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# ON THE IDENTIFICATION OF OPTIMAL PARTITION AND OPTIMAL SOLUTIONS FOR SEMIDEFINITE OPTIMIZATION

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**Abstract.** The concept of optimal partition was originally introduced for linear optimization and linear complementary problems and subsequently extended to semidefinite optimization. For linear optimization and sufficient linear complementary problems, the optimal partition and a maximally complementary optimal solution can be identified in strongly polynomial time. In this paper, under no assumption on strict complementarity, we formalize the optimal partition concept for semidefinite optimization and present a methodology for an  $\epsilon$ -feasible maximally complementary solution. The magnitude of the eigenvalues belonging to each partition is quantified using a condition number and the degree of singularity of the problem. Using the bounds from the optimal partition, a rounding procedure is applied to obtain an approximate maximally complementary solution from a central solution.

**Key words.** Semidefinite optimization, Strict complementarity, Optimal partition, Interior point methods, Degree of singularity, Maximally complementary optimal solution

**AMS subject classifications.** 90C22, 90C51

**1. Introduction.** Semidefinite optimization (SDO) is known as a generalization of linear optimization (LO), where the cone of symmetric positive semidefinite matrices substitutes for the nonnegative orthant. In SDO, one minimizes/maximizes the linear objective function

$$C \bullet X := \text{tr}(CX),$$

where  $C$  and  $X$  are  $n \times n$  symmetric matrices, over the intersection of the positive semidefinite cone and a set of affine constraints. Mathematically, an SDO problem is written as

$$(P) \quad p^* := \min\{C \bullet X \mid A^i \bullet X = b_i, \quad i = 1, \dots, m, \quad X \succeq 0\},$$

where matrices  $A_i$  for  $i = 1, \dots, m$  are  $n \times n$  symmetric matrices,  $b \in \mathbb{R}^m$ , and  $X \succeq 0$  indicates that  $X$  is positive semidefinite. The dual SDO problem is given by

$$(D) \quad d^* := \max\{b^T y \mid \sum_{i=1}^m y_i A^i + S = C, \quad S \succeq 0, \quad y \in \mathbb{R}^m\}.$$

Let  $\mathcal{P}$  and  $\mathcal{D}$  denote the primal and dual feasible sets, respectively, as follows

$$\begin{aligned} \mathcal{P} &:= \{X \mid A^i \bullet X = b_i, \quad i = 1, \dots, m, \quad X \succeq 0\}, \\ \mathcal{D} &:= \{(y, S) \mid \sum_{i=1}^m y_i A^i + S = C, \quad S \succeq 0\}. \end{aligned}$$

In light of this notation, the primal and dual optimal sets are defined as

$$\begin{aligned} \mathcal{P}^* &:= \{X \mid X \in \mathcal{P}, \quad C \bullet X = p^*\}, \\ \mathcal{D}^* &:= \{(y, S) \mid (y, S) \in \mathcal{D}, \quad b^T y = d^*\}. \end{aligned}$$

We assume that the matrices  $A^i$  for  $i = 1, \dots, m$  are linearly independent. It is also assumed that the interior point condition holds, that is, there exists  $(X, y, S) \in \mathcal{P} \times \mathcal{D}$ , where  $X, S \succ 0$ . The first assumption guarantees that  $y$  is uniquely determined for a given dual solution  $S$ , and the latter ensures that the primal and dual optimal sets are nonempty, and that strong duality holds.

SDO problems are frequently used in many applications, e.g., control theory, structural optimization, statistics, robust optimization, eigenvalue optimization, pattern recognition, and combinatorial optimization.

Second-order conic optimization (SOCO) problems can be embedded in SDO formulation. See [31] and [32] for a detailed description of the problems. Analogous to LO, using interior point methods (IPMs), SDO problems can be solved in polynomial time, though they require significantly more computational effort per iteration. The Extension of IPMs from LO to SDO was pioneered by Nesterov and Nemirovski [20], and Alizadeh [1].

The main idea of primal-dual path following IPMs is to follow the central path, which is defined as the set of solutions of

$$\begin{aligned}
 & A^i \bullet X = b_i, \quad i = 1, \dots, m, \\
 (1) \quad & \sum_{i=1}^m A^i y_i + S = C, \\
 & XS = \mu I, \\
 & X, S \succeq 0.
 \end{aligned}$$

For a given  $\mu > 0$ , the central solution  $(X(\mu), y(\mu), S(\mu))$  to this system exists and is uniquely defined under the full rank and the interior point assumptions. For  $0 \leq \mu \leq \bar{\mu}$ , where  $\bar{\mu} > 0$ , the set of solutions of (1) is bounded, and thus the trajectory of the central solutions has limit points in the relative interior of the optimal set [6, 12]. A proof was given by Halická et al. [8] stating that the central path converges to a maximally complementary optimal solution<sup>1</sup>.

Since the domain and range of (1) are not identical, a direct application of the Newton method to (1) leads to a non-symmetric  $X$ . To get around this problem, many variants of IPMs have been introduced for SDO based on how the search direction and the neighborhood of the central path is defined. To name a few, we can mention the AHO [2], HRVW/KSH/M [9, 15, 18] and the NT search directions [21, 22]. A search direction using a least squares solution of the overdetermined system (1) was proposed by de Klerk et al. [13] and Kruk et al. [16]. See [19] and [24] for a survey of IPMs for SDO.

Even though a primal-dual path following IPM can be easily extended to SDO, the superlinear convergence of an IPM cannot be easily established for SDO, unless the strict complementarity assumption holds<sup>2</sup>. This is mainly due to the analyticity and limiting behavior of the central path in SDO as extensively studied by Goldfarb and Scheinberg [6], Halická [7], Luo, Sturm and Zhang [17], and Sturm and Zhang [29]. Goldfarb and Scheinberg [6] showed that all the limit points of the central path converge to the same solution in the optimal set. They also showed, under the strict complementarity assumption, that the first order derivatives of the central path converge as  $\mu \rightarrow 0$ . However, the first order derivatives are unbounded if the strict complementarity fails to hold. Sturm [28] showed that the Hoffman error bound can be extended to SDO when strict complementarity holds. However, this bound can be as bad as  $\Omega(\mu^{2-n})$  when strict complementarity fails (see Example 3.3 in [14] which is adopted from [28]).

Numerical experiments verify that the condition number of the system of search directions is increasing drastically, leading to ill-posed systems, during the final iterations of IPMs [24]. It would be helpful if we could avoid this ill-conditioning, by switching over to a rounding procedure, when  $\mu$  is sufficiently small. This motivates to exploit the optimal partition information. The notion of optimal partition was originally introduced for LO and linear complementarity problems (LCPs). Ye [33] proposed a finite termination strategy for IPMs which generates a strictly complementary optimal solution from a primal-dual solution sufficiently close to the optimal set. Roos, Terlaky, and Vial [27] present a rounding procedure which uses optimal partition information to identify a strictly complementary solution. Illés, Roos, and Terlaky [11] proposed a strongly polynomial rounding procedure to generate a maximally complementary optimal solution for sufficient LCPs. The concept of optimal partition was extended to SDO by [6, 12] and general convex optimization by Yildirim [34]. Bonnans and Ramirez [5] established another algebraic definition of

<sup>1</sup>An optimal solution pair  $(X^*, y^*, S^*)$  is maximally complementary if  $\text{rank}(X^* + S^*)$  is maximal over all the optimal solutions.

<sup>2</sup>A optimal solution pair  $(X^*, y^*, S^*)$  is strictly complementary if  $\text{rank}(X^* + S^*) = n$ .

the optimal partition for SOCO. Recently, Terlaky and Wang [30] have studied the identification of optimal partition for SOCO.

In this paper, we formalize the identification of the optimal partition for SDO when strict complementarity may fail. The magnitude of the eigenvalues belonging to each partition is quantified by using a condition number and the degree of singularity of the problem. We also propose a methodology for the identification of an  $\epsilon$ -feasible maximally complementary solution, which uses estimates of the optimal partition. The rounding procedure then obtains an approximate maximally complementary solution from a central solution by solving two least square problems. The rest of this paper is organized as follows. In Section 2 we formally define the optimal partition and maximally complementary concepts in details. In Section 3, we analyze the magnitude of the eigenvalues of the solutions on the central path based on a condition number and the distance to the optimal set. In Section 4, we present a rounding procedure to generate an approximate maximally complementary solution and provide feasibility bounds for primal-dual SDO. In Section 5, we extend the optimal partition results to the solutions in a neighborhood of the central path and provide complexity bounds.

