial^2 \pi^{BO,1}(Q_1,Q_2)}{\partial Q_1^2} < \frac{\partial^2 \pi^{BO,1}(Q_1,Q_2)}{\partial Q_1 \partial Q_2} \leq 0$. Therefore, store 1's best response function to the order quantity of store 2 (Q_2) is a decreasing function and the absolute value of its slope of the best response function is less than 1.

Theorem 5. *There exists a unique Nash equilibrium of order quantities in the backorder case.*

It is natural to compare the magnitude of the Nash equilibrium of order quantities under backorders and lost sales. Suppose the system shifts from lost sales to backorders. On the one hand, stores do not need as much inventory as before to protect against lost sales since the other store may provide inventory via transshipment. On the other hand, stores may want to increase their inventory so that they may sell more to the other store when they experience backorders at the end of the first period. In Section 7, we find that Q^{d*} under lost sales does not differ from Q^{c*} that much. The numerical study by Güllü et al. [16] reveals a similar finding for the backorder case.

9.3 Proofs of Statements

Theorem 1

Proof. We prove the theorem for store 1; the proof for store 2 is similar. We first assume that $x_2 = -x_1$ and prove that the policy described in the theorem is optimal. Then we discuss the case in which $x_2 \neq -x_1$ and prove that store 1's strategy is unchanged; therefore, the strategy is dominant.

Let

$$u(x|I_1) = pI_1 + (p^t - p)x + (l - p)E[(I_1 - D^2 - x)^+]$$

and $v(x|I_1) = u(x|I_1) - c^t x$. It is clear that $u(0|I_1) = v(0|I_1)$, and both $u(x|I_1)$ and $v(x|I_1)$ are strictly concave due to our assumptions on the demand and the relationships among the parameters. Using the standard newsboy model, it can be shown that $I_1 - G_1^{-1}(\alpha_1^l)$ is a maximizer of $u(x|I_1)$ and $I_1 - G_1^{-1}(\alpha_1^h)$ is a maximizer of $v(x|I_1)$. It is easy to see that

$$I_1 - G_1^{-1}(\alpha_1^l) \geq I_1 - G_1^{-1}(\alpha_1^h). \tag{21}$$

When $x_2 = -x_1$, (1) can be reduced to

$$\begin{aligned} \pi_1^2(x_1, I_1, -x_1) &= u(-x_1^-|I_1) + v(x_1^+|I_1) - v(0|I_1) \\ &= u(x_1|I_1) - c^t x_1^+. \end{aligned} \tag{22}$$

Therefore, $\pi_1^2(x_1, I_1, -x_1)$ is strictly concave in x_1.

We now consider the three cases for I_1 given in the theorem. For each case, we use the results proven above to confirm the optimality of the strategy specified in the theorem.

Case 1. When $I_1 < G_1^{-1}(\alpha_1^l)$, then $I_1 - G_1^{-1}(\alpha_1^l)$, the maximizer of $u(x_1|I_1)$ is negative. We will show that this quantity also maximizes π_1^2 by showing first that it is optimal among all negative values of x_1 and then that it dominates all non-negative values. For any $x_1 < 0$, we have

$$\begin{aligned}\pi_1^2(I_1 - G_1^{-1}(\alpha_1^l), I_1, -(I_1 - G_1^{-1}(\alpha_1^l))) &= u(I_1 - G_1^{-1}(\alpha_1^l)|I_1)\\ &\geq u(x_1|I_1)\\ &= \pi_1^2(x_1, I_1, -x_1).\end{aligned}$$

(The two equalities follow from (22). The inequality follows from the fact that $I_1 - G_1^{-1}(\alpha_1^l)$ is the maximizer of u.) Similarly, when $x_1 \geq 0$,

$$\begin{aligned}\pi_1^2(I_1 - G_1^{-1}(\alpha_1^l), I_1, -(I_1 - G_1^{-1}(\alpha_1^l))) &= u(I_1 - G_1^{-1}(\alpha_1^l)|I_1)\\ &> u(0|I_1)\\ &= v(0|I_1)\\ &\geq v(x_1|I_1)\\ &= \pi_1^2(x_1, I_1, -x_1).\end{aligned}$$

(The last inequality follows from the strict concavity of v, (21), and the fact that x_1 is non-negative while its maximizer, $I_1 - G_1^{-1}(\alpha_1^h)$, is negative.) Therefore, $x_1^* = I_1 - G_1^{-1}(\alpha_1^l)$.

Case 2. The proof that $x^* = I_1 - G_1^{-1}(\alpha_1^h)$ when $I_1 > G_1^{-1}(\alpha_1^h)$ is similar.

Case 3. Suppose $I_1 \in [G_1^{-1}(\alpha_1^l), G_1^{-1}(\alpha_1^h)]$. If $x_1 < 0$, then $\pi_1^2(x_1, I_1, -x_1) = u(x_1|I_1)$. Since $x_1 < 0 \leq I_1 - G_1^{-1}(\alpha_1^l)$, the maximizer of $u(x_1|I_1)$, and since u is strictly concave, $\pi_1^2(x_1, I_1, -x_1) < \pi_1^2(0, I_1, 0)$. Similarly, if $x_1 > 0$, then $\pi_1^2(x_1, I_1, -x_1) = v(x_1|I_1)$. Since $x_1 > 0 \geq I_1 - G_1^{-1}(\alpha_1^h)$, the maximizer of $v(x_1|I_1)$, and since v is strictly concave, $\pi_1^2(x_1, I_1, -x_1) < \pi_1^2(0, I_1, 0)$. Therefore, $x_1^* = 0$.

Now suppose that $x_2 \neq -x_1$. We claim that the same x_1^* is still optimal. Consider case 2 ($I_1 > G_1^{-1}(\alpha_1^h)$).

In Table 3, we list six possible, mutually exhaustive scenarios when store 1 chooses $x_1 \neq x_1^* = I_1 - G_1^{-1}(\alpha_1^h)$. The first two columns represent the range of proposed transshipment amounts by store 1 and store 2, respectively. The next two columns represent the realized transshipment amounts from store 1 to store 2 when store 1 chooses its proposed transshipment amount according to column 1 and when it chooses x_1^*, respectively. Column 5 represents the profit relationship when store 1 chooses the other proposed transshipment amount versus x_1^*. The last column shows the ordering of the realized transshipment amounts and x_1^*.

