ISE Industrial and Systems Engineering

q-Factor Optimization for 2-WEC Arrays

LAWRENCE V. SNYDER

Dept. of Industrial and Systems Engineering
Lehigh University, Bethlehem, PA

COR@L Technical Report 14T-001

LEHIGH UNIVERSITY

COR@L COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

q-Factor Optimization for 2-WEC Arrays

Lawrence V. Snyder*[1]

[1]Dept. of Industrial and Systems Engineering , Lehigh University, Bethlehem, PA

February 7, 2014

Abstract

We derive an analytic solution for the wave energy converter (WEC) layout problem with $N = 2$ devices, assuming regular (sinusoidal) incident waves, under the point-absorber approximation.

Assume WEC 1 is located at the origin and WEC 2 is located at (d, α). We assume there is a minimum-separation constraint of the form

$$d \geq d_0\lambda, \tag{1}$$

where λ is the incident wavelength and $d_0 > 0$ is a constant. Then $(d_1, \alpha_1) = (0, 0)$; $(d_2, \alpha_2) = (d, \alpha)$; $d_{12} = d_{21} = d$. Let $z \equiv kd\cos(\beta - \alpha)$. Then

$$L = \begin{bmatrix} 1 \\ e^{ikd\cos(\beta-\alpha)} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{iz} \end{bmatrix} \qquad L^* = \begin{bmatrix} 1 & e^{-iz} \end{bmatrix} \tag{2}$$

$$J = \begin{bmatrix} 1 & J_0(kd) \\ J_0(kd) & 1 \end{bmatrix} \qquad J^{-1} = \frac{1}{1 - J_0(kd)^2} \begin{bmatrix} 1 & -J_0(kd) \\ -J_0(kd) & 1 \end{bmatrix} \tag{3}$$

since $J_0(0) = 1$.

Therefore:

$$\begin{aligned} q &= \frac{1}{2}\mathbf{L}^*\mathbf{J}^{-1}\mathbf{L} \\ &= \frac{1}{2(1 - J_0(kd)^2)} \begin{bmatrix} 1 & e^{-iz} \end{bmatrix} \begin{bmatrix} 1 & -J_0(kd) \\ -J_0(kd) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ e^{iz} \end{bmatrix} \\ &= \frac{1}{2(1 - J_0(kd)^2)} \begin{bmatrix} 1 - J_0(kd)e^{-iz} & -J_0(kd) + e^{-iz} \end{bmatrix} \begin{bmatrix} 1 \\ e^{iz} \end{bmatrix} \\ &= \frac{2 - J_0(kd)(e^{-iz} + e^{iz})}{2(1 - J_0(kd)^2)} \\ &= \frac{2 - J_0(kd)(2\cos z)}{2(1 - J_0(kd)^2)} \end{aligned}$$

*larry.snyder@lehigh.edu

since $e^{-iz} + e^{iz} = (\cos z + i \sin z) + (\cos z - i \sin z) = 2\cos z$

$$= \frac{1 - J_0(kd)\cos(kd\cos(\beta - \alpha))}{1 - J_0(kd)^2}$$

Now, we can choose any α, so we can get any $\beta - \alpha \in [-\pi, \pi]$ ($\mod 2\pi$), so we can get any value of $\cos(\beta-\alpha)$ in $[-1, 1]$, so we can get any value of $\cos(kd\cos(\beta - \alpha))$ in $[-1, 1]$, assuming $kd \geq \pi$ (which is almost certainly true).

Note that $J_0(0) = 1$ and for $x > 0$, $J_0(x)$ resembles a cosine function whose amplitude decreases as x increases. (See Figure 1.) Let j_n be the nth local optimizer (min or max) of $J_0(\cdot)$. Maximizers have odd indices n while minimizers have even indices. Approximate values are given in Table 1. (Ignore the last three columns for now.)

Figure 1: Bessel function of the first kind of order 0.