Throughout this paper, the eigenvalues of an arbitrary matrix  $X$  is represented by  $\lambda_i(X)$ . Further,  $\lambda_{\min}(X)$  and  $\lambda_{\max}(X)$  stand for the minimal and maximal eigenvalues of  $X$ , respectively. The subscript  $[i]$  in our notation means the  $i^{\text{th}}$  largest component of a vector. For instance,  $\lambda_{[i]}(X)$  denotes the  $i^{\text{th}}$  largest eigenvalue of  $X$  so that

$$\lambda_{[1]}(X) \geq \lambda_{[2]}(X) \geq \dots \geq \lambda_{[n]}(X).$$

The following two lemmas will be useful in providing optimal partition bounds.

LEMMA 1 (Theorem 2.3.8 in [32]). *Let  $X$  be an  $n \times n$  symmetric matrix. Then,*

$$\begin{aligned} \lambda_{[n-k+1]}(X) + \dots + \lambda_{[n]}(X) &= \min_{U \in \mathbb{R}^{n \times k}} \text{tr}(U^T X U), \\ \text{s.t. } U^T U &= I_k. \end{aligned}$$

LEMMA 2 (Problem III.6.14 in [4]). *Let  $X$  and  $S$  be two  $n \times n$  symmetric matrices. Then,*

$$\lambda_{[1]}(X)\lambda_{[n]}(S) + \dots + \lambda_{[n]}(X)\lambda_{[1]}(S) \leq X \bullet S \leq \lambda_{[1]}(X)\lambda_{[1]}(S) + \dots + \lambda_{[n]}(X)\lambda_{[n]}(S).$$

Further, if  $X$  and  $S$  are positive definite, then for  $k = 1, \dots, n$  we have

$$\sum_{i=1}^k \left( \begin{array}{c} \lambda_{[1]}(X)\lambda_{[n]}(S) \\ \vdots \\ \lambda_{[n]}(X)\lambda_{[1]}(S) \end{array} \right)_{[n-i+1]} \geq \sum_{i=1}^k \lambda_{[n-i+1]}(XS).$$

**2. Optimal partition in terms of subspaces.** Consider the set of optimality conditions for  $(P)$  and  $(D)$ . Since the interior point condition holds, for optimality the KKT conditions [23] are necessary and sufficient for  $(P)$  and  $(D)$ , which are written as

$$\begin{aligned} A^i \bullet X &= b_i, \quad i = 1, \dots, m, \\ (2) \quad \sum_{i=1}^m A^i y_i + S &= C, \\ XS &= 0, \quad X, S \succeq 0. \end{aligned}$$

A solution  $(X, y, S)$  which satisfies  $XS = 0$  is called complementary.

DEFINITION 3 (Definition 2.7 in [14]). *Let  $(X^*, S^*) \in \mathcal{P}^* \times \mathcal{D}^*$ . Then,  $(X^*, S^*)$  is a maximally complementary optimal pair if  $\text{rank}(X^* + S^*)$  is maximal over the optimal set.*

Equivalently,  $(X^*, S^*)$  is a maximally complementary optimal pair if  $(X^*, S^*) \in \text{ri}(\mathcal{P}^* \times \mathcal{D}^*)$ . An optimal pair  $(X^*, S^*)$  is strictly complementary if  $X^* + S^* \succ 0$ . In contrast to LO, strict complementarity may fail in SDO; that is we might have  $\text{rank}(X^*) + \text{rank}(S^*) < n$ .

It follows from [Definition 3](#) that all  $(X^*, S^*) \in \text{ri}(\mathcal{P}^* \times \mathcal{D}^*)$  have the same range space. Let  $\mathcal{B} := \mathcal{R}(X^*)$  and  $\mathcal{N} := \mathcal{R}(S^*)$ . All this means that  $\mathcal{R}(X) \subseteq \mathcal{B}$  for all  $X \in \mathcal{P}^*$ , and  $\mathcal{R}(S) \subseteq \mathcal{N}$  for all  $S \in \mathcal{D}^*$ , where  $\mathcal{R}(\cdot)$  denotes the range space. Since  $X^* \in \mathcal{P}^*$  and  $S^* \in \mathcal{D}^*$  commute, they have the same orthonormal eigenvector basis  $Q(X^*) = Q(S^*) = Q^*$ . Hence, we have

$$(3) \quad \begin{aligned} X^* &= Q^* \Lambda(X^*) (Q^*)^T, & S^* &= Q^* \Lambda(S^*) (Q^*)^T, \\ \mathcal{R}(X^*) &= \mathcal{R}(Q^* \Lambda(X^*)), & \mathcal{R}(S^*) &= \mathcal{R}(Q^* \Lambda(S^*)), \end{aligned}$$

where  $\Lambda(X^*)$  and  $\Lambda(S^*)$  are diagonal matrices containing the eigenvalues of  $X^*$  and  $S^*$ , respectively. All this states that the range spaces are spanned by the eigenvectors associated with the positive eigenvalues.

In case of strict complementarity, the subspaces  $\mathcal{B}$  and  $\mathcal{N}$  span  $\mathbb{R}^n$ . In the other case, when  $\text{rank}(X^*) + \text{rank}(S^*) < n$ , there exists a subspace  $\mathcal{T}$ , which is the orthogonal complement to  $\mathcal{B} + \mathcal{N}$ . Then  $\mathbb{R}^n$  is partitioned into three mutually orthogonal subspaces  $\mathcal{B}$ ,  $\mathcal{N}$ , and  $\mathcal{T}$ . The subspaces of  $\mathcal{B}$ ,  $\mathcal{N}$  and  $\mathcal{T}$  form an optimal partition of an SDO problem, which is invariant over the interior of the optimal set (i.e., identical for every maximally complementary optimal solution).

As indicated by [\(3\)](#), the optimal partition is defined based upon the subspaces related to the optimal set. We can represent the subspaces using their orthonormal basis, which arises from a maximally complementary optimal solution. Hence, for a primal-dual optimal solution, according to the eigenvalues of  $X^*$  and  $S^*$ , the index set  $\{1, \dots, n\}$  can be partitioned into three sets

$$(4) \quad \begin{aligned} B &:= \{i \mid \lambda_i(X^*) > 0 \text{ for a primal optimal solution } X^*\}, \\ N &:= \{i \mid \lambda_i(S^*) > 0 \text{ for a dual optimal solution } S^*\}, \\ T &:= \{1, \dots, n\} \setminus \{B \cup N\}, \end{aligned}$$

where there exists a one-to-one correspondence between the indices in  $B$ ,  $N$ , and  $T$  and the eigenvectors of  $X^*$  and  $S^*$ , i.e., every  $i \in B$  corresponds to an eigenvector  $q_i^*$  with  $\lambda_i(X^*) > 0$ , every  $i \in N$  corresponds to an eigenvector  $q_i^*$  with  $\lambda_i(S^*) > 0$ , and every  $i \in T$  corresponds to the eigenvector  $q_i^*$  where  $\lambda_i(X^*) = \lambda_i(S^*) = 0$ . Therefore,  $B$ ,  $N$ , and  $T$  can be equivalently referred to as the optimal partition for an SDO problem. We resort to the definition [\(4\)](#) throughout this paper.

Let  $Q_B^*$ ,  $Q_N^*$ , and  $Q_T^*$  denote the orthonormal basis corresponding to the sets  $B$ ,  $N$ , and  $T$ , respectively. Therefore,  $\mathcal{R}(Q_B^*) = \mathcal{B}$ ,  $\mathcal{R}(Q_N^*) = \mathcal{N}$ , and  $\mathcal{R}(Q_T^*) = \mathcal{T}$ . In what follows, we recall an important theorem from [\[14\]](#).

**THEOREM 4** (Theorem 2.7 in [\[14\]](#)). *Every primal-dual optimal solution  $(X^*, S^*)$  can be represented as*

$$X^* = Q_B^* U_{X^*} (Q_B^*)^T, \quad S^* = Q_N^* U_{S^*} (Q_N^*)^T,$$

where  $U_{X^*}$  and  $U_{S^*}$  are symmetric  $|B| \times |B|$  and  $|N| \times |N|$  positive semidefinite matrices, respectively. If  $(X^*, S^*)$  is a maximally complementary optimal solution, then we have  $U_{X^*} \succ 0$  and  $U_{S^*} \succ 0$ .