Table 5: Scenarios with $I_1 > G_1^{-1}(\alpha_1^h)$ for Theorem 1

Store 1's trans. decision	Store 2's trans. decision	Actual trans. amount	Optimal trans. amount	Profit order	Reason
$x_1 > I_1 - G_1^{-1}(\alpha_1^h)$	$x_2 \geq -(I_1 - G_1^{-1}(\alpha_1^h))$	x_2^-	x_2^-	$=$	$x_2^- = x_2^- < x_1^*$
$x_1 > I_1 - G_1^{-1}(\alpha_1^h)$	$x_2 < -(I_1 - G_1^{-1}(\alpha_1^h))$	$\min(x_1, x_2^-)$	x_1^*	$<$	$\min(x_1, x_2^-) > x_1^* = x_1^*$
$0 \leq x_1 < I_1 - G_1^{-1}(\alpha_1^h)$	$x_2 \geq -x_1$	x_2^-	x_2^-	$=$	$x_2^- = x_2^- < x_1^*$
$0 \leq x_1 < I_1 - G_1^{-1}(\alpha_1^h)$	$x_2 < -x_1$	x_1	$\min(x_1^*, -x_2)$	$<$	$x_1 < \min(x_1^*, -x_2) \leq x_1^*$
$x_1 < 0$	$x_2 \leq 0$	0	$\min(x_1^*, -x_2)$	$\leq$	$0 \leq \min(x_1^*, -x_2) \leq x_1^*$
$x_1 < 0$	$x_2 > 0$	$\max(x_1, -x_2)$	0	$\leq$	$\max(x_1, -x_2) \leq 0 \leq x_1^*$

From column 6 in Table 3, we can see that the optimal realized transshipment amount is always between the proposed realized transshipment amount and x_1^*. Since $\pi_1^2(z_{12}, I_1, -z_{12})$ is strictly concave in z_{12}, choosing $x_1 = x_1^*$ never generates a lower profit in any scenario. Moreover, in some cases, its profit is strictly higher. Therefore, $x_1 = x_1^*$ weakly dominates when $I_1 > G_1^{-1}(\alpha_1^h)$.

The cases where $I_1 < G_1^{-1}(\alpha_1^l)$ and $I_1 \in [G_1^{-1}(\alpha_1^l), G_1^{-1}(\alpha_1^h)]$ are similar. □

Lemma 1

Proof. Parts 1 to 3 are straightforward to verify. Part 4 follows immediately from parts 1 to 3. The partial derivative or subgradient is bounded since the partial derivative of each term in (4) is bounded. □

Lemma 2

Proof. Part 1:

It is obvious that $(p-c)Q$ and $pE[(Q-D^1)^+]$ is continuous and differentiable. In Lemma 1, we show that $\pi^2(I, x_2)$ is differentiable almost everywhere. So is $\pi^2((Q-D^1)^+, x_2)$. Since we assume the demand distribution is well defined, that is the set where $\pi^2((Q-D^1)^+, x_2)$ is nondifferentiable has probability measure 0, the partial derivative of $E[\pi^2((Q-D^1)^+, x_2)]$ is also well defined. And $\frac{\partial E[\pi^2((Q-D^1)^+, x_2)]}{\partial Q} = E[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q}]$ by the finiteness of the partial derivative, shown in Lemma 1.

Part 2: $(x_2 \leq 0)$

The difficulty is that evaluating $\frac{\partial \tilde{\pi}^1(Q, x_2)}{\partial Q}$ relies on evaluating $\frac{\partial E[\pi^2((Q-D^1)^+, x_2)]}{\partial Q}$. In order to facilitate the analysis, we generalize the method used by Netessine and Rudi [27]. Suppose the functions k_i satisfy $-\infty = x_1 + k_0(x_2) \leq x_1 + k_1(x_2) ... \leq x_1 + k_n(x_2) = \infty$ and that the function $f(x_1, x_2, d)$ is continuous in the vector (x_1, x_2, d) and differentiable almost everywhere. Let $1_{\{D \in X\}}$ be the indicator function. One can show

$$\frac{\partial E\left[\sum_{i=0}^{n} f(x_1,x_2,D) 1_{\{x_1+k_i(x_2)\le D< x_1+k_{i+1}(x_2)\}}\right]}{\partial x_1} = E\left[\sum_{i=0}^{n} \frac{\partial f(x_1,x_2,D)}{\partial x_1} 1_{\{x_1+k_i(x_2)\le D< x_1+k_{i+1}(x_2)\}}\right]. \tag{23}$$

To apply (23), we let $x_1 = Q$ and $f(x_1, x_2, d) = \pi^2((Q-d))^+, x_2)$ and define the breakpoint functions as $k_1(x_2) = -G^{-1}(\alpha^h) + x_2$, $k_2(x_2) = -G^{-1}(\alpha^h)$, and $k_3(x_2) = 0$. These functions satisfy the condition given above since $x_2 \le 0$. Then, applying (5) and (23), we get

$$\begin{aligned}
\frac{\partial \tilde{\pi}^1(Q,x_2)}{\partial Q} &= (p-c) - pF(Q) \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{D^1 \le Q - G^{-1}(\alpha^h)+x_2\}}\right] \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{Q-G^{-1}(\alpha^h)+x_2 < D^1 \le Q - G^{-1}(\alpha^h)\}}\right] \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{Q-G^{-1}(\alpha^h) < D^1 \le Q\}}\right] \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{D^1 > Q\}}\right]
\end{aligned}$$

In what follows, we apply (4), setting $I = Q - D^1$.

If $D^1 > Q$ and $x_2 \le 0$, we have

$$\pi^2((Q-D^1)^+, x_2) = \pi^2(0, x_2) = 0.$$

Therefore,

$$E[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{D^1 > Q\}}] = 0. \tag{24}$$

If $Q - G^{-1}(\alpha^h) < D^1 \le Q$ and $x_2 \le 0$, we have

$$\begin{aligned}
\pi^2((Q-D^1)^+, x_2) &= \pi^2(Q-D^1, x_2) \\
&= p(Q-D^1) + (l-p)E[(Q-D^1-D^2)^+].
\end{aligned}$$

$(Q-D^1-D^2)^+ = Q-D^1-D^2$ if $D^1+D^2 \le Q$ and 0 otherwise. We have $E[((Q-D^1-D^2)^+)'] = E[1_{\{D^1+D^2\le Q\}}]$. This can be verified using Leibniz's rule. Therefore,

$$\begin{aligned}
& E\left[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{Q-G^{-1}(\alpha^h) < D^1 \le Q\}}\right] \qquad (25)\\
= \;& E[1_{\{Q-G^{-1}(\alpha^h) < D^1 \le Q\}}(p + (l-p)1_{\{D^1+D^2 \le Q\}})] \\
= \;& p\Pr(Q - G^{-1}(\alpha^h) < D^1 \le Q) + (l-p)\Pr(Q-G^{-1}(\alpha^h) < D^1 \le Q, D^1+D^2 \le Q) \\
= \;& p\Pr(Q - G^{-1}(\alpha^h) < D^1 \le Q) + (l-p)\Pr(Q-G^{-1}(\alpha^h) < D^1 \le Q - D^2).
\end{aligned}$$

