On the Volumetric Path

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Abstract

We consider the logarithmic and the volumetric barrier functions used in interior point methods. In the case of the logarithmic barrier function, the anaytic center of a level set is the point at which the central path intersects that level set. We prove that this also holds for the volumetric path. For the central path, it is also true that the analytic center of the optimal level set is the limit point of the central path. The only known case where this last property with the logarithmic barrier function fails occurs in case of semidefinite optimization in the absence of strict complementarity. By an example we show that this property does not hold even for a linear optimization problem in canonical form for the volumetric path.

1 Introduction

Let P be a polyhedral set of the form $P = \{x : Ax \geq b\}$ where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. We assume that the feasible set P has a nonempty interior and is bounded. Boundedness implies $\text{rank}(A) = n$ and $m > n$. Let $F(x) = -\sum_{i=1}^m \log(a_i^T x - b_i)$ be the logarithmic barrier function, and $H(x) = \nabla^2 F(x)$ be the Hessian of $F(x)$. As $F(x)$ is a strictly convex function, the Hessian $H(x)$ is positive definite. Define $V(x) = \log \det H(x)$. The function $V(x)$ is called the volumetric barrier function for $x \in P$ and is known to be strictly convex as well [8]. In [4] Atkinson and Vaidya introduces a cutting plane algorithm using the volumetric barrier function. This algorithm is further improved in [1, 2]. Anstreicher([3]) extends the volumetric barrier to the semidefinite case.

Given the linear optimization problem in the canonical form with $c \neq 0$:

$$\text{(LP)} \quad \min \quad c^T x \qquad \text{s.t} \quad Ax \geq b$$

for $\mu > 0$, consider the nonlinear optimization problem using the volumetric barrier function:

$$\begin{array}{ll} \min & c^T x + \mu V(x) \\ \text{s.t} & Ax > b \ . \end{array} \tag{1}$$

Let $G_\mu(x) = c^T x + \mu V(x)$ and $x(\mu)$ be the (unique) optimal solution of $G_\mu(x)$. The optimal points $x(\mu)$ parameterized by μ form a differentiable path called the volumetric path. As $\mu \to 0$, $x(\mu)$ converges to an optimal solution of (LP).

2 Properties of the volumetric path

In the next two propositions we prove certain fundamental properties of the volumetric path. The next proposition deals with the monotonicity of the objective on the volumetric path [7].

Proposition 2.1. *For $\mu_1 < \mu_2$, we have the following:*

1. $x(\mu_1) \neq x(\mu_2)$.
2. $c^T x(\mu_1) < c^T x(\mu_2)$.

Proof.

1. Let $\mu_1 < \mu_2$ and suppose, to the contrary, that $x(\mu_1) = x(\mu_2) = \bar{x}$. The first order optimality conditions for (1) give

$$\nabla(c^T x + \mu_1 V(x)) = 0 \Longrightarrow c + \mu_1 \nabla V(x) = 0$$

$$\nabla(c^T x + \mu_1 V(x)) = 0 \Longrightarrow c + \mu_2 \nabla V(x) = 0,$$

that implies

$$\nabla V(x) = \frac{-1}{\mu_1} c = \frac{-1}{\mu_2} c,$$

which is a contradiction for $c \neq 0$.

2. Let x^1 and x^2 be the optimal solutions of G_{μ_1} and G_{μ_2}, respectively. Since G_μ is strictly convex, for $\mu > 0$, we have

$$G_{\mu_1}(x^1) < G_{\mu_1}(x^2)$$

$$G_{\mu_2}(x^2) < G_{\mu_2}(x^1),$$

which implies

$$c^T x^1 + \mu_1 V(x^1) < c^T x^2 + \mu_1 V(x^2) \tag{2}$$

$$c^T x^2 + \mu_2 V(x^2) < c^T x^1 + \mu_2 V(x^1). \tag{3}$$

By multiplying the inequalities (2) by μ_2, and (3) by μ_1, respectively, and adding the resulting inequalities, after cancellations one gets

$$\mu_2 c^T x^1 + \mu_1 c^T x^2 < \mu_2 c^T x^2 + \mu_1 c^T x^1,$$

which implies $c^T x^1 < c^T x^2$.

□

Next we examine the relationship between the points $x(\mu)$, $\mu > 0$ on the volumetric path and the so-called volumetric center of the level sets $c^T x = \alpha$.