An orthogonal transformation of  $(X^*, S^*) \in \text{ri}(\mathcal{P}^* \times \mathcal{D}^*)$  with respect to  $Q^*$  reveals the optimal partition as

$$(5) \quad (Q^*)^T X^* Q^* = \begin{bmatrix} U_{X^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (Q^*)^T S^* Q^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & U_{S^*} \end{bmatrix},$$

where  $U_{X^*} \succ 0$  and  $U_{S^*} \succ 0$ . By [Theorem 4](#),  $(X^*, S^*)$  is indeed a maximally complementary optimal solution. As a consequence, the following lemma is in order.

**LEMMA 5** ([\[8\]](#)). *In an optimal partition, we have*

1.  $\lambda_i(X^*) = 0$ ,  $\forall i \in N \cup T$  for all  $(X^*, y^*, S^*) \in \mathcal{P}^* \times \mathcal{D}^*$ .
2.  $\lambda_i(S^*) = 0$ ,  $\forall i \in B \cup T$  for all  $(X^*, y^*, S^*) \in \mathcal{P}^* \times \mathcal{D}^*$ .

**3. Condition numbers and bounds.** To derive bounds for the magnitudes of the eigenvalues on the central path as  $\mu \rightarrow 0$ , we define some condition numbers as

$$(6) \quad \begin{aligned} \sigma_B &:= \min_{i \in B} \max_{X^* \in \mathcal{P}^*} \lambda_i(X^*), \\ \sigma_N &:= \min_{i \in N} \max_{(y^*, S^*) \in \mathcal{D}^*} \lambda_i(S^*), \\ \sigma &:= \min\{\sigma_B, \sigma_N\}. \end{aligned}$$

Observe that  $\sigma_B := \infty$  if  $B = \emptyset$ , and  $\sigma_N := \infty$  if  $N = \emptyset$ .

LEMMA 6. *The condition number  $\sigma$  is a positive constant value.*

*Proof.* By the interior point assumption,  $\mathcal{P}^* \times \mathcal{D}^*$  is nonempty and compact. Thus  $\sigma$  is well-defined since at least either of  $B$  or  $N$  has to be nonempty. Assume that  $B \neq \emptyset$ . Then there exists  $X^* \in \mathcal{P}^*$  so that  $\lambda_i(X^*) > 0$  for some  $i \in B$ . By the compactness of  $\mathcal{P}^*$ , there exists  $\bar{X} \in \mathcal{P}^*$  so that

$$\max_{X \in \mathcal{P}^*} \lambda_i(X) = \lambda_i(\bar{X}) \geq \lambda_i(X^*) > 0, \quad i \in B,$$

which by (6) implies that  $\sigma_B > 0$ . A similar argument can be made to show that  $\sigma_N > 0$  if  $N \neq \emptyset$ .  $\square$

*Remark 7.* Using the first order theory of reals [26], we can provide a lower bound for the condition number  $\sigma$ , which will be used to measure the magnitude of the eigenvalues of a central solution. See [Appendix A](#) for details.

Consider an orthogonal transformation of a central solution  $(X(\mu), S(\mu))$  with respect to  $Q^* = [Q_B^*, Q_T^*, Q_N^*]$  denoted by

$$(7) \quad \hat{X}(\mu) := \begin{bmatrix} \hat{X}_B(\mu) & \hat{X}_{BT}(\mu) & \hat{X}_{BN}(\mu) \\ \hat{X}_{TB}(\mu) & \hat{X}_T(\mu) & \hat{X}_{TN}(\mu) \\ \hat{X}_{NB}(\mu) & \hat{X}_{NT}(\mu) & \hat{X}_N(\mu) \end{bmatrix}, \quad \hat{S}(\mu) := \begin{bmatrix} \hat{S}_B(\mu) & \hat{S}_{BT}(\mu) & \hat{S}_{BN}(\mu) \\ \hat{S}_{TB}(\mu) & \hat{S}_T(\mu) & \hat{S}_{TN}(\mu) \\ \hat{S}_{NB}(\mu) & \hat{S}_{NT}(\mu) & \hat{S}_N(\mu) \end{bmatrix},$$

where  $\hat{X}(\mu) := (Q^*)^T X(\mu) Q^*$  and  $\hat{S}(\mu) := (Q^*)^T S(\mu) Q^*$ . In what follows, we resort to an important error bound for a linear matrix inequality (LMI) system from [28]. [Lemma 8](#) specifies an upper bound for the distance of a central solution from the optimal set.

LEMMA 8. *Let  $(X(\mu), S(\mu))$  be a central solution for some  $\mu > 0$ . Then, it follows from the error bound for an LMI system that*

$$\|X(\mu) - X^*\| \leq c(n\mu)^\gamma, \quad \|S(\mu) - S^*\| \leq c(n\mu)^\gamma,$$

where  $\gamma = 2^{-d}$ , in which  $d$  denotes the degree of singularity<sup>3</sup> of the minimal subspace containing the optimal set, and  $c$  is a constant independent of  $\mu$ .

*Proof.* The bound can be established easily by applying the error bound for LMI system as stated in [Lemma B.1](#). Note that the optimality conditions for an SDO problem are equivalent to

$$(8) \quad \begin{aligned} A^i \bullet X &= b_i, \quad i = 1, \dots, m, \\ \sum_{i=1}^m A^i y_i + S &= C, \\ C \bullet X - b^T y &\leq 0, \quad X, S \succeq 0. \end{aligned}$$

<sup>3</sup>The degree of singularity is defined as the minimum number of facial reduction steps to get the minimal face of the positive semidefinite cone which contains the primal-dual optimal set.

As defined by (1), the central solutions for  $0 < \mu < \bar{\mu}$ , with  $\bar{\mu} > 0$ , form a bounded trajectory. Thus, for a given solution  $(X(\mu), y(\mu), S(\mu))$ , the backward error<sup>4</sup> is equal to  $n\mu$  with respect to the LMI system (8). Hence, it can be deduced from Lemma B.1 that

$$\sqrt{\|X(\mu) - X^*\|^2 + \|y(\mu) - y^*\|^2 + \|S(\mu) - S^*\|^2} \leq c(n\mu)^\gamma,$$

where  $\gamma = 2^{-d}$ , in which  $d$  denotes the degree of singularity of the minimal subspace containing the optimal set (8), see [28] for details.  $\square$

From the orthogonal transformation in (7), we can derive

$$\lim_{\mu \rightarrow 0} \hat{X}_B(\mu) = U_{X^*}, \quad \text{and} \quad \lim_{\mu \rightarrow 0} \hat{S}_N(\mu) = U_{S^*},$$

and

$$\lim_{\mu \rightarrow 0} (Q_{T \cup N}^*)^T X(\mu) Q_{T \cup N}^* = 0, \quad \lim_{\mu \rightarrow 0} (Q_{B \cup T}^*)^T S(\mu) Q_{B \cup T}^* = 0,$$

where  $Q_{T \cup N}^*$  and  $Q_{B \cup T}^*$  denotes the columns of  $Q^*$  corresponding to  $T \cup N$  and  $B \cup T$ , respectively. The following lemma establishes upper bounds on some vanishing blocks of the central solution  $(\hat{X}(\mu), \hat{S}(\mu))$  which are needed in our analysis.