If $Q - G^{-1}(\alpha^h) + x_2 < D^1 \leq Q - G^{-1}(\alpha^h)$ and $x_2 \leq 0$, we have

$$\begin{aligned}
& \pi^2((Q-D^1)^+, x_2) \\
= \ & \pi^2(Q-D^1, x_2) \\
= \ & p(Q-D^1) + (p^t - c^t - p)(Q - D^1 - G^{-1}(\alpha^h)) + (l-p)E[(Q - D^1 - D^2 - (Q - D^1 - G^{-1}(\alpha^h)))^+] \\
= \ & p(Q-D^1) + (p^t - c^t - p)(Q - D^1 - G^{-1}(\alpha^h)) + (l-p)E[(G^{-1}(\alpha^h) - D^2)^+] \\
= \ & (p^t - c^t)Q - (p^t - c^t)(D^1 + G^{-1}(\alpha^h)) + pG^{-1}(\alpha^h) + (l-p)E[(G^{-1}(\alpha^h) - D^2)^+].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{Q - G^{-1}(\alpha^h) + x_2 < D^1 \leq Q - G^{-1}(\alpha^h)\}}] \\
= \ & E[(p^t - c^t) 1_{\{Q - G^{-1}(\alpha^h) + x_2 < D^1 \leq Q - G^{-1}(\alpha^h)\}}] \\
= \ & (p^t - c^t) \Pr(Q - G^{-1}(\alpha^h) + x_2 < D^1 \leq Q - G^{-1}(\alpha^h)).
\end{aligned} \tag{26}$$

If $D^1 \leq Q - G^{-1}(\alpha^h) + x_2$ and $x_2 \leq 0$, we have

$$\begin{aligned}
& \pi^2((Q-D^1)^+, x_2) \\
= \ & \pi^2(Q-D^1, x_2) \\
= \ & p(Q-D^1) - (p^t - c^t - p)x_2 + (l-p)E[(Q + x_2 - D^1 - D^2)^+].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{D^1 \leq Q - G^{-1}(\alpha^h) + x_2\}}] \\
= \ & E[1_{\{D^1 \leq Q - G^{-1}(\alpha^h) + x_2}(p + (l-p)1_{\{D^1 + D^2 \leq Q + x_2\}})] \\
= \ & p \Pr(D^1 \leq Q - G^{-1}(\alpha^h) + x_2) + (l-p)\Pr(D^1 \leq Q - G^{-1}(\alpha^h) + x_2, D^1 + D^2 \leq Q + x_2).
\end{aligned} \tag{27}$$

Combining (24)–(27), we get

$$
\begin{aligned}
& \frac{\partial \tilde{\pi}^1(Q,x_2)}{\partial Q} \\
= \; & (p-c) - pF(Q) \\
& + p \Pr(Q - G^{-1}(\alpha^h) < D^1 \le Q) + (l-p)\Pr(Q - G^{-1}(\alpha^h) < D^1 \le Q - D^2) \\
& + (p^t - c^t)\Pr(Q - G^{-1}(\alpha^h) + x_2 < D^1 \le Q - G^{-1}(\alpha^h)) \\
& + p\Pr(D^1 \le Q - G^{-1}(\alpha^h) + x_2) + (l-p)\Pr(D^1 \le Q - G^{-1}(\alpha^h) + x_2, D^1 + D^2 \le Q + x_2) \\
= \; & (p-c) - pF(Q) \\
& + p\left[\Pr(Q - G^{-1}(\alpha^h) < D^1 \le Q) + \Pr(D^1 \le Q - G^{-1}(\alpha^h) + x_2)\right] \\
& + (l-p)\left[\Pr(Q - G^{-1}(\alpha^h) < D^1 \le Q - D^2) + \Pr(D^1 \le Q - G^{-1}(\alpha^h) + x_2, D^1 + D^2 \le Q + x_2)\right] \\
& + (p^t - c^t)\left[\Pr(Q - G^{-1}(\alpha^h) + x_2 < D^1 \le Q - G^{-1}(\alpha^h))\right] \\
= \; & (p-c) + (p^t - c^t - p)\Pr(Q - G^{-1}(\alpha^h) + x_2 < D^1 \le Q - G^{-1}(\alpha^h)) \\
& + (l-p)\left[\Pr(Q - G^{-1}(\alpha^h) < D^1 \le Q - D^2) + \Pr(D^1 \le Q - G^{-1}(\alpha^h) + x_2, D^1 + D^2 \le Q + x_2)\right],
\end{aligned}
$$

as desired.

Part 3: $(0 < x_2 \le G^{-1}(\alpha^l))$

Similarly, we have

$$
\begin{aligned}
\frac{\partial \tilde{\pi}^1(Q,x_2)}{\partial Q} &= (p-c) - pF(Q) \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{D^1 \le Q - G^{-1}(\alpha^l)\}}\right] \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{Q - G^{-1}(\alpha^l) < D^1 \le Q - G^{-1}(\alpha^l) + x_2\}}\right] \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{Q - G^{-1}(\alpha^l) + x_2 < D^1 \le Q\}}\right] \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{D^1 > Q\}}\right].
\end{aligned}
$$

If $D^1 > Q$ and $0 < x_2 \le G^{-1}(\alpha^l)$, we have

$$
\begin{aligned}
\pi^2((Q-D^1)^+, x_2) &= \pi^2(0, x_2) \\
&= -(p^t - p)x_2 + (l-p)E[(-D^2 + x_2)^+].
\end{aligned}
$$

Therefore,

$$
E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{D^1 > Q\}}\right] = 0. \tag{28}
$$

If $Q - G^{-1}(\alpha^l) + x_2 < D^1 \le Q$ and $0 < x_2 \le G^{-1}(\alpha^l)$, we have

$$
\begin{aligned}
\pi^2((Q-D^1)^+, x_2) &= \pi^2(Q - D^1, x_2) \\
&= p(Q - D^1) - (p^t - p)x_2 + (l-p)E[(Q - D^1 - D^2 + x_2)^+].
\end{aligned}
$$

Therefore,

$$\begin{aligned}
& E\left[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{Q-G^{-1}(\alpha^l)+x_2<D^1\le Q\}}\right] \quad (29)\\
= & \; E\left[1_{\{Q-G^{-1}(\alpha^l)+x_2<D^1\le Q\}}(p+(l-p)1_{\{Q-D^1-D^2+x_2\ge 0\}})\right]\\
= & \; p\Pr(Q-G^{-1}(\alpha^l)+x_2<D^1\le Q)\\
& +(l-p)\Pr(Q-G^{-1}(\alpha^l)+x_2<D^1\le Q, Q-D^1-D^2+x_2\ge 0).
\end{aligned}$$

