q-Factor Optimization for 2-WEC Arrays

LAWRENCE V. SNYDER
Dept. of Industrial and Systems Engineering
Lehigh University, Bethlehem, PA

COR@L Technical Report 14T-001
q-Factor Optimization for 2-WEC Arrays

LAWRENCE V. SNYDER*1

1Dept. of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA

February 7, 2014

Abstract

We derive an analytic solution for the wave energy converter (WEC) layout problem with $N = 2$ devices, assuming regular (sinusoidal) incident waves, under the point-absorber approximation.

Assume WEC 1 is located at the origin and WEC 2 is located at $(d, \alpha)$. We assume there is a minimum-separation constraint of the form

$$d \geq d_0 \lambda,$$

where $\lambda$ is the incident wavelength and $d_0 > 0$ is a constant. Then $(d_1, \alpha_1) = (0, 0); (d_2, \alpha_2) = (d, \alpha); d_{12} = d_{21} = d$. Let $z \equiv kd \cos(\beta - \alpha)$. Then

$$L = \begin{bmatrix} 1 \\ e^{ikd \cos(\beta - \alpha)} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{iz} \end{bmatrix} \quad L^* = \begin{bmatrix} 1 & e^{-iz} \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & J_0(kd) \\ J_0(kd) & 1 \end{bmatrix} \quad J^{-1} = \frac{1}{1 - J_0(kd)^2} \begin{bmatrix} 1 & -J_0(kd) \\ -J_0(kd) & 1 \end{bmatrix}$$

since $J_0(0) = 1$.

Therefore:

$$q = \frac{1}{2} L^* J^{-1} L$$

$$= \frac{1}{2(1 - J_0(kd)^2)} \begin{bmatrix} 1 & e^{-iz} \end{bmatrix} \left[ \begin{bmatrix} 1 \\ -J_0(kd) \end{bmatrix} - J_0(kd) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] \begin{bmatrix} 1 \\ e^{iz} \end{bmatrix}$$

$$= \frac{1}{2(1 - J_0(kd)^2)} \begin{bmatrix} 1 - J_0(kd)e^{-iz} \\ J_0(kd)e^{-iz} - J_0(kd) + e^{-iz} \end{bmatrix} \begin{bmatrix} 1 \\ e^{iz} \end{bmatrix}$$

$$= \frac{2 - J_0(kd)(e^{-iz} + e^{iz})}{2(1 - J_0(kd)^2)}$$

$$= \frac{2 - J_0(kd)(2 \cos z)}{2(1 - J_0(kd)^2)}$$

* larry.snyder@lehigh.edu
since $e^{-iz} + e^{iz} = (\cos z + i \sin z) + (\cos z - i \sin z) = 2 \cos z$

$$= \frac{1 - J_0(kd) \cos(kd \cos(\beta - \alpha))}{1 - J_0(kd)^2}$$

Now, we can choose any $\alpha$, so we can get any $\beta - \alpha \in [-\pi, \pi] \mod 2\pi$, so we can get any value of $\cos(\beta - \alpha)$ in $[-1, 1]$, so we can get any value of $\cos(kd \cos(\beta - \alpha))$ in $[-1, 1]$, assuming $kd \geq \pi$ (which is almost certainly true).

Note that $J_0(0) = 1$ and for $x > 0$, $J_0(x)$ resembles a cosine function whose amplitude decreases as $x$ increases. (See Figure 1.) Let $j_n$ be the $n$th local optimizer (min or max) of $J_0(\cdot)$. Maximizers have odd indices $n$ while minimizers have even indices. Approximate values are given in Table 1. (Ignore the last three columns for now.)