LEMMA 9. For a central solution  $(\hat{X}(\mu), \hat{S}(\mu))$ , we have

$$\begin{aligned} \text{tr}(\hat{X}_N(\mu)) &\leq \frac{n\mu}{\sigma}, & \text{tr}(\hat{S}_B(\mu)) &\leq \frac{n\mu}{\sigma}, \\ \|(Q_{T \cup N}^*)^T X(\mu) Q_{T \cup N}^*\| &\leq c(n\mu)^\gamma, & \|(Q_{B \cup T}^*)^T S(\mu) Q_{B \cup T}^*\| &\leq c(n\mu)^\gamma. \end{aligned}$$

*Proof.* Due to the definition of  $\sigma$  in (6) and the compactness of  $\mathcal{P}^* \times \mathcal{D}^*$ , we can find an optimal solution  $(X^*, y^*, S^*)$ , where

$$(9) \quad \lambda_{\min}(U_{X^*}) \geq \sigma, \quad \lambda_{\min}(U_{S^*}) \geq \sigma.$$

Recall from the optimality conditions that

$$(X(\mu) - X^*) \bullet (S(\mu) - S^*) = 0,$$

which by (1) gives

$$X(\mu) \bullet S^* + X^* \bullet S(\mu) = n\mu.$$

Since the inner product is invariant with respect to an orthogonal transformation, we get

$$X(\mu) \bullet S^* + X^* \bullet S(\mu) = \hat{X}_N(\mu) \bullet U_{S^*} + U_{X^*} \bullet \hat{S}_B(\mu) = n\mu.$$

Therefore, it follows from  $\hat{X}_N(\mu) \succ 0$  and (9) that

$$\lambda_{\min}(U_{S^*}) \text{tr}(\hat{X}_N(\mu)) \leq n\mu, \quad \text{which implies} \quad \text{tr}(\hat{X}_N(\mu)) \leq \frac{n\mu}{\sigma}.$$

In a similar manner, it follows from  $\hat{S}_B(\mu) \succ 0$  that  $\text{tr}(\hat{S}_B(\mu)) \leq \frac{n\mu}{\sigma}$ . Subsequently, it can be deduced from Lemma 8 that

$$\begin{aligned} \|(Q_{T \cup N}^*)^T X(\mu) Q_{T \cup N}^*\| &= \left\| \begin{bmatrix} \hat{X}_T(\mu) & \hat{X}_{TN}(\mu) \\ \hat{X}_{NT}(\mu) & \hat{X}_N(\mu) \end{bmatrix} \right\| \leq \|X(\mu) - X^*\| \leq c(n\mu)^\gamma, \\ \|(Q_{B \cup T}^*)^T S(\mu) Q_{B \cup T}^*\| &= \left\| \begin{bmatrix} \hat{S}_B(\mu) & \hat{S}_{BT}(\mu) \\ \hat{S}_{TB}(\mu) & \hat{S}_T(\mu) \end{bmatrix} \right\| \leq \|S(\mu) - S^*\| \leq c(n\mu)^\gamma, \end{aligned}$$

which completes the proof.  $\square$

<sup>4</sup>The backward error is simply defined as the amount of infeasibility of the system (8) with respect to a central solution  $(X(\mu), y(\mu), S(\mu))$ . Here, the backward error is just  $n\mu$ , since a central solution only violates the complementarity constraint but satisfies the primal-dual and cone feasibility constraints.

Note that the eigenvalues and eigenvectors of a central solution vary continuously as  $\mu \rightarrow 0$ . In other words, compared to its counterpart in LO, the optimal partition of  $\mathbb{R}^n$  to the three subspaces  $\mathcal{B}$ ,  $\mathcal{N}$ , and  $\mathcal{T}$  cannot be identified exactly from a central solution sufficiently close to the optimal set. Nevertheless, it can be observed from [Lemma 5](#) that for any primal optimal solution, the  $|N| + |T|$  smallest eigenvalues will be zero, and for any dual optimal solution, the  $|B| + |T|$  smallest eigenvalues will be zero. In fact, associated with a central solution  $(X(\mu), y(\mu), S(\mu))$  there exist three disjoint index sets defined as:

- A set of indices corresponding to the eigenvectors with  $\lambda_i(X(\mu))$  converging to a positive value and  $\lambda_i(S(\mu))$  converging to 0;
- A set of indices corresponding to the eigenvectors with  $\lambda_i(S(\mu))$  converging to a positive value and  $\lambda_i(X(\mu))$  converging to 0;
- A set of indices corresponding to the eigenvectors with both  $\lambda_i(X(\mu))$  and  $\lambda_i(S(\mu))$  converging to 0.

As  $\mu \rightarrow 0$ , and the subspaces  $\mathcal{B}$ ,  $\mathcal{N}$ , and  $\mathcal{T}$  converge, the above index sets become identical to  $B$ ,  $N$ , and  $T$ , respectively. [Theorem 10](#) and [Theorem 11](#) characterize these index sets and specify the conditions for the identification of  $B$ ,  $N$ , and  $T$ .

**THEOREM 10.** *For a central solution  $(X(\mu), y(\mu), S(\mu))$  with  $\mu > 0$ , it holds that*

1. *For every  $i \in B$  we have*

$$(10) \quad \lambda_i(S(\mu)) \leq \frac{n\mu}{\sigma}, \quad \lambda_i(X(\mu)) \geq \frac{\sigma}{n},$$

2. *For every  $i \in N$  we have*

$$(11) \quad \lambda_i(S(\mu)) \geq \frac{\sigma}{n}, \quad \lambda_i(X(\mu)) \leq \frac{n\mu}{\sigma},$$

3. *For every  $i \in T$  we have*

$$(12) \quad \frac{\mu}{c\sqrt{n}(n\mu)^\gamma} \leq \lambda_i(X(\mu)), \lambda_i(S(\mu)) \leq c\sqrt{n}(n\mu)^\gamma.$$

*Proof.* Note that  $\hat{S}_B(\mu) = (Q_B^*)^T S(\mu) Q_B^*$  and  $\hat{X}_N(\mu) = (Q_N^*)^T X(\mu) Q_N^*$  as defined in [\(7\)](#). Then, it follows from [Lemma 1](#) that

$$\begin{aligned} \lambda_{[n-|B|+1]}(S(\mu)) + \dots + \lambda_{[n]}(S(\mu)) &\leq \text{tr}(\hat{S}_B(\mu)) \leq \frac{n\mu}{\sigma}, \\ \lambda_{[n-|N|+1]}(X(\mu)) + \dots + \lambda_{[n]}(X(\mu)) &\leq \text{tr}(\hat{X}_N(\mu)) \leq \frac{n\mu}{\sigma}. \end{aligned}$$

Therefore, noting that  $\lambda_i(X(\mu)), \lambda_i(S(\mu)) > 0$ , we get

$$(13) \quad \begin{aligned} \lambda_{[n-i+1]}(S(\mu)) &\leq \frac{n\mu}{\sigma}, \quad i = 1, \dots, |B|, \\ \lambda_{[n-i+1]}(X(\mu)) &\leq \frac{n\mu}{\sigma}, \quad i = 1, \dots, |N|. \end{aligned}$$

Further, taking the perturbed complementarity condition  $\lambda_i(X(\mu))\lambda_i(S(\mu)) = \mu$  into consideration, we can derive

$$(14) \quad \begin{aligned} \lambda_{[i]}(X(\mu)) &\geq \frac{\sigma}{n}, \quad i = 1, \dots, |B|, \\ \lambda_{[i]}(S(\mu)) &\geq \frac{\sigma}{n}, \quad i = 1, \dots, |N|. \end{aligned}$$

It follows from the second part of [Lemma 9](#), and from the fact that for any  $X$  we have  $\text{tr}(X) \leq \sqrt{n}\|X\|$ , that

$$\begin{aligned} \frac{1}{\sqrt{n}} \left( \lambda_{[n-|N|-|T|+1]}(X(\mu)) + \dots + \lambda_{[n]}(X(\mu)) \right) &\leq \left\| (Q_{T \cup N}^*)^T X(\mu) Q_{T \cup N}^* \right\| \leq c(n\mu)^\gamma, \\ \frac{1}{\sqrt{n}} \left( \lambda_{[n-|B|-|T|+1]}(S(\mu)) + \dots + \lambda_{[n]}(S(\mu)) \right) &\leq \left\| (Q_{B \cup T}^*)^T S(\mu) Q_{B \cup T}^* \right\| \leq c(n\mu)^\gamma, \end{aligned}$$



which, together with the perturbed complementarity condition  $\lambda_i(X(\mu))\lambda_i(S(\mu)) = \mu$ , implies

$$(15) \quad \begin{aligned} \lambda_{[n-i+1]}(X(\mu)) &\leq c\sqrt{n}(n\mu)^\gamma, \quad \lambda_{[i]}(S(\mu)) \geq \frac{\mu}{c\sqrt{n}(n\mu)^\gamma}, \quad i = 1, \dots, |N| + |T|, \\ \lambda_{[n-i+1]}(S(\mu)) &\leq c\sqrt{n}(n\mu)^\gamma, \quad \lambda_{[i]}(X(\mu)) \geq \frac{\mu}{c\sqrt{n}(n\mu)^\gamma}, \quad i = 1, \dots, |B| + |T|. \end{aligned}$$

From the inequalities (13) and (14), we can deduct that the  $|B|$  largest eigenvalues of  $X(\mu)$  stay positive while the smallest  $|B|$  eigenvalues of  $S(\mu)$  will converge to 0. Similarly, the largest  $|N|$  eigenvalues of  $S(\mu)$  will remain positive while the last  $|N|$  eigenvalues of  $X(\mu)$  converge to 0 as  $\mu \rightarrow 0$ . The inequalities (15) also hint that there should exist a set of  $|T|$  eigenvalues of  $X(\mu)$  and  $S(\mu)$  which stay within the interval  $\left[\frac{\mu}{c\sqrt{n}(n\mu)^\gamma}, c\sqrt{n}(n\mu)^\gamma\right]$ , and thus both converge to 0 as  $\mu \rightarrow 0$ . This completes the proof.  $\square$

The following theorem states that if  $\mu$  is sufficiently small, then the optimal partition can be identified.