If $Q-G^{-1}(\alpha^l)<D^1\le Q-G^{-1}(\alpha^l)+x_2$ and $0<x_2\le G^{-1}(\alpha^l)$, we have

$$\begin{aligned}
& \pi^2((Q-D^1)^+,x_2)\\
= & \; \pi^2(Q-D^1,x_2)\\
= & \; p(Q-D^1)+(p^t-p)(Q-G^{-1}(\alpha^l)-D^1)+(l-p)E[(Q-D^1-D^2-(Q-D^1-G^{-1}(\alpha^l)))^+]\\
= & \; p^tQ-pD^1-(p^t-p)(G^{-1}(\alpha^l)+D^1)+(l-p)E[(G^{-1}(\alpha^l)-D^2)^+].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E\left[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{Q-G^{-1}(\alpha^l)<D^1\le Q-G^{-1}(\alpha^l)+x_2\}}\right] \quad (30)\\
= & \; E\left[1_{\{Q-G^{-1}(\alpha^l)<D^1\le Q-G^{-1}(\alpha^l)+x_2\}}p^t\right]\\
= & \; p^t\Pr(Q-G^{-1}(\alpha^l)<D^1\le Q-G^{-1}(\alpha^l)+x_2).
\end{aligned}$$

If $D^1\le Q-G^{-1}(\alpha^l)$ and $0<x_2\le G^{-1}(\alpha^l)$, we have

$$\begin{aligned}
\pi^2((Q-D^1)^+,x_2) &= \pi^2(Q-D^1,x_2)\\
&= p(Q-D^1)+(l-p)E[(Q-D^1-D^2)^+].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E\left[\frac{\partial \pi^2((Q-D^1)^+,x_2)}{\partial Q} 1_{\{D^1\le Q-G^{-1}(\alpha^l)\}}\right] \quad (31)\\
= & \; E\left[1_{\{D^1\le Q-G^{-1}(\alpha^l)\}}(p+(l-p)1_{\{Q-D^1-D^2\ge 0\}})\right]\\
= & \; p\Pr(D^1\le Q-G^{-1}(\alpha^l))+(l-p)\Pr(D^1\le Q-G^{-1}(\alpha^l), Q-D^1-D^2\ge 0).
\end{aligned}$$

Combining (28)–(31), we get

$$\begin{aligned}
&\frac{\partial \tilde{\pi}^1(Q,x_2)}{\partial Q} \\
= \;& (p-c) - pF(Q) \\
&+p\Pr(Q - G^{-1}(\alpha^l) + x_2 < D^1 \le Q) \\
&+(l-p)\Pr(Q - G^{-1}(\alpha^l) + x_2 < D^1 \le Q, Q - D^1 - D^2 + x_2 \ge 0) \\
&+p^t \Pr(Q - G^{-1}(\alpha^l) < D^1 \le Q - G^{-1}(\alpha^l) + x_2) \\
&+p\Pr(D^1 \le Q - G^{-1}(\alpha^l)) + (l-p)\Pr(D^1 \le Q - G^{-1}(\alpha^l), Q - D^1 - D^2 \ge 0) \\
= \;& (p-c) - pF(Q) \\
&+p\left[\Pr(Q - G^{-1}(\alpha^l) + x_2 < D^1 \le Q) + \Pr(D^1 \le Q - G^{-1}(\alpha^l))\right] \\
&+(l-p)\left[\Pr(Q - G^{-1}(\alpha^l) + x_2 < D^1 \le Q, Q - D^1 - D^2 + x_2 \ge 0)\right. \\
&\left.+\Pr(D^1 \le Q - G^{-1}(\alpha^l), Q - D^1 - D^2 \ge 0)\right] \\
&+p^t \Pr(Q - G^{-1}(\alpha^l) < D^1 \le Q - G^{-1}(\alpha^l) + x_2) \\
= \;& (p-c) + (p^t - p)\Pr(Q - G^{-1}(\alpha^l) < D^1 \le Q - G^{-1}(\alpha^l) + x_2) \\
&+(l-p)\left[\Pr(Q - G^{-1}(\alpha^l) + x_2 < D^1 \le Q, Q - D^1 - D^2 + x_2 \ge 0)\right. \\
&\left.+\Pr(D^1 \le Q - G^{-1}(\alpha^l), Q - D^1 - D^2 \ge 0)\right],
\end{aligned}$$

as desired.

Part 4: $(x_2 > G^{-1}(\alpha^l))$

Similarly, we have

$$\begin{aligned}
\frac{\partial \tilde{\pi}^1(Q,x_2)}{\partial Q} &= (p-c) - pF(Q) \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{D^1 \le Q - G^{-1}(\alpha^l)\}}\right] \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{Q - G^{-1}(\alpha^l) < D^1 \le Q\}}\right] \\
&+ E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{D^1 > Q\}}\right].
\end{aligned}$$

If $D^1 > Q$ and $x_2 > G^{-1}(\alpha^l)$, we have

$$\begin{aligned}
\pi^2((Q-D^1)^+, x_2) &= \pi^2(0, x_2) \\
&= -(p^t - p)G^{-1}(\alpha^l) + (l-p)E[(-D^2 + G^{-1}(\alpha^l))^+].
\end{aligned}$$

Therefore,

$$E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{D^1 > Q\}}\right] = 0. \tag{32}$$

If $Q - G^{-1}(\alpha^l) < D^1 \leq Q$ and $x_2 > G^{-1}(\alpha^l)$, we have

$$\begin{aligned}
& \pi^2((Q-D^1)^+, x_2) \\
= \; & \pi^2(Q-D^1, x_2) \\
= \; & p(Q-D^1) + (p^t - p)(Q - G^{-1}(\alpha^l) - D^1) + (l-p)E[(Q - D^1 - D^2 - (Q - D^1 - G^{-1}(\alpha^l)))^+] \\
= \; & p^t Q - pD^1 - (p^t - p)(G^{-1}(\alpha^l) + D^1) + (l-p)E[(G^{-1}(\alpha^l) - D^2)^+].
\end{aligned}$$

Therefore,

$$\begin{aligned}
E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{Q-G^{-1}(\alpha^l)<D^1\leq Q\}}\right] &= E\left[1_{\{Q-G^{-1}(\alpha^l)<D^1\leq Q\}} p^t\right] \\
&= p^t \Pr(Q - G^{-1}(\alpha^l) < D^1 \leq Q).
\end{aligned} \tag{33}$$

If $D^1 \leq Q - G^{-1}(\alpha^l)$ and $x_2 > G^{-1}(\alpha^l)$, we have

$$\begin{aligned}
\pi^2((Q-D^1)^+, x_2) &= \pi^2(Q-D^1, x_2) \\
&= p(Q-D^1) + (l-p)E[(Q-D^1-D^2)^+].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E\left[\frac{\partial \pi^2((Q-D^1)^+, x_2)}{\partial Q} 1_{\{D^1\leq Q-G^{-1}(\alpha^l)\}}\right] \\
= \; & E\left[1_{\{D^1\leq Q-G^{-1}(\alpha^l)\}}(p + (l-p)1_{\{Q-D^1-D^2\geq 0\}})\right] \\
= \; & p \Pr(D^1 \leq Q - G^{-1}(\alpha^l)) + (l-p)\Pr(D^1 \leq Q - G^{-1}(\alpha^l), Q - D^1 - D^2 \geq 0).
\end{aligned} \tag{34}$$

Combining (32)–(34), we get

$$\begin{aligned}
& \frac{\partial_1 \tilde{\pi}(Q, x_2)}{\partial Q} \\
= \; & (p-c) - pF(Q) \\
& + p^t \Pr(Q - G^{-1}(\alpha^l) < D^1 \leq Q) \\
& + p \Pr(D^1 \leq Q - G^{-1}(\alpha^l)) + (l-p)\Pr(D^1 \leq Q - G^{-1}(\alpha^l), Q - D^1 - D^2 \geq 0) \\
= \; & (p-c) + (p^t - p)\Pr(Q - G^{-1}(\alpha^l) < D^1 \leq Q) \\
& + (l-p)\Pr(D^1 \leq Q - G^{-1}(\alpha^l), Q - D^1 - D^2 \geq 0),
\end{aligned}$$

as desired.