Definition 2.2.

1. *A level set* $\mathcal{L}_\alpha$ *for (P) is the set* $\{x \in \mathbb{R}^n : c^T x = \alpha,\ Ax \geq b\}$
2. *The volumetric center* $\hat{x}$ *of a (bounded) level set* $\mathcal{L}_\alpha$ *is defined to be the (unique) minimizer of the volumetric function* $V(x)$ *over* $\mathcal{L}_\alpha$.

Proposition 2.3. *Let* $\mu > 0$ *and* $\hat{x} = x(\mu)$ *be the optimal solution of (1) with* $c^T \hat{x} = \alpha$ *for some* α*. Then* $\hat{x}$ *is the volumetric center of* $\mathcal{L}_\alpha$*.*

Proof. Consider the following problems:

$$(\S) \quad \begin{array}{ll} \min & V(x) \\ \text{s.t} & c^T x = \alpha \\ & Ax > b. \end{array} \qquad\qquad (\S\S) \quad \begin{array}{ll} \min & \frac{c^T x}{\mu} + V(x) \\ \text{s.t} & Ax > b. \end{array}$$

Let $\bar{x}$ and $\hat{x}$ be the optimal solutions of (§) and (§§), respectively. The first order optimality conditions for (§) give

$$\nabla V(\bar{x}) + \lambda c = 0, \quad c^T \bar{x} = \alpha\ , \tag{4}$$

where λ is the (unique) Lagrange multiplier, and for (§§) give

$$\frac{c}{\mu} + \nabla V(\hat{x}) = 0\ . \tag{5}$$

Since by assumption $c^T \hat{x} = \alpha$, $\hat{x}$ satisfies (4) with the choice of $\lambda = \frac{1}{\mu}$. Since (§) and (§§) have unique optimal solutions, it follows that $\bar{x} = \hat{x}$. This completes the proof.

□

3 The limit of the volumetric path

Let $x^* = \lim_{\mu \to 0} x(\mu)$ be an optimal solution of (LP) with the corresponding optimal objective value $\alpha^* = c^T x^*$. From Proposition 2.3, one sees that as α decreases to α^*, the volumetric centers of the level sets $\mathcal{L}_{\alpha^*}$ converge to x^*. Thus a natural question arises about whether x^* is the volumetric center ofthe optimal level set $\mathcal{L}_{\alpha^*}$. Observe that since certain constraints have to be active in the optimal level set $\mathcal{L}_{\alpha^*}$, the volumetric barrier function $V(x)$ is not defined on $\mathcal{L}_{\alpha^*}$. Hence in order to define the volumetric center of $\mathcal{L}_{\alpha^*}$, one needs to identify the constraints that are inactive at x^*, i.e. the constraints which hold with strict inequality in the relative interior of $\mathcal{L}_{\alpha^*}$. Let I be the set of inactive constraints of $Ax \geq b$ in the relative interior of the optimal level set $\mathcal{L}_{\alpha^*}$. Let $\overline{F}(x) = -\sum_{i \in I} \log(a_i^T x - b_i)$ and $\overline{V}(x)$ be defined accordingly. The volumetric center of the optimal level set $\mathcal{L}_{\alpha^*}$ is defined as the unique minimizer of

$$\begin{array}{ll} \min & \overline{V}(x) \\ \text{s.t} & c^T x = \alpha^* \\ & a_i^T x = b_i, \; i \notin I \\ & a_i^T x > b_i, \; i \in I \end{array} \tag{6}$$

It is known that for the logarithmic barrier function, the central path converges to the analytic center of the optimal level set [7]. In particular, for a linear optimization problem in the standard form, the volumetric barrier function reduces to the logarithmic barrier function, hence in this case the volumetric path converges to the volumetric center of the optimal level set also. A natural question to ask is whether this extends to the problems in the form (LP). As the following example illustrates, this fact fails to hold for (LP).