**THEOREM 11.** *If  $\mu$  satisfies*

$$(16) \quad \mu < \min \left\{ \frac{1}{n} \left( \frac{\sigma}{cn^{\frac{3}{2}}} \right)^{\frac{1}{\gamma}}, (c^2 n^{2\gamma+1})^{\frac{1}{1-2\gamma}} \right\},$$

then we can identify the partition  $B$ ,  $N$ , and  $T$ .

*Proof.* Using the bounds in (10), (11) and (12), we can identify  $B$ ,  $N$ , and  $T$  if

$$\frac{n\mu}{\sigma} < \frac{\mu}{c\sqrt{n}(n\mu)^\gamma} < c\sqrt{n}(n\mu)^\gamma < \frac{\sigma}{n},$$

which is equivalent to

$$\mu < \min \left\{ \frac{1}{n} \left( \frac{\sigma}{cn^{\frac{3}{2}}} \right)^{\frac{1}{\gamma}}, (c^2 n^{2\gamma+1})^{\frac{1}{1-2\gamma}} \right\}.$$

The result follows by noting that  $q_i(\mu) \rightarrow q_i^*$  for each  $i$ .  $\square$

*Remark 12.* In the case when the degree of singularity is 1, then we have  $\gamma = \frac{1}{2}$ , and therefore we can simplify (16) to

$$\mu < \frac{\sigma^2}{c^2 n^4}.$$

In case of LO, the  $T$  part does not exist, and thus the bound (16) reduces to  $\mu < \frac{\sigma^2}{n^2}$ .

**4. An approximate maximally complementary solution.** Even though an SDO problem is polynomially solvable using a primal-dual path-following IPM, an exact solution of an SDO cannot be obtained even with rational data, since the eigenvectors corresponding to  $B$ ,  $N$  and  $T$  vary continuously. Nevertheless, from the current central solution  $(X(\mu), y(\mu), S(\mu))$  for a sufficiently small  $\mu$ , we can make a projection onto the boundary of the positive semidefinite cone to generate a solution with zero complementary gap but with  $\epsilon$  primal-dual feasibility, which is a so called approximate maximally complementary solution in this paper.

Suppose that a central solution  $(X(\mu), y(\mu), S(\mu))$  is given, where  $\mu$  satisfies (16), i.e.,  $\mu$  is sufficiently small to identify the partition  $B$ ,  $N$ , and  $T$ . The eigenvectors of  $X(\mu)$  and  $S(\mu)$  can be rearranged so that  $Q(\mu) := [Q_B(\mu), Q_T(\mu), Q_N(\mu)]$ . Let  $\hat{X}^* := Q(\mu)^T X^* Q(\mu)$ , and  $\hat{S}^* := Q(\mu)^T S^* Q(\mu)$ , where

$$\hat{X}^* := \begin{bmatrix} \hat{X}_B^* & \hat{X}_{BT}^* & \hat{X}_{BN}^* \\ \hat{X}_{TB}^* & \hat{X}_T^* & \hat{X}_{TN}^* \\ \hat{X}_{NB}^* & \hat{X}_{NT}^* & \hat{X}_N^* \end{bmatrix}, \quad \hat{S}^* := \begin{bmatrix} \hat{S}_B^* & \hat{S}_{BT}^* & \hat{S}_{BN}^* \\ \hat{S}_{TB}^* & \hat{S}_T^* & \hat{S}_{TN}^* \\ \hat{S}_{NB}^* & \hat{S}_{NT}^* & \hat{S}_N^* \end{bmatrix}.$$

From the primal feasibility constraints and the above orthogonal transformation, we have

$$\bar{A}^i \bullet \hat{X}^* = b_i, \quad \bar{A}^i \bullet \Lambda(X(\mu)) = b_i, \quad i = 1, \dots, m,$$

in which  $\bar{A}^i := Q(\mu)^T A^i Q(\mu)$  and  $\Lambda(X(\mu)) := Q(\mu)^T X(\mu) Q(\mu)$ , where

$$\bar{A}^i := \begin{bmatrix} \bar{A}_B^i & \bar{A}_{BT}^i & \bar{A}_{BN}^i \\ \bar{A}_{TB}^i & \bar{A}_T^i & \bar{A}_{TN}^i \\ \bar{A}_{NB}^i & \bar{A}_{NT}^i & \bar{A}_N^i \end{bmatrix}, \quad \Lambda(X(\mu)) := \begin{bmatrix} \Lambda_B(X(\mu)) & 0 & 0 \\ 0 & \Lambda_T(X(\mu)) & 0 \\ 0 & 0 & \Lambda_N(X(\mu)) \end{bmatrix}.$$

Subtracting the second equation from the first one, for  $i = 1, \dots, m$ , we get

$$(17) \quad \bar{A}_B^i \bullet \Delta X_B(\mu) = \bar{A}_T^i \bullet \Lambda_T(X(\mu)) + \bar{A}_N^i \bullet \Lambda_N(X(\mu)) + \xi_i,$$

where the residual term  $\xi_i$  is defined as

$$\begin{aligned} \xi_i &= -\bar{A}_N^i \bullet \hat{X}_N^* - \bar{A}_T^i \bullet \hat{X}_T^* - 2(\bar{A}_{BT}^i \bullet \hat{X}_{BT}^* + \bar{A}_{BN}^i \bullet \hat{X}_{BN}^* + \bar{A}_{TN}^i \bullet \hat{X}_{TN}^*), \\ \Delta X_B(\mu) &= \hat{X}_B^* - \Lambda_B(X(\mu)). \end{aligned}$$

Analogously for the dual constraints, we get

$$\sum_{i=1}^m y_i^* \bar{A}^i + \hat{S}^* = \bar{C}, \quad \sum_{i=1}^m y_i(\mu) \bar{A}^i + \Lambda(S(\mu)) = \bar{C},$$

where  $\bar{C} = Q(\mu)^T C Q(\mu)$ , and

$$\Lambda(S(\mu)) = \begin{bmatrix} \Lambda_B(S(\mu)) & 0 & 0 \\ 0 & \Lambda_T(S(\mu)) & 0 \\ 0 & 0 & \Lambda_N(S(\mu)) \end{bmatrix}.$$

Subtracting the second equation from the first one, we get

$$(18) \quad \sum_{i=1}^m \Delta y_i(\mu) \bar{A}^i + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta S_N(\mu) \end{bmatrix} = \begin{bmatrix} \Lambda_B(S(\mu)) & 0 & 0 \\ 0 & \Lambda_T(S(\mu)) & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \hat{S}_B^* & \hat{S}_{BT}^* & \hat{S}_{BN}^* \\ \hat{S}_{TB}^* & \hat{S}_T^* & \hat{S}_{TN}^* \\ \hat{S}_{NB}^* & \hat{S}_{NT}^* & 0 \end{bmatrix},$$

where  $\Delta y_i(\mu) = y_i^* - y_i(\mu)$ ,  $\Delta S_N(\mu) = \hat{S}_N^* - \Lambda_N(S(\mu))$ , and the rightest matrix in (18) is referred to as the residual matrix.

Now, we aim to find a solution  $(\tilde{X}, \tilde{y}, \tilde{S})$  closest to a central solution, which has zero complementarity gap and  $\epsilon$ -feasibility. In contrast to the LO case, the system of equations in (17) and (18) may not be solvable if we drop the residual terms. Instead, we solve two least squares problems to obtain search directions for the primal-dual solutions. In [Subsection 4.3](#), we show that for sufficiently small  $\mu$ , a full step along the search directions results in  $\tilde{X}, \tilde{S} \succeq 0$ .