![Figure 1: Bessel function of the first kind of order 0.](image)

Table 1: Local optima of Bessel function, and resulting optimal $q$ (for any $k$, $\beta$) and optimal locations of WEC 2 (for $k = 0.2$, $\beta = 0$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$j_n$</th>
<th>$J_0(j_n)$</th>
<th>$q^*$</th>
<th>$d^*$</th>
<th>$\alpha^*$</th>
<th>$x^*$</th>
<th>$y^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000</td>
<td>1.0000</td>
<td>$\infty$</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>3.8317</td>
<td>-0.4028</td>
<td>1.6744</td>
<td>19.1585</td>
<td>-1.5708</td>
<td>0.0000</td>
<td>-19.1585</td>
</tr>
<tr>
<td>3</td>
<td>7.0156</td>
<td>0.3001</td>
<td>1.4288</td>
<td>35.0780</td>
<td>-1.1065</td>
<td>15.7080</td>
<td>-31.3644</td>
</tr>
<tr>
<td>4</td>
<td>10.1735</td>
<td>-0.2497</td>
<td>1.3328</td>
<td>50.8675</td>
<td>-1.5708</td>
<td>0.0000</td>
<td>-50.8675</td>
</tr>
<tr>
<td>5</td>
<td>13.3237</td>
<td>0.2184</td>
<td>1.2794</td>
<td>66.6185</td>
<td>-1.3328</td>
<td>15.7080</td>
<td>-64.7401</td>
</tr>
<tr>
<td>6</td>
<td>16.4706</td>
<td>-0.1965</td>
<td>1.2445</td>
<td>82.3530</td>
<td>-1.5708</td>
<td>0.0000</td>
<td>-82.3530</td>
</tr>
<tr>
<td>7</td>
<td>19.6158</td>
<td>0.1801</td>
<td>1.2196</td>
<td>98.0790</td>
<td>-1.4099</td>
<td>15.7080</td>
<td>-96.8130</td>
</tr>
<tr>
<td>8</td>
<td>22.7601</td>
<td>-0.1672</td>
<td>1.2007</td>
<td>113.8005</td>
<td>-1.5708</td>
<td>0.0000</td>
<td>-113.8005</td>
</tr>
</tbody>
</table>

(Local optima were found using Matlab fminsearch with default settings.)

Consider a fixed value of $d$. We consider two cases:

Case 1: $J_0(kd) \geq 0$. Then $q$ is maximized when $\alpha$ is chosen such that $\cos(kd \cos(\beta - \alpha)) = -1$, in which case

$$q = \frac{1 + J_0(kd)}{1 - J_0(kd)^2} = \frac{1}{1 - J_0(kd)}.$$  \hspace{1cm} (4)

We want $J_0(kd)$ to be as small as possible. Since local mins become progressively larger as $d$ increases, we want the smallest local argmin possible, which means choosing the smallest $d \geq d_0 \lambda$ so that $kd$ is a local argmin of
\( J_0(\cdot) \), unless \( J_0(kd_0\lambda) \) is already smaller than \( J_0 \) of this argmin, in which case we should set \( d = d_0\lambda \). Therefore,

\[
d^* = \begin{cases} 
\frac{j^*}{k}, & \text{if } J_0(j^*) \leq J_0(kd_0\lambda) \\
d_0\lambda, & \text{otherwise},
\end{cases}
\]

where \( n^* = 2m^* \) and \( m^* \) is the smallest integer \( m \) such that \( j_{2m} \geq kd_0\lambda \). (In other words, \( n^* \) is the smallest argmin greater than or equal to \( kd_0\lambda \).) We want \( \cos(kd\cos(\beta - \alpha^*)) = -1 \), i.e.,

\[
\alpha^* = \beta - \arccos \left( \frac{\arccos(-1)}{kd} \right) = \beta - \arccos \left( \frac{\pi}{kd} \right).
\]

In the special case of \( \beta = 0 \), this solution translates to the following rectangular coordinates (assuming the first case in (5) holds):

\[
x^* = d^* \cos(\alpha^*) \\
y^* = d^* \sin(\alpha^*) \\
= \frac{j^*}{k} \cdot \frac{\pi}{k} = \frac{j^*}{k} \\
= \frac{j_n^*}{k} \sin \left( - \arccos \left( \frac{\pi}{kd^*} \right) \right) \\
= \frac{-j_n^*}{k} \sin \left( \arccos \left( \frac{\pi}{j_n^*} \right) \right) \\
= \frac{-j_n^*}{k} \sqrt{1 - \left( \frac{\pi}{j_n^*} \right)^2} \quad (\sin(\arccos x) = \sqrt{1 - x^2}) \\
= \frac{-j_n^*}{k} \sqrt{1 - \frac{\pi^2}{j_n^*^2}} 
\]