Part 5:

It is straightforward to check that $\frac{\partial_1 \tilde{\pi}(Q,x_2)}{\partial Q}$ is continuous in the vector (Q, x_2). The partial derivative $\frac{\partial_1^2 \tilde{\pi}(Q,x_2)}{\partial Q^2}$ exists due to the results of parts 2 to 4. It is clear that the partial derivative of $\frac{\partial_1 \tilde{\pi}(Q,x_2)}{\partial Q}$ with respect to x_2 exists when $x_2 \in (-\infty, 0) \cup (0, G^{-1}(\alpha^l)) \cup (G^{-1}(\alpha^l), \infty)$. Moreover, the left- and right-side partial derivatives of $\frac{\partial_1 \tilde{\pi}(Q,x_2)}{\partial Q}$ with respect to x_2 are equal when $x_2 = 0$ and $G^{-1}(\alpha^l)$. Therefore

$\frac{\partial_1^2 \tilde{\pi}(Q,x_2)}{\partial Q \partial x_2}$ also exists. Since the pdf function for the demand is finite, the partial derivatives of $\frac{\partial_1 \tilde{\pi}(Q,x_2)}{\partial Q}$ are bounded.

Part 6:

We need to verify that $\frac{\partial^2 \tilde{\pi}^1(Q,x_2)}{\partial Q^2} \leq 0$ for all x_2.

If $x_2 \leq 0$, by applying Leibniz's rule, we get

$$
\begin{aligned}
& \frac{\partial^2 \tilde{\pi}^1(Q,x_2)}{\partial Q^2} \\
= \; & (p^t - c^t - p)(f(Q - G^{-1}(\alpha^h)) - f(Q - G^{-1}(\alpha^h) + x_2)) \\
& + (l-p)\frac{\partial \int_{Q-G^{-1}(\alpha^h)}^{Q} \int_0^{Q-d^1} g(d^2) f(d^1) dd^2 dd^1}{\partial Q} \\
& + (l-p)\frac{\partial \int_0^{Q-G^{-1}(\alpha^h)+x_2} \int_0^{Q-d^1+x_2} g(d^2) f(d^1) dd^2 dd^1}{\partial Q} \\
= \; & (p^t - c^t - p)(f(Q - G^{-1}(\alpha^h)) - f(Q - G^{-1}(\alpha^h) + x_2)) \\
& + (l-p)\frac{\partial \int_{Q-G^{-1}(\alpha^h)}^{Q} G(Q-d^1) f(d^1) dd^1}{\partial Q} \\
& + (l-p)\frac{\partial \int_0^{Q-G^{-1}(\alpha^h)+x_2} G(Q-d^1+x_2) f(d^1) dd^1}{\partial Q} \\
= \; & (p^t - c^t - p)(f(Q - G^{-1}(\alpha^h)) - f(Q - G^{-1}(\alpha^h) + x_2)) \\
& + (l-p)\left[\int_{Q-G^{-1}(\alpha^h)}^{Q} g(Q-d^1) f(d^1) dd^1 + G(0) f(Q) - G(G^{-1}(\alpha^h)) f(Q - G^{-1}(\alpha^h))\right] \\
& + (l-p)\left[\int_0^{Q-G^{-1}(\alpha^h)+x_2} g(Q-d^1+x_2) f(d^1) dd^1 + G(G^{-1}(\alpha^h)) f(Q - G^{-1}(\alpha^h) + x_2)\right] \\
= \; & (p^t - c^t - p)(f(Q - G^{-1}(\alpha^h)) - f(Q - G^{-1}(\alpha^h) + x_2)) \\
& + (l-p)\left[\int_{Q-G^{-1}(\alpha^h)}^{Q} g(Q-d^1) f(d^1) dd^1 - \alpha^h f(Q - G^{-1}(\alpha^h))\right] \\
& + (l-p)\left[\int_0^{Q-G^{-1}(\alpha^h)+x_2} g(Q-d^1+x_2) f(d^1) dd^1 + \alpha^h f(Q - G^{-1}(\alpha^h) + x_2)\right] \\
= \; & (l-p)\int_{Q-G^{-1}(\alpha^h)}^{Q} g(Q-d^1) f(d^1) dd^1 + (l-p)\int_0^{Q-G^{-1}(\alpha^h)+x_2} g(Q-d^1+x_2) f(d^1) dd^1 \\
< \; & 0
\end{aligned}
$$

The fourth equality is due to $G(0) = 0$ based on our assumptions on the demand distribution. The last equality holds because $\alpha^h = \frac{p-p^t+c^t}{p-l}$.

If $0 < x_2 \leq G^{-1}(\alpha^l)$, by applying Leibniz's rule, we get

$$\begin{aligned}
&\frac{\partial^2 \tilde{\pi}^1(Q,x_2)}{\partial Q^2} \\
= \quad & (p^t - p)(f(Q - G^{-1}(\alpha^l) + x_2) - f(Q - G^{-1}(\alpha^l))) \\
& + (l-p)\frac{\partial \int_0^{Q-G^{-1}(\alpha^l)} \int_0^{Q-d^1} g(d^2) f(d^1) dd^2 dd^1}{\partial Q} \\
& + (l-p)\frac{\partial \int_{Q-G^{-1}(\alpha^l)+x_2}^{Q} \int_0^{Q-d^1+x_2} g(d^2) f(d^1) dd^2 dd^1}{\partial Q} \\
= \quad & (p^t - p)(f(Q - G^{-1}(\alpha^l) + x_2) - f(Q - G^{-1}(\alpha^l))) \\
& + (l-p)\frac{\partial \int_0^{Q-G^{-1}(\alpha^l)} G(Q-d^1) f(d^1) dd^1}{\partial Q} \\
& + (l-p)\frac{\partial \int_{Q-G^{-1}(\alpha^l)+x_2}^{Q} G(Q-d^1+x_2) f(d^1) dd^1}{\partial Q} \\
= \quad & (p^t - p)(f(Q - G^{-1}(\alpha^l) + x_2) - f(Q - G^{-1}(\alpha^l))) \\
& + (l-p)\left[\int_0^{Q-G^{-1}(\alpha^l)} g(Q-d^1) f(d^1) dd^1 + G(G^{-1}(\alpha^l)) f(Q - G^{-1}(\alpha^l))\right] \\
& + (l-p)\left[\int_{Q-G^{-1}(\alpha^l)+x_2}^{Q} g(Q-d^1+x_2) f(d^1) dd^1\right. \\
& \left. + G(x_2) f(Q) - G(G^{-1}(\alpha^l)) f(Q - G^{-1}(\alpha^l) + x_2)\right] \\
= \quad & (l-p)\left[\int_0^{Q-G^{-1}(\alpha^l)} g(Q-d^1) f(d^1) dd^1\right. \\
& \left. + \int_{Q-G^{-1}(\alpha^l)+x_2}^{Q} g(Q-d^1+x_2) f(d^1) dd^1 + G(x_2) f(Q))\right] \\
< \quad & 0
\end{aligned}$$