**4.1. Primal least squares problem.** For the primal problem we solve

$$(19) \quad \begin{aligned} \min \quad & \|e\|^2 + \|\Delta X\|^2 \\ \text{s.t.} \quad & \bar{A}_B^i \bullet \Delta X - e_i = \bar{A}_T^i \bullet \Lambda_T(X(\mu)) + \bar{A}_N^i \bullet \Lambda_N(X(\mu)), \quad i = 1, \dots, m, \end{aligned}$$

where  $\|\cdot\|$  denotes the Frobenius norm. We may assume that  $\bar{A}_B^i \neq 0$  for  $i = 1, \dots, m$ . Otherwise, the optimal solution to the auxiliary problem is  $\Delta x^* = 0$ , and thus the effect of the vanishing terms is absorbed in the error of primal infeasibility.

The optimal solution  $(e^*, \Delta X^*)$  to this auxiliary problem yields  $\tilde{X}_B = \Lambda_B(X(\mu)) + \Delta X^*$  so that

$$\bar{A}_B^i \bullet \tilde{X}_B = b_i + e_i^*, \quad i = 1, \dots, m.$$

Thus,  $\tilde{X}_B$  has  $\epsilon_p$  infeasibility for the primal constraints, where

$$(20) \quad \epsilon_p := \|e^*\|.$$

Let

$$\mathcal{A}^v := \begin{bmatrix} \text{vec}(A^1)^T \\ \text{vec}(A^2)^T \\ \vdots \\ \text{vec}(A^m)^T \end{bmatrix}, \quad \mathcal{A}_B^s := \begin{bmatrix} \text{svec}(\bar{A}_B^1)^T \\ \text{svec}(\bar{A}_B^2)^T \\ \vdots \\ \text{svec}(\bar{A}_B^m)^T \end{bmatrix}, \quad r(n) := \frac{n(n+1)}{2},$$

in which the operator  $\text{vec} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n^2}$  is the concatenation of the columns of a matrix, and  $\text{svec} : \mathbb{S}^n \rightarrow \mathbb{R}^{r(n)}$  is the concatenation of the upper triangular part of a matrix, where the off-diagonal entries are multiplied by  $\sqrt{2}$ . By the assumption,  $\mathcal{A}^v$  is a full row rank matrix, but  $\mathcal{A}_B^s$  might be rank deficient. Then, the auxiliary problem (19) reduces to

$$(21) \quad \min \|\mathcal{A}_B^s \Delta x - \eta\|^2 + \|\Delta x\|^2,$$

where  $\Delta x = \text{svec}(\Delta X)$ , and  $\eta_i = \bar{A}_T^i \bullet \Lambda_T(X(\mu)) + \bar{A}_N^i \bullet \Lambda_N(X(\mu))$  denotes the vanishing term for  $i = 1, \dots, m$ , which should be zero for all optimal solutions. Lemma 13 establishes upper bounds on  $\|e^*\|$  and  $\|\Delta X^*\|$ . The bounds are functions of the parameter  $\pi_p$  defined as

$$\pi_p := \prod_{k=1}^{r(|B|)} \left\| \left( (\mathcal{A}_B^s)^T \mathcal{A}_B^s + I \right)_{.k} \right\|,$$

where  $\left( (\mathcal{A}_B^s)^T \mathcal{A}_B^s + I \right)_{.k}$  denotes the  $k^{\text{th}}$  column of  $(\mathcal{A}_B^s)^T \mathcal{A}_B^s + I$ . Using the upper bounds in Lemma 13, we show in Theorem 15 that  $\tilde{X}_B \succ 0$  for sufficiently small  $\mu$ .

LEMMA 13. *Let  $(e^*, \Delta X^*)$  be the unique optimal solution to (19). Then we have*

$$\begin{aligned} \|\Delta X^*\| &\leq 2\pi_p \sqrt{r(|B|)} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|N|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}, \\ \epsilon_p &\leq 2\|\mathcal{A}^v\| \left( \pi_p \sqrt{r(|B|)} \|\mathcal{A}^v\|^2 + 1 \right) \max \left\{ \frac{n\sqrt{|N|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}. \end{aligned}$$

where  $\epsilon_p$  refers to primal infeasibility (20).

*Proof.* The optimality conditions for (21) are given by

$$(22) \quad \left( (\mathcal{A}_B^s)^T \mathcal{A}_B^s + I \right) \Delta x = (\mathcal{A}_B^s)^T \eta,$$

where  $(\mathcal{A}_B^s)^T \mathcal{A}_B^s + I \succ 0$ , and  $I$  is the identity matrix of size  $r(|B|)$ . All this means that the system of equations (22) has a unique solution. The solution to (22) can be computed using Cramer's rule [10]:

$$\Delta x_j^* = \frac{\det \left( \left( (\mathcal{A}_B^s)^T \mathcal{A}_B^s + I \right)^{(j)} \right)}{\det \left( (\mathcal{A}_B^s)^T \mathcal{A}_B^s + I \right)}, \quad j = 1, \dots, r(|B|),$$

in which the nominator arises from substituting the  $j^{\text{th}}$  column in  $(\mathcal{A}_B^s)^T \mathcal{A}_B^s + I$  by  $(\mathcal{A}_B^s)^T \eta$ . Noting that<sup>5</sup>  $\det((\mathcal{A}_B^s)^T \mathcal{A}_B^s + I) \geq 1$ , we can deduce from Hadamard's inequality [10] that

$$|\Delta x_j^*| \leq \left| \det \left( \left( (\mathcal{A}_B^s)^T \mathcal{A}_B^s + I \right)^{(j)} \right) \right| \leq \|(\mathcal{A}_B^s)^T \eta\| \prod_{k=1, k \neq j}^{r(|B|)} \left\| \left( (\mathcal{A}_B^s)^T \mathcal{A}_B^s + I \right)_{.k} \right\|,$$

<sup>5</sup>This is true regardless of data type, since the eigenvalues of  $(\mathcal{A}_B^s)^T \mathcal{A}_B^s + I$  are at least 1. However, we depend on the integrality of data to derive a lower bound for the condition number  $\sigma$ , as clarified in Appendix A.

for  $j = 1, \dots, r(|B|)$ . Since the diagonal entries of  $(\mathcal{A}_B^s)^T \mathcal{A}_B^s + I$  are greater than or equal to 1, the norm of each column is at least 1, and thus the bound for all  $j = 1, \dots, r(|B|)$  can be generalized to

$$(23) \quad |\Delta x_j^*| \leq \|(\mathcal{A}_B^s)^T \eta\| \prod_{k=1, k \neq j}^{r(|B|)} \|((\mathcal{A}_B^s)^T \mathcal{A}_B^s + I)_{.k}\| \leq \pi_p \|(\mathcal{A}_B^s)^T \eta\|.$$

Noting that  $\|\bar{A}_N^i\| \leq \|\bar{A}^i\| = \|A^i\|$  and  $\|\bar{A}_T^i\| \leq \|\bar{A}^i\| = \|A^i\|$ , we can observe from (11) and (12) that

$$\begin{aligned} |\bar{A}_N^i \bullet \Lambda_N(X(\mu))| &\leq \frac{n\sqrt{|N|}\mu}{\sigma} \|A^i\|, & i = 1, \dots, m, \\ |\bar{A}_T^i \bullet \Lambda_T(X(\mu))| &\leq c\sqrt{n|T|}(n\mu)^\gamma \|A^i\|, & i = 1, \dots, m, \end{aligned}$$

which yields the upper bound

$$(24) \quad |\eta_i| \leq 2\|A^i\| \max \left\{ \frac{n\sqrt{|N|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}, \quad i = 1, \dots, m.$$

Consequently, from the bounds in (23) and (24) we can conclude that

$$|\Delta x_j^*| \leq \pi_p \|(\mathcal{A}_B^s)^T \eta\| \leq 2\pi_p \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|N|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}, \quad j = 1, \dots, r(|B|),$$

where we have used the inequality  $\|\mathcal{A}_B^s\| \leq \|\mathcal{A}^v\|$ . As a result, we get

$$\begin{aligned} \|e^*\| &= \|\mathcal{A}_B^s \Delta x^* - \eta\| \leq \|\mathcal{A}_B^s\| \|\Delta x^*\| + \|\eta\| \\ &\leq 2\|\mathcal{A}^v\| \left( \pi_p \sqrt{r(|B|)} \|\mathcal{A}^v\|^2 + 1 \right) \max \left\{ \frac{n\sqrt{|N|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}. \end{aligned}$$