Example 3.1.
Let the rows of the matrix $A \in \mathbb{R}^{5\times 2}$ be given by the vectors $a_1^T = (1, 0)$, $a_2^T = (-0.1, -1)$, $a_3^T = (-1, 0)$, $a_4^T = (-0.1, 1)$, $a_5^T = (0, 1)$ with the objective vector $c^T = (0, 1)$ and $b^T = (0, -1, -1, 0, 0.1)$. The optimal objective value is $\alpha^* = 0.1$.
For a polyhedral set of the form $P = \{x : Ax \geq b\}$, the Hessian $H(x) = \nabla^2 F(x)$ of the logarithmic barrier function is computed as $H(x) = \sum_{i=1}^m H_i(x)$, where $H_i(x) = a_i a_i^T/(a_i^T x - b_i)^2$ [8]. For our example $m = 5$ and
$H_1(x) = \frac{1}{x_1^2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $H_2(x) = \frac{1}{(1-0.1x_1-x_2)^2}\begin{bmatrix} 0.01 & 0.1 \\ 0.1 & 1 \end{bmatrix}$, $H_3(x) = \frac{1}{(1-x_1)^2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,
$H_4(x) = \frac{1}{(-0.1x_1+x_2)^2}\begin{bmatrix} 0.01 & -0.1 \\ -0.1 & 1 \end{bmatrix}$, $H_5(x) = \frac{1}{(x_2-0.1)^2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

From Proposition 2.3, the volumetric path converges to

$$x^* = \lim_{\epsilon \to 0} x(\epsilon)$$

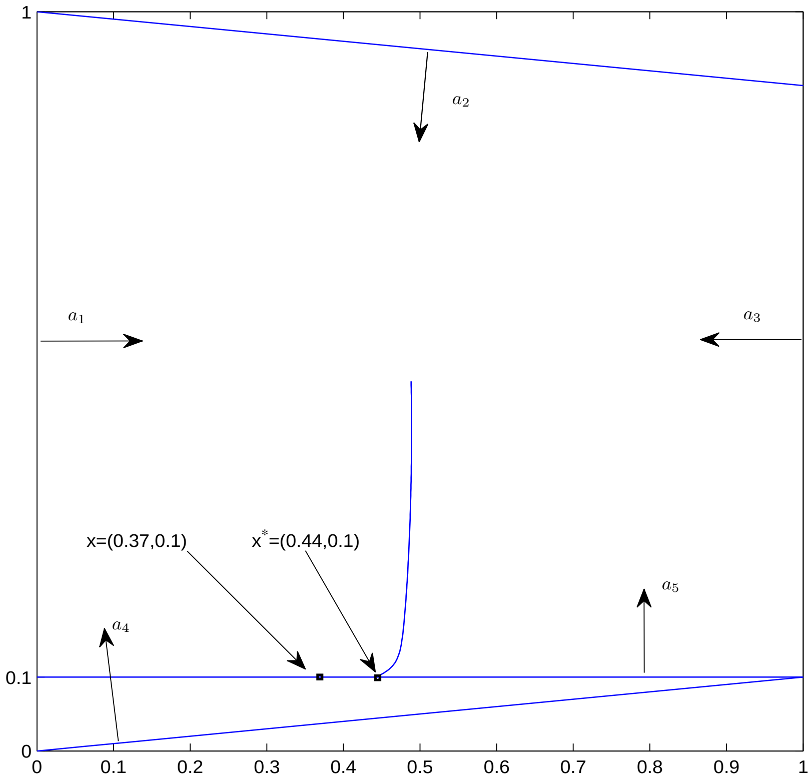

Figure 1: The volumetric center of the optimal level set is (0.37,0.1), while the volumetric path converges to (0.44, 0.1).

where

$$x(\epsilon) = \arg\min \quad \log \det H(x)$$
$$x_2 = 0.1 + \epsilon$$

Now, log det $H(x)$ is computed as

$$\log[(60000\epsilon^4 x_1^2 - 60000\epsilon^4 x_1 + 30000\epsilon^4 - 64000\epsilon^3 x_1^2 + 64000\epsilon^3 x_1 - 32000\epsilon^3 + 600\epsilon^2 x_1^4$$
$$-1200\epsilon^2 x_1^3 + 26200\epsilon^2 x_1^2 - 25600\epsilon^2 x_1 + 12800\epsilon^2 + 160\epsilon x_1^4 - 3200\epsilon x_1^3 + 6080\epsilon x_1^2$$
$$-4480\epsilon x_1 + 1440\epsilon + 4x_1^6 - 66x_1^5 + 401x_1^4 - 800x_1^3 + 722x_1^2 - 342x_1 + 81)/$$
$$(\epsilon^2 x_1^2 (x_1 - 1)^2 (10\epsilon - x_1 + 1)^2 (10\epsilon + x_1 - 9)^2)]\ .$$