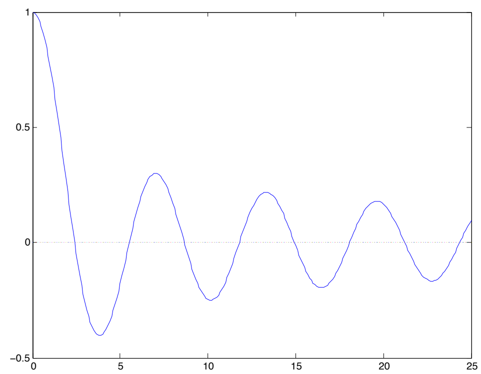

Table 1: Local optima of Bessel function, and resulting optimal q (for any k, β) and optimal locations of WEC 2 (for $k = 0.2$, $\beta = 0$).

n	j_n	$J_0(j_n)$	q^*	d^*	α^*	x^*	y^*
1	0.0000	1.0000	∞	—	—	—	—
2	3.8317	-0.4028	1.6744	19.1585	−1.5708	0.0000	−19.1585
3	7.0156	0.3001	1.4288	35.0780	−1.1065	15.7080	−31.3644
4	10.1735	−0.2497	1.3328	50.8675	−1.5708	0.0000	−50.8675
5	13.3237	0.2184	1.2794	66.6185	−1.3328	15.7080	−64.7401
6	16.4706	−0.1965	1.2445	82.3530	−1.5708	0.0000	−82.3530
7	19.6158	0.1801	1.2196	98.0790	−1.4099	15.7080	−96.8130
8	22.7601	−0.1672	1.2007	113.8005	−1.5708	0.0000	−113.8005

(Local optima were found using Matlab `fminsearch` with default settings.)

Consider a fixed value of d. We consider two cases:

Case 1: $J_0(kd) \geq 0$. Then q is maximized when α is chosen such that $\cos(kd\cos(\beta - \alpha)) = -1$, in which case

$$q = \frac{1 + J_0(kd)}{1 - J_0(kd)^2} = \frac{1}{1 - J_0(kd)}. \tag{4}$$

We want $J_0(kd)$ to be as small as possible. Since local mins become progressively larger as d increases, we want the smallest local argmin possible, which means choosing the smallest $d \geq d_0\lambda$ so that kd is a local argmin of

$J_0(\cdot)$, *unless* $J_0(kd_0\lambda)$ is already smaller than J_0 of this argmin, in which case we should set $d = d_0\lambda$. Therefore,

$$d^* = \begin{cases} \frac{j_n^*}{k}, & \text{if } J_0(j_{n^*}) \leq J_0(kd_0\lambda) \\ d_0\lambda, & \text{otherwise,} \end{cases} \tag{5}$$

where $n^* = 2m^*$ and m^* is the smallest integer m such that $j_{2m} \geq kd_0\lambda$. (In other words, n^* is the smallest argmin greater than or equal to $kd_0\lambda$.) We want $\cos(kd\cos(\beta - \alpha^*)) = -1$, i.e.,

$$\alpha^* = \beta - \arccos\left(\frac{\arccos(-1)}{kd}\right) = \beta - \arccos\left(\frac{\pi}{kd}\right). \tag{6}$$

In the special case of $\beta = 0$, this solution translates to the following rectangular coordinates (assuming the first case in (5) holds):

$$\begin{aligned} x^* &= d^*\cos(\alpha^*) \\ &= \frac{j_n^*}{k} \cdot \frac{\pi}{j_n^*} = \frac{\pi}{k} \\ y^* &= d^*\sin(\alpha^*) \\ &= \frac{j_{n^*}}{k}\sin\left(-\arccos\left(\frac{\pi}{kd^*}\right)\right) \\ &= -\frac{j_{n^*}}{k}\sin\left(\arccos\left(\frac{\pi}{j_{n^*}}\right)\right) \\ &= -\frac{j_{n^*}}{k}\sqrt{1 - \left(\frac{\pi}{j_{n^*}}\right)^2} \qquad (\sin(\arccos x) = \sqrt{1 - x^2}) \\ &= -\frac{j_{n^*}}{k}\sqrt{1 - \frac{\pi^2}{j_{n^*}^2}} \end{aligned}$$