This completes the second part of the proof.  $\square$

**4.2. Dual least squares problem.** Let  $E$  denote the residual matrix as

$$E := \begin{bmatrix} E_B & E_{BT} & E_{BN} \\ E_{TB} & E_T & E_{TN} \\ E_{NB} & E_{NT} & 0 \end{bmatrix},$$

which is defined in accordance with the residual matrix in (18). Then, the auxiliary problem for an approximate dual solution is formulated as

$$(25) \quad \begin{aligned} \min \quad & \|E\|^2 + \|\Delta S\|^2 + \|\Delta y\|^2 \\ \text{s.t.} \quad & \sum_{i=1}^m \Delta y_i \bar{A}^i + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta S \end{bmatrix} - E = \begin{bmatrix} \Lambda_B(S(\mu)) & 0 & 0 \\ 0 & \Lambda_T(S(\mu)) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The optimal solution  $(E^*, \Delta y^*, \Delta S^*)$  gives  $\tilde{y}_i = y_i(\mu) + \Delta y_i^*$  for  $i = 1, \dots, m$  and  $\tilde{S}_N = \Lambda_N(S(\mu)) + \Delta S^*$  with  $\epsilon_d$  infeasibility for the dual constraints, where

$$(26) \quad \epsilon_d := \|E_B^*\| + \|E_T^*\| + \|E_{BT}^*\| + \|E_{BN}^*\| + \|E_{TN}^*\|.$$

For the sake of clarity in what follows, the auxiliary problem (25) is represented in vector form. To do so, we consider the  $\text{vec}(\cdot)$  operator for each block of  $\bar{A}^i$ , i.e.,  $\mathcal{A}_B^v$ ,  $\mathcal{A}_T^v$ ,  $\mathcal{A}_N^v$ ,  $\mathcal{A}_{BT}^v$ ,  $\mathcal{A}_{BN}^v$ , and  $\mathcal{A}_{TN}^v$ . Therefore, the auxiliary problem (25) can be simplified to the following least squares problem

$$(27) \quad \begin{aligned} \min \quad & \|(\mathcal{A}_B^v)^T \Delta y - \zeta_B\|^2 + \|(\mathcal{A}_T^v)^T \Delta y - \zeta_T\|^2 + \|(\mathcal{A}_N^v)^T \Delta y\|^2 + \|\Delta y\|^2 \\ & + \|(\mathcal{A}_{BT}^v)^T \Delta y\|^2 + \|(\mathcal{A}_{BN}^v)^T \Delta y\|^2 + \|(\mathcal{A}_{TN}^v)^T \Delta y\|^2, \end{aligned}$$

where  $\zeta_B = \text{vec}(\Lambda_B(S(\mu)))$  and  $\zeta_T = \text{vec}(\Lambda_T(S(\mu)))$ . Lemma 14 establishes upper bounds for  $\epsilon_d$  and  $\|\Delta S^*\|$ . These bounds are functions of the parameter  $\pi_d$  defined as

$$\pi_d := \prod_{k=1}^m \|\mathcal{H}_{.k}\|,$$

where

$$\begin{aligned} \mathcal{H} := & \mathcal{A}_B^v(\mathcal{A}_B^v)^T + \mathcal{A}_T^v(\mathcal{A}_T^v)^T + \mathcal{A}_N^v(\mathcal{A}_N^v)^T + \mathcal{A}_{BT}^v(\mathcal{A}_{BT}^v)^T + \mathcal{A}_{BN}^v(\mathcal{A}_{BN}^v)^T \\ & + \mathcal{A}_{TN}^v(\mathcal{A}_{TN}^v)^T + I, \end{aligned}$$

and  $\mathcal{H}_{.k}$  denotes the  $k^{\text{th}}$  column of  $\mathcal{H}$ . Observe that  $\mathcal{H}$  is positive definite. Theorem 15 proves that  $\tilde{S}_N \succ 0$  for sufficiently small  $\mu$ .

LEMMA 14. Problem (25) has a unique optimal solution  $(E^*, \Delta y^*, \Delta S^*)$ , which satisfies

$$\begin{aligned} \|\Delta S^*\| & \leq 2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}, \\ \epsilon_d & \leq (10\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 + 2) \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}. \end{aligned}$$

*Proof.* For the sake of brevity, we define

$$\varphi := \mathcal{A}_B^v \zeta_B + \mathcal{A}_T^v \zeta_T.$$

The optimality conditions for (27) can be written as

$$\mathcal{H} \Delta y = \varphi,$$

where  $\mathcal{H} \succ 0$ , and thus (27) has a unique solution. This solution can be computed by using Cramer's rule as follows

$$\Delta y_i^* = \frac{\det(\mathcal{H}^{(i)})}{\det(\mathcal{H})}, \quad i = 1, \dots, m,$$

where the nominator arises from substituting the  $i^{\text{th}}$  column in  $\mathcal{H}$  by  $\varphi$ . Note that  $\lambda_{\min}(\mathcal{H}) \geq 1$ , which implies  $\det(\mathcal{H}) \geq 1$ . Therefore, we get

$$|\Delta y_i^*| \leq |\det(\mathcal{H}^{(i)})| \leq \|\varphi\| \prod_{k=1, k \neq i}^m \|\mathcal{H}_{.k}\| \leq \pi_d \|\varphi\|, \quad i = 1, \dots, m,$$

where the latter follows from Hadamard's inequality. Note that  $\prod_{k=1, k \neq i}^m \|\mathcal{H}_{.k}\| \leq \pi_d$ , since the diagonal entries of  $\mathcal{H}$  are at least 1. Furthermore, we have from (10) and (12) that

$$\|\varphi\| \leq \|\mathcal{A}_B^v \zeta_B\| + \|\mathcal{A}_T^v \zeta_T\| \leq 2\|\mathcal{A}^v\| \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\},$$

which leads to

$$(28) \quad |\Delta y_i^*| \leq 2\pi_d \|\mathcal{A}^v\| \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}, \quad i = 1, \dots, m.$$

Consequently, from (28) it follows that

$$\|\Delta S^*\| = \|(\mathcal{A}_N^v)^T \Delta y\| \leq 2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}.$$

Using (28), we can also bound the components of the residual matrix as follows

$$\begin{aligned}
\|E_B^*\| &= \|(\mathcal{A}_B^v)^T \Delta y^* - \zeta_B\| \leq 2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\} \\
&\quad + \frac{n\sqrt{|B|}\mu}{\sigma} \leq (2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 + 1) \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}, \\
\|E_T^*\| &= \|(\mathcal{A}_T^v)^T \Delta y^* - \zeta_T\| \leq 2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\} \\
&\quad + c\sqrt{n|T|}(n\mu)^\gamma \leq (2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 + 1) \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}, \\
\|E_{BT}^*\|, \|E_{BN}^*\|, \|E_{TN}^*\| &\leq 2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}.
\end{aligned}$$

The bound for  $\epsilon_d$  follows from the bounds for the components of the residual matrix.  $\square$

**4.3. Cone feasibility.** As specified by Lemma 13 and Lemma 14,

$$\tilde{X} = Q(\mu) \begin{bmatrix} \tilde{X}_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q(\mu)^T, \quad \tilde{S} = Q(\mu) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{S}_N \end{bmatrix} Q(\mu)^T,$$

and  $\tilde{y}$  is a complementary solution for the primal-dual SDO problems. This primal-dual pair has  $\epsilon = \max\{\epsilon_p, \epsilon_d\}$  infeasibility w.r.t the linear constraints. However, to be an  $\epsilon$ -feasible maximally complementary solution,  $(\tilde{X}, \tilde{y}, \tilde{S})$  needs to be feasible with respect to the positive semidefinite cone as well. Theorem 15 shows that for a sufficiently small  $\mu$ , this rounding procedure yields a primal-dual solution with  $\tilde{X}_B, \tilde{S}_N \succ 0$ .