The last equality holds because $\alpha^l = \frac{p-p^t}{p-l}$.

Finally, if $x_2 > G^{-1}(\alpha^l)$, by applying Leibniz's rule, we can get

$$\begin{aligned}
&\frac{\partial^2 \tilde{\pi}^1(Q,x_2)}{\partial Q^2} \\
= \quad & (p^t - p)(F(Q) - f(Q - G^{-1}(\alpha^l)) \\
& + (l-p)\frac{\partial \int_0^{Q-G^{-1}(\alpha^l)} \int_0^{Q-d^1} g(d^2) f(d^1) dd^2 dd^1}{\partial Q} \\
= \quad & (p^t - p)(F(Q) - f(Q - G^{-1}(\alpha^l)) \\
& + (l-p)\left[\int_0^{Q-G^{-1}(\alpha^l)} g(Q-d^1) f(d^1) dd^1 + G(G^{-1}(\alpha^l)) f(Q - G^{-1}(\alpha^l))\right] \\
= \quad & (p^t - p)F(Q) + (l-p)\int_0^{Q-G^{-1}(\alpha^l)} g(Q-d^1) f(d^1) dd^1 \\
< \quad & 0
\end{aligned}$$

The last equality holds because $\alpha^l = \frac{p-p^t}{p-l}$.

Part 7:

If $x_2 \le 0$,

$$
\begin{aligned}
& \frac{\partial^2 \tilde{\pi}^1(Q, x_2)}{\partial Q \partial x_2} \\
= & -(p^t - c^t - p) f(Q - G^{-1}(\alpha^h) + x_2) \\
& +(l-p)\left[\int_0^{Q-G^{-1}(\alpha^h)+x_2} g(Q - d^1 + x_2) f(d^1) dd^1 + G(G^{-1}(\alpha^h)) f(Q - G^{-1}(\alpha^h) + x_2)\right] \\
= & (l-p) \int_0^{Q-G^{-1}(\alpha^h)+x_2} g(Q - d^1 + x_2) f(d^1) dd^1 \\
< & \ 0
\end{aligned}
$$

and

$$
\frac{\partial^2 \tilde{\pi}^1(Q, x_2)}{\partial Q^2} - \frac{\partial^2 \tilde{\pi}(Q, x_2)}{\partial Q \partial x_2} = (l-p) \int_{Q-G^{-1}(\alpha^h)}^{Q} g(Q - d^1) f(d^1) dd^1 < 0.
$$

If $0 < x_2 \le G^{-1}(\alpha^l)$,

$$
\begin{aligned}
& \frac{\partial^2 \tilde{\pi}^1(Q, x_2)}{\partial Q \partial x_2} \\
= & (p^t - p) f(Q - G^{-1}(\alpha^l) + x_2) + (l-p) \frac{\partial \int_{Q-G^{-1}(\alpha^l)+x_2}^{Q} G(Q - d^1 + x_2) f(d^1) dd^1}{\partial x_2} \\
= & (l-p) \int_{Q-G^{-1}(\alpha^l)+x_2}^{Q} g(Q - d^1 + x_2) f(d^1) dd^1 < 0
\end{aligned}
$$

and

$$
\frac{\partial^2 \tilde{\pi}^1(Q, x_2)}{\partial Q^2} - \frac{\partial^2 \tilde{\pi}(Q, x_2)}{\partial Q \partial x_2} = (l-p) \left[\int_0^{Q-G^{-1}(\alpha^l)} g(Q - d^1) f(d^1) dd^1 + G(x_2) f(Q)\right] < 0.
$$

If $x_2 > G^{-1}(\alpha^l)$,

$$
\frac{\partial^2 \tilde{\pi}(Q, x_2)}{\partial Q \partial x_2} = 0.
$$

□

Lemma 3

Proof. Since $x_2(Q_2, d_2^1)$ is a continuous function and is differentiable almost everywhere, parts 1 and 2 directly follow from the preservation of continuity, differentiability and concavity under expectation.

By Lemma 2.5 and the definition of $x_2(Q_2, d_2^1)$, both $\frac{\partial \tilde{\pi}^1(Q_1, x_2)}{\partial Q_1}$ and $x_2(Q_2, d_2^1)$ are continuous, differentiable almost everywhere, and their partial derivatives are bounded everywhere. Therefore, the same is true of $\frac{\partial \tilde{\pi}^1(Q_1, x_2(Q_2, d_2^1))}{\partial Q_1}$. Therefore, we can change the second-order differential and expectation operations:

$$
\begin{aligned}
\frac{\partial^2 E[\tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))]}{\partial Q_1^2} &= E\left[\frac{\partial^2 \tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))}{\partial Q_1^2}\right] \\
\frac{\partial^2 E[\tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))]}{\partial Q_1 \partial Q_2} &= E\left[\frac{\partial^2 \tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))}{\partial Q_1 \partial Q_2}\right]
\end{aligned}
$$

For each realization of d_2^1 and Q_2, we have

$$\begin{aligned}
\frac{\partial^2 \tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))}{\partial^2 Q_1} &< \frac{\partial^2 \tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))}{\partial Q_1 \partial x_2} \\
&\leq \frac{\partial^2 \tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))}{\partial Q_1 \partial x_2} \frac{\partial x_2}{\partial Q_2} \\
&= \frac{\partial^2 \tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))}{\partial Q_1 \partial Q_2} \\
&\leq 0
\end{aligned}$$

The first inequality follows from Lemma 2.7. The second inequality is due to the fact that $0 \leq \frac{dx_2(Q_2, D_2^1)}{dQ_2} \leq 1$. The last inequality is due to the fact that $\frac{dx_2}{dQ_2} \geq 0$ and $\frac{\partial^2 \tilde{\pi}^1(Q_1, x_2(Q_2, D_2^1))}{\partial Q_1 \partial x_2} \leq 0$. Then $\frac{\partial^2 \pi^1(Q_1, Q_2)}{\partial Q_1^2} < \frac{\partial^2 \pi^1(Q_1, Q_2)}{\partial Q_1 \partial Q_2} \leq 0$ immediately follows. Let $Q_1^*(Q_2)$ be the best response function for store 1. Then by the implicit function theorem, we have

$$\frac{\partial Q_1^*}{\partial Q_2} = -\frac{\frac{\partial^2 \pi^1(Q_1, Q_2)}{\partial Q_1 \partial Q_2}}{\frac{\partial^2 \pi^1(Q_1, Q_2)}{\partial Q_1^2}} \in (-1, 0).$$

Therefore, the best response function of the order-up-to level of store 1 is a decreasing function with respect to Q_2, and the absolute value of its slope is less than 1.