Let $h(x_1, \epsilon) = \log(\epsilon^2 \det H(x))$. Clearly for $\epsilon > 0$ fixed, the minimizer of the function log det $H(x)$ is the same as the minimizer of $h(x_1, \epsilon)$. Note that

$$\lim_{\epsilon \to 0} h(x_1, \epsilon) = \log \frac{(4x_1^6 - 66x_1^5 + 401x_1^4 - 800x_1^3 + 722x_1^2 - 342x_1 + 81)}{x_1^2(x_1-1)^4(x_1-9)^2}$$
$$= \log \frac{(4x_1^4 - 58x_1^3 + 281x_1^2 - 180x_1 + 81)}{x_1^2(x_1-1)^2(x_1-9)^2}$$

Denote this limit by $g(x_1)$. We will argue that the first coordinate of the limit point of the volumetric path $x^* = \lim_{\epsilon \to 0} x(\epsilon)$ is the minimizer of $g(x_1)$.

First the unique minimizer of $g(x_1)$ can be computed as $x^* = 0.44248$. Let $g_n(x_1) = h(x_1, \frac{1}{n})$. We will show that $\lim_{n\to\infty} \arg\min\ g_n(x_1) = x^*$. Suppose by contradiction that $\lim_{n\to\infty} \arg\min\ g_n(x_1) = \bar{x} \neq x^*$ for some $\bar{x}$. Choose an interval $[a, b] \subseteq [0, 1]$ containing x^* such that $\bar{x} \notin [a, b]$. Since $g(x_1)$ has minimum at x^*, one can choose an ϵ with $0 < \epsilon < \min\{\frac{g(a)-g(x^*)}{2}, \frac{g(b)-g(x^*)}{2}\}$. Since $g_n(x_1)$ converges uniformly to $g(x_1)$ on the compact interval $[a, b]$, there exists a number $N \in \mathbb{N}$ such that $n \geq N$ implies $g(x) - \epsilon < g_n(x) < g(x) + \epsilon$ for all $x \in [a, b]$. For $n \geq N$ we have,

$$\begin{aligned} g_n(x^*) < g(x^*) + \epsilon < g(a) - \epsilon < g_n(a) \\ g_n(x^*) < g(x^*) + \epsilon < g(b) - \epsilon < g_n(b) \end{aligned} \tag{7}$$

Fix $n \geq N$. If $g_n(x_1)$ has a minimizer x not in $[a, b]$, (7) would imply that the points $g_n(a), g_n(b), g_n(x^*)$ and $g_n(x)$ contradict the convexity property of $g_n(x_1)$. This shows that for any $n \geq N$, the unique minimizer of $g_n(x_1)$ must lie in the interval $[a, b]$. Hence this would be a contradiction to the assumption that $\lim_{n\to\infty} \arg\min\ g_n(x_1) = \bar{x} \notin [a, b]$. Thus we obtain $\lim_{n\to\infty} \arg\min\ g_n(x_1) = x^* = 0.44248$ as the limit point of the volumetric path.

On the other hand, at the optimal objective value $\alpha^* = 0.1$ the constraint a_5 is active, and the volumetric center of the optimal level set defined by (6) is the unique solution of

$$\begin{aligned} \min \quad & \log\det \overline{H}(x) \\ & x_2 = 0.1, \end{aligned} \tag{8}$$

where $\overline{H}(x) = \sum_{i=1}^{4} H_i(x)$. The optimal solution of (8) is computed as $x = (0.37087, 0.1)$, whose first coordinate is not equal to x^*. Thus this counterexample demonstrates that the limit of the volumetric path is not necessarily the volumetric center of the optimal level set.

4 Conclusion

In this article, we investigate certain properties of the volumetric path for a linear optimization problem (LP) in the standard dual form. These properties are all known to be true for the logarithmic barrier function. We show that the objective function is monotone on the volumetric path . Next we prove that the intersection of the volumetric path with a level set is the volumetric center of that level set. While for the logaritmic barrier function, the limit of the central path is known to be the analytic center of the optimal level set, this property does not hold for the case of semidefinite optimization when strict complementarity fails to hold [5, 6]. The main result of the article is that the above property fails to hold for the volumetric barrier function even for linear programs. Thus we conclude that the limit of the volumetric path is not necessarily the volumetric center of the optimal set even for the case of linear optimization where strict complementarity always holds.

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