**THEOREM 15.** Let  $\vartheta_1 := 2n^2 \|\mathcal{A}^v\|^2$ ,  $\vartheta_2 := 2cn\sqrt{n|T|} \|\mathcal{A}^v\|^2$ , and let

$$\tilde{\mu} := \min \left\{ \frac{\sigma^2}{\vartheta_1 \max\{\pi_p \sqrt{r(|B|)|N|}, \pi_d \sqrt{m|B|}\}}, \frac{1}{n} \left( \frac{\sigma}{\vartheta_2 \max\{\pi_p \sqrt{r(|B|)}, \pi_d \sqrt{m}\}} \right)^{\frac{1}{\gamma}} \right\}.$$

If  $\mu \leq \tilde{\mu}$ , then we have  $\tilde{X}_B, \tilde{S}_N \succ 0$ .

*Proof.* We only need to show that for  $\mu \leq \tilde{\mu}$  the rounding procedure results in  $\tilde{X}_B, \tilde{S}_N \succ 0$ . Noting that

$$|\lambda_{\min}(\Delta X^*)| \leq \|\Delta X^*\|, \quad |\lambda_{\min}(\Delta S^*)| \leq \|\Delta S^*\|,$$

we can conclude from Lemma 13 and Lemma 14 that

$$\begin{aligned}
\lambda_{\min}(\tilde{X}_B) &\geq \lambda_{\min}(\Lambda_B(X(\mu))) + \lambda_{\min}(\Delta X^*) \\
&\geq \frac{\sigma}{n} - 2\pi_p \sqrt{r(|B|)} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|N|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}, \\
\lambda_{\min}(\tilde{S}_N) &\geq \lambda_{\min}(\Lambda_N(S(\mu))) + \lambda_{\min}(\Delta S^*) \\
&\geq \frac{\sigma}{n} - 2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\}.
\end{aligned}$$

Consequently,  $\tilde{X}_B, \tilde{S}_N \succ 0$  holds if

$$2\pi_p \sqrt{r(|B|)} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|N|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\} < \frac{\sigma}{n},$$

$$2\pi_d \sqrt{m} \|\mathcal{A}^v\|^2 \max \left\{ \frac{n\sqrt{|B|}\mu}{\sigma}, c\sqrt{n|T|}(n\mu)^\gamma \right\} < \frac{\sigma}{n}.$$

These inequalities hold if  $\mu \leq \tilde{\mu}$ . The proof is complete.  $\square$

*Remark 16.* Computing an  $\epsilon$ -feasible maximally complementary solution requires  $\mathcal{O}(\max\{|B|^6, m^3\})$  arithmetic operations. In fact, solving (19) and (25) is equivalent to solving two linear systems of equations, using the Gauss elimination method, with  $r(|B|)$  and  $m$  variables, respectively.

**4.4. Rounding procedure.** Using the bounds given in Lemma 13, Lemma 14, and Theorem 15, we can outline a simple procedure which yields an approximate maximally complementary solution.

---

**Algorithm 1** Rounding procedure for SDO

---

**Parameters**

Desired tolerance for primal infeasibility according to the bound for  $\|e^*\|$ .

Desired tolerance for dual infeasibility according to the bound for  $\|E^*\|$ .

**Input**

A central solution  $(X(\mu), y(\mu), S(\mu))$ , where  $\mu \leq \tilde{\mu}$ .

**Do**

Solve the primal least squares problem (19) to get  $\tilde{X}$ .

Solve the dual least squares problem (25) to get  $(\tilde{y}, \tilde{S})$ .

**return** Approximate maximally complementary solution  $(\tilde{X}, \tilde{y}, \tilde{S})$ .

---

**5. Identification of optimal partition for approximate centers.** Thus far, we assumed that the solution given by IPMs is exactly on the central path. In reality, however, path-following IPMs operate in a specified vicinity of the central path by getting approximate solutions to (1). Consider a solution  $(X, y, S) \in \text{ri}(\mathcal{P} \times \mathcal{D})$  given by a primal-dual path-following IPM. The proximity of  $(X, y, S)$  to the central path can be measured (i.e., Section 6.4 in [14]) by

$$\kappa(XS) = \frac{\lambda_{\max}(XS)}{\lambda_{\min}(XS)}, \quad (X, y, S) \in \text{ri}(\mathcal{P} \times \mathcal{D}).$$

It then follows that  $\kappa(XS) \geq 1$ , and the equality holds only when  $(X, y, S)$  is on the central path. The neighborhood of the central path is defined by

$$(29) \quad \mathcal{N}_\kappa(\tau) = \{(X, y, S) \in \text{ri}(\mathcal{P} \times \mathcal{D}) \mid \kappa(XS) \leq \tau\},$$

where  $\tau > 1$ . There exists  $\tau_2 \geq 1 \geq \tau_1 > 0$  so that  $\tau_2 = \tau\tau_1$ , and then for  $(X, y, S) \in \mathcal{N}_\kappa(\tau)$

$$(30) \quad \tau_1 \lambda_{\min}(XS) \leq \lambda_i(XS) \leq \tau_2 \lambda_{\min}(XS), \quad i = 1, \dots, n.$$

The following lemma provides a lower bound for  $\lambda_i(XS)$ .

LEMMA 17. *Let  $(X, y, S) \in \mathcal{N}_\kappa(\tau)$ . Then, we have*

$$\lambda_{[i]}(X) \lambda_{[n-i+1]}(S) \geq \lambda_{\min}(XS), \quad i = 1, \dots, n.$$

*Proof.* The proof is straightforward from the variational principles for eigenvalues [4]. It follows from the result of Lemma 2 that

$$\sum_{i=1}^k \left( \begin{array}{c} \lambda_{[n]}(X) \lambda_{[1]}(S) \\ \vdots \\ \lambda_{[1]}(X) \lambda_{[n]}(S) \end{array} \right)_{[n-i+1]} \geq \sum_{i=1}^k \lambda_{[n-i+1]}(XS), \quad k = 1, \dots, n.$$

For the special case  $k = 1$ , there holds that

$$\min \{ \lambda_{[n]}(X)\lambda_{[1]}(S), \lambda_{[n-1]}(X)\lambda_{[2]}(S), \dots, \lambda_{[1]}(X)\lambda_{[n]}(S) \} \geq \lambda_{\min}(XS),$$

which completes the proof.  $\square$

The following theorem generalizes the bounds derived in [Theorem 10](#) to a solution  $(X, y, S)$  in the given neighborhood of the central path.

**THEOREM 18.** *Let  $(X, y, S) \in \mathcal{N}_\kappa(\tau)$ , and  $\mu = \frac{X \bullet S}{n}$ . Then, it holds that:*

1. *For every  $i \in B$  we have*

$$\lambda_i(S) \leq \frac{n\mu}{\sigma}, \quad \lambda_i(X) \geq \frac{\sigma}{n\tau}.$$

2. *For every  $i \in N$  we have*

$$\lambda_i(S) \geq \frac{\sigma}{n\tau}, \quad \lambda_i(X) \leq \frac{n\mu}{\sigma}.$$

3. *For every  $i \in T$  we have*

$$\frac{\mu}{c\sqrt{n}\tau(n\mu)^\gamma} \leq \lambda_i(X), \lambda_i(S) \leq c\sqrt{n}(n\mu)^\gamma.$$

If  $\mu$  satisfies

$$(31) \quad \mu < \min \left\{ \frac{1}{n} \left( \frac{\sigma}{cn^{\frac{3}{2}}\tau} \right)^{\frac{1}{\gamma}}, (c^2 n^{2\gamma+1} \tau)^{\frac{1}{1-2\gamma}} \right\},$$

then we can identify the partition  $B$ ,  $N$  and  $T$ .

*Proof.* The proof technique can be traced back to [Theorem 10](#) fairly easily. Let  $(X^*, y^*, S^*) \in \mathcal{P}^* \times \mathcal{D}^*$  which satisfies [\(9\)](#), and  $(X, y, S) \in \mathcal{N}_\kappa(\tau)$ . From the orthogonality between  $(X - X^*)$  and  $(S - S^*)$ , it follows that

$$X \bullet S^* + X^* \bullet S = X \bullet S.$$

In a similar way as in [\(7\)](#), letting  $(\hat{X}, \hat{S})$  be an orthogonal transformation of  $(X, S)$  with respect to  $Q^*$ , we get

$$\begin{aligned} \lambda_{\min}(U_{X^*}) \operatorname{tr}(\hat{S}_B) &\leq X \bullet S \Rightarrow \operatorname{tr}(\hat{S}_B) \leq \frac{n\mu}{\sigma}, \\ \lambda_{\min}(U_{S^*}) \operatorname{tr}(\hat{X}_N) &\leq X \bullet S \Rightarrow