Case 2: \( J_0(kd) < 0 \). Then \( q \) is maximized when \( \alpha \) is chosen such that \( \cos(kd\cos(\beta - \alpha)) = 1 \), in which case

\[
q = \frac{1 - J_0(kd)}{1 - J_0(kd)^2} = \frac{1}{1 + J_0(kd)}.
\]

We want \( J_0(kd) \) to be as large as possible. Since local maxes become progressively smaller as \( d \) increases, we want the smallest local argmax possible, which means choosing the smallest \( d \geq d_0\lambda \) so that \( kd \) is a local argmax of \( J_0(\cdot) \), unless \( J_0(kd_0\lambda) \) is already larger than \( J_0 \) of this argmax, in which case we should set \( d = d_0\lambda \). Therefore,

\[
d^* = \begin{cases} 
\frac{j^*}{k}, & \text{if } J_0(j^*) \geq J_0(kd_0\lambda) \\
d_0\lambda, & \text{otherwise},
\end{cases}
\]

where \( n^* = 2m^* + 1 \) and \( m^* \) is the smallest integer \( m \) such that \( j_{2m+1} \geq kd_0\lambda \). (In other words, \( n^* \) is the smallest argmax greater than or equal to \( kd_0\lambda \).) We want \( \cos(kd\cos(\beta - \alpha^*)) = 1 \), i.e.,

\[
\alpha^* = \beta - \arccos \left( \frac{\arccos(1)}{kd} \right) = \beta - \frac{\pi}{2}.
\]
In the special case of $\beta = 0$, this solution translates to the following rectangular coordinates (assuming the first case in (8) holds):

\[
x^* = d^* \cos(\alpha^*) \\
= \frac{j_n^*}{k} \cos \left( -\frac{\pi}{2} \right) = 0 \\
y^* = d^* \sin(\alpha^*) \\
= \frac{j_n^*}{k} \sin \left( -\frac{\pi}{2} \right) = -\frac{j_n^*}{k}
\]

**Theorem 1** Let $n^*$ be the smallest integer $n$ such that $j_n \geq kd_0 \lambda$. Then the optimal solution $(d^*, \alpha^*)$ and the corresponding $q^*$, are as follows:

\[
d^* = \begin{cases} 
\frac{j_n^*}{k}, & \text{if } |J_0(j_n^*)| \geq |J_0(kd_0 \lambda)| \\
d_0 \lambda, & \text{otherwise}
\end{cases}
\]

\[
\alpha^* = \begin{cases} 
\beta - \arccos \left( \frac{\pi}{kd^*} \right), & \text{if } J_0(j_n^*) \geq 0, \\
\beta - \frac{\pi}{2}, & \text{if } J_0(j_n^*) < 0
\end{cases}
\]

\[
q^* = \frac{1}{1 - |J_0(kd^*)|}
\]

In the special case of $\beta = 0$, and assuming $|J_0(j_n^*)| \geq |J_0(kd_0 \lambda)|$, the translation of the optimal solution to rectangular coordinates $(x^*, y^*)$ is:

\[
x^* = \begin{cases} 
\frac{\pi}{k}, & \text{if } J_0(j_n^*) \geq 0, \\
0, & \text{if } J_0(j_n^*) < 0
\end{cases}
\]

\[
y^* = \begin{cases} 
-\frac{j_n^*}{k} \sqrt{1 - \frac{\pi^2}{j_n^*}}, & \text{if } J_0(j_n^*) \geq 0, \\
-\frac{j_n^*}{k}, & \text{if } J_0(j_n^*) < 0
\end{cases}
\]

Numerical values for the solutions under different values of $n^*$, assuming $\beta = 0$ and $k = 0.2$, are given in the last set of columns in Table 1.