Case 2: $J_0(kd) < 0$. Then q is maximized when α is chosen such that $\cos(kd\cos(\beta - \alpha)) = 1$, in which case

$$q = \frac{1 - J_0(kd)}{1 - J_0(kd)^2} = \frac{1}{1 + J_0(kd)}. \tag{7}$$

We want $J_0(kd)$ to be as large as possible. Since local maxes become progressively smaller as d increases, we want the smallest local argmax possible, which means choosing the smallest $d \geq d_0\lambda$ so that kd is a local argmax of $J_0(\cdot)$, *unless* $J_0(kd_0\lambda)$ is already larger than J_0 of this argmax, in which case we should set $d = d_0\lambda$. Therefore,

$$d^* = \begin{cases} \frac{j_{n^*}}{k}, & \text{if } J_0(j_{n^*}) \geq J_0(kd_0\lambda) \\ d_0\lambda, & \text{otherwise,} \end{cases} \tag{8}$$

where $n^* = 2m^* + 1$ and m^* is the smallest integer m such that $j_{2m+1} \geq kd_0\lambda$. (In other words, n^* is the smallest argmax greater than or equal to $kd_0\lambda$.) We want $\cos(kd\cos(\beta - \alpha^*)) = 1$, i.e.,

$$\alpha^* = \beta - \arccos\left(\frac{\arccos(1)}{kd}\right) = \beta - \frac{\pi}{2}. \tag{9}$$

In the special case of $\beta = 0$, this solution translates to the following rectangular coordinates (assuming the first case in (8) holds):

$$
\begin{aligned}
x^* &= d^* \cos(\alpha^*) \\
&= \frac{j_{n^*}}{k} \cos\left(-\frac{\pi}{2}\right) = 0 \\
y^* &= d^* \sin(\alpha^*) \\
&= \frac{j_{n^*}}{k} \sin\left(-\frac{\pi}{2}\right) \\
&= -\frac{j_{n^*}}{k}
\end{aligned}
$$

Theorem 1 *Let n^* be the smallest integer n such that $j_n \geq kd_0\lambda$. Then the optimal solution (d^*, α^*) and the corresponding q^*, are as follows:*

$$
\begin{aligned}
d^* &= \begin{cases} \frac{j_{n^*}}{k}, & \text{if } |J_0(j_{n^*})| \geq |J_0(kd_0\lambda)| \\ d_0\lambda, & \text{otherwise} \end{cases} \\
\alpha^* &= \begin{cases} \beta - \arccos\left(\frac{\pi}{kd^*}\right), & \text{if } J_0(j_{n^*}) \geq 0, \\ \beta - \frac{\pi}{2}, & \text{if } J_0(j_{n^*}) < 0 \end{cases} \\
q^* &= \frac{1}{1 - |J_0(kd^*)|}
\end{aligned}
$$

In the special case of $\beta = 0$, and assuming $|J_0(j_{n^*})| \geq |J_0(kd_0\lambda)|$, the translation of the optimal solution to rectangular coordinates (x^*, y^*) is:

$$
\begin{aligned}
x^* &= \begin{cases} \frac{\pi}{k}, & \text{if } J_0(j_{n^*}) \geq 0, \\ 0, & \text{if } J_0(j_{n^*}) < 0 \end{cases} \\
y^* &= \begin{cases} -\frac{j_{n^*}}{k}\sqrt{1 - \frac{\pi^2}{j_{n^*}^2}}, & \text{if } J_0(j_{n^*}) \geq 0, \\ -\frac{j_{n^*}}{k}, & \text{if } J_0(j_{n^*}) < 0 \end{cases}
\end{aligned}
$$

Numerical values for the solutions under different values of n^*, assuming $\beta = 0$ and $k = 0.2$, are given in the last set of columns in Table 1.