Theorem 2

To prove Theorem 2, we note that all possible candidates for the Nash-equilibrium order quantity for each store are bounded above by the optimal newsboy order quantity that would result if that store were to serve the customers from both stores. Therefore, the decision space for each store is compact.

Also we note that Lemma 3 applies to store 2. Combined with the result from Part 2 in Lemma 3, this implies there exists a Nash equilibrium of order quantities [13, page 34]. The fact that the best response function of the order quantity of store i is a decreasing function with respect to $Q_{\neg i}$ and the absolute value of its slope is less than 1 guarantees the satisfaction of the contraction mapping theorem [6]. Therefore the Nash equilibrium of order quantities is unique. □

Proposition 1

Proof. Theorem 2 still applies to the formulation in (8) since the feasible set in (8) is still convex. Then there exists a unique NE under (8). Thus, we only need to show that the solutions given in this proposition under the four initial conditions are NEs. Then their uniqueness is guaranteed automatically.

1. Because $\pi_{\neg i}(x, V_i^0)$ is a concave function and $B_{\neg i}(V_i^0)$ is its maximizer, $\min(V_{\neg i}^0, B_{\neg i}(V_i^0))$ is a best response for store $\neg i$ when store i chooses V_i^0. Now we need to show that V_i^0 is a best response for store i when store $\neg i$ chooses $\min(V_{\neg i}^0, B_{\neg i}(V_i^0))$. Note that for any $x > y$,

$$x - y > B_i(y) - B_i(x) > 0 \quad (35)$$

$$x - y > B_{\neg i}(y) - B_{\neg i}(x) > 0 \quad (36)$$

since $B_i(\cdot)$ and $B_{\neg i}(\cdot)$ are decreasing functions with slope in $(-1,0)$ by Lemma 3. Then we have

$$\begin{aligned} Q_i^{0*} - V_i^0 &> B_{\neg i}(V_i^0) - B_{\neg i}(Q_i^{0*}) \\ &= B_{\neg i}(V_i^0) - Q_{\neg i}^{0*} \\ &\geq \min(V_{\neg i}^0, B_{\neg i}(V_i^0)) - Q_{\neg i}^{0*} \\ &> B_i(Q_{\neg i}^{0*}) - B_i(\min(V_{\neg i}^0, B_{\neg i}(V_i^0))) \\ &= Q_i^{0*} - B_i(\min(V_{\neg i}^0, B_{\neg i}(V_i^0))) \end{aligned}$$

The first line follows from (36) and the second line follows from the fact that $B_{\neg i}(Q_i^{0*}) = Q_{\neg i}^{0*}$ since $(Q_i^{0*}, Q_{\neg i}^{0*})$ is a NE pair. A similar argument holds for the remaining inequalities. Therefore, V_i^0 is the best response for store i if store $\neg i$ chooses $\min(V_{\neg i}^0, B_{\neg i}(V_i^0))$, since $\pi_i(x, \min(V_{\neg i}^0, B_{\neg i}(V_i^0)))$ is a concave function, $B_i(\min(V_{\neg i}^0, B_{\neg i}(V_i^0)))$ is its maximizer and $B_i(\min(V_{\neg i}^0, B_{\neg i}(V_i^0))) > V_i^0 \geq Q_i$.

2. Follows from part 1 by symmetry.

3. Since π_i^1 is concave with respect to S_i, respectively, the optimality condition is unchanged when $V_1^0 \geq Q_1^{0*}$ and $V_2^0 \geq Q_2^{0*}$. Therefore, (Q_1, Q_2)=(Q_1^{0*}, Q_2^{0*}).

4. When $V_1^0 < Q_1^{0*}$ and $V_2^0 < Q_2^{0*}$, no store would deviate from ordering (V_1^0, V_2^0) since $\pi_i^1(Q_i, V_{\neg i}^0)$ is concave function. So it is a NE. □

Theorem 3

Proof. Follows immediately from Proposition 1 and the fact that $B_i(x)$ is decreasing in x. □

Proposition 2

1. We prove the inequalities from left to right.

    - We can treat MS as CS with both stores taking the higher sales price and salvage value, lower purchase cost, and zero transshipment cost. In the MS system, however, the two stores can transship their inventory whenever they want. Therefore, the first inequality holds.
    - For every pair of orders (Q_1, Q_2) at the two stores, the distribution of (I_1, I_2) is the same under CS and DS since the demand realizations in the first subperiod do not depend on the type of system. Therefore, the profit in the first subperiod is the same under CS and DS. For each (I_1, I_2), we have $\pi^{c2}(I_1, I_2) \geq \pi_1^{d2}(I_1, I_2) + \pi_2^{d2}(I_1, I_2)$ since the solution under CS is the global maximizer. Therefore, the second inequality holds.
    - In DS, store i has recourse in the second subperiod, so its total profit cannot be worse than in SS, in which it has no recourse. Therefore, the last inequality holds.

2. Follows from the proof of part 2.

Theorem 4

Proof. Part 1: Although the optimal transfer policy in the CS system is unknown, we show that no CBC transfer policy can be optimal in the CS system. Suppose a CBC transfer policy is optimal for the CS system. Then let γ be store 1's transship-up-to level and β be store 2's transship-down-to level under the optimal CS system. Moreover, suppose that $p_1 \leq p_2$ WLOG.

Based on the assumptions of the unboundedness of the demand distribution, I_1 equals zero with positive probability. Moreover, suppose $I_2 = k < Q_2^{c*}$ such that it is optimal for store 2 to transfer a positive quantity to store 1 under the CS system. (Since there is no zero probability of transshipment between the two stores, we can always find such k.)

Let z be the actual transshipment quantity from store 2 to store 1. When $(I_1, I_2) = (0, k)$, the total profit of the CS system in the second sub-period is as follows:

$$\begin{aligned}\pi^{c2}(z,0,k) = \quad & p_1 E[\min(z, D_1^2)] + l_1 E[(z - D_1^2)^+] \\ & + p_2 E[\min(k-z, D_2^2)] + l_1 E[(k - z - D_2^2)^+] \\ & - c^t z\end{aligned}$$

Then we have,

$$f(z|k) = \frac{\partial \pi^{c2}(z,0,k)}{\partial z} \quad = \quad p_1 - (p_1 - l_1)G_1(z) - p_2 + (p_2 - l_2)G_2(k-z) - c^t$$

Define $z^*(0,k)$ to be the optimal transfer quantity from store 2 to store 1. Using the definition of k and the fact that $p_1 \leq p_2$, we know that $0 < z^*(0,k) < k$ and $f(z^*(0,k)|k) = 0$. As a result, we can see either $\gamma = z^*(0,k)$ or $\beta = k - z^*(0,k)$ due to the optimality of the CBC policy under the CS system. We show that neither case is possible.

If $\gamma = z^*(0,k)$, then $\beta \leq k - z^*(0,k)$ due to the CBC policy and the fact that $0 < z^*(0,k) < k$. Suppose that we have $I_1 = 0$ and $I_2 = \frac{k+Q_2^{c*}}{2} > k$. Then, store 1 will propose to transship in $z^*(0,k)$ and store 2 will propose to transship out $I_2 - \beta > k - \beta \geq z^*(0,k)$. Therefore, the realized transshipment amount $z^*\left(0, \frac{k+Q_2^{c*}}{2}\right) = z^*(0,k)$. However,

$$\begin{aligned}& \frac{\partial \pi^{c2}(z, 0, \frac{k+Q_2^{c*}}{2})}{\partial z}\Big|_{z=z^*(0,k)} \\ = \quad & p_1 - (p_1 - l_1)G_1(z^*(0,k)) - p_2 + (p_2 - l_2)G_2\left(\frac{k+Q_2^{c*}}{2} - z^*(0,k)\right) - c^t \\ > \quad & p_1 - (p_1 - l_1)G_1(z^*(0,k)) - p_2 + (p_2 - l_2)G_2\left(k - z^*(0,k)\right) - c^t \\ = \quad & 0\end{aligned}$$

The inequality is due to the fact that $\frac{k+Q_2^{c*}}{2} > k$ and the fact that $G_2(x)$ is strictly increasing for $x \geq 0$. This indicates that $z^*\left(0, \frac{k+Q_2^{c*}}{2}\right) \neq z^*(0,k)$. Therefore, we have $\gamma \neq z^*(0,k)$.

Now we must have $\beta = k - z^*(0,k)$. Similarly, we have $z^*\left(0, \frac{k+\beta}{2}\right) = \frac{k-\beta}{2}$ when $I_1 = 0$ and $I_2 = \frac{k+\beta}{2}$. Then we have

$$
\begin{aligned}
& \frac{\partial \pi^{c2}(z, 0, \frac{k+\beta}{2})}{\partial z}|_{z=\frac{k-\beta}{2}} \\
= \ & p_1 - (p_1 - l_1)G_1\left(\frac{k-\beta}{2}\right) - p_2 + (p_2 - l_2)G_2(\beta) - c^t \\
> \ & p_1 - (p_1 - l_1)G_1(z^*(0,k)) - p_2 + (p_2 - l_2)G_2(\beta) - c^t \\
= \ & 0
\end{aligned}
$$

Similarly, we have $\beta \neq k - z^*(0,k)$. This is a contradiction. Therefore, the CBC transfer policy cannot be optimal under centralized transshipments. It is strictly dominated by the optimal centralized transfer policy.

Part 2:

For each pair of order quantities at both stores, there is a strictly positive probability that the CS system performs better than the DS system in the second subperiod while the first-subperiod total profit is the same for the both systems, no matter how p^t is set in the DS system. Therefore, $\pi^{c*} > \pi^{d*}(p^t)$ for any p^t. □

Proposition 3

Proof. Part 1

By changing the decision space from the transshipment amount (z) to the inventory level after transshipment (L_1, L_2), the optimal total profit in the second subperiod can be obtained by solving the following problem:

$$
\begin{aligned}
\pi^{c2}(I_1, I_2) = \quad & \max_{L_1, L_2} \quad E[p(\min(L_1, D_1^2) + \min(L_2, D_2^2)) \\
& \qquad\quad +l((L_1 - D_1^2)^+ + (L_2 - D_2^2)^+)] \\
& \text{s.t.} \quad L_1 + L_1 = I_1 + I_2 \\
& \qquad\quad L_1, L_1 \in R_+^2
\end{aligned} \tag{37}
$$

Note that $(I_1, I_2) \in R_+^2$. It is straightforward to show that $L_1 = L_2 = \frac{I_1+I_2}{2}$ is a maximal point for (37). Suppose there is a maximal point $(\frac{I_1+I_2}{2} - \beta, \frac{I_1+I_2}{2} + \beta)$ for some $\beta \in R$; then $(\frac{I_1+I_2}{2} + \beta, \frac{I_1+I_2}{2} - \beta)$ is also a maximal point by symmetry. Then all feasible points lying between those two points are also maximal points due to the concavity of the objective function. Since the objective function is strictly concave due to the demand assumptions, there is only one maximal point. Therefore $\beta = 0$ and $z^*(I_1, I_2) = \frac{I_1-I_2}{2}$.

Part 2

Using part 1, $\pi^{c2}(I_1, I_2)$ can be simplified as follows:

$$\begin{aligned}\pi^{c2}(I_1, I_2) &= E\left[p\left(\min\left(\frac{I_1+I_2}{2}, D_1^2\right) + \min\left(\frac{I_1+I_2}{2}, D_2^2\right)\right)\right. && (38)\\ &\quad \left. +l\left(\left(\frac{I_1+I_2}{2} - D_1^2\right)^+ + \left(\frac{I_1+I_2}{2} - D_2^2\right)^+\right)\right] \\ &= p(I_1+I_2) - 2(p-l)E\left[\left(\frac{I_1+I_2}{2} - D_1^2\right)^+\right] && (39)\end{aligned}$$

After setting $(I_1, I_2) = ([Q_1^c - D_1^1]^+, [Q_2^c - D_2^1]^+)$ and substitute $\pi^{c2}(I_1, I_2)$ from (39) to (14), we get

$$\begin{aligned}\pi^{c1}(Q_1^c, Q_2^c) &= (p-c)(Q_1^c + Q_2^c) - p(I_1+I_2) + \pi^{c2}(I_1, I_2) \\ &= (p-c)(Q_1^c + Q_2^c) - 2(p-l)E\left[\left(\frac{[Q_1^c - D_1^1]^+ + [Q_2^c - D_2^1]^+}{2} - D_1^2\right)^+\right]\end{aligned}$$

From this it is straightforward to show that $\pi^{c1}(Q_1^c, Q_2^c)$ is strictly concave in (Q_1^c, Q_2^c). □

Lemma 4

Proof. The proof is similar to that of Lemma 2. □

Lemma 5

Proof. The proof is similar to that of Lemma 3. □

Theorem 5

Proof. The proof is similar to that of Theorem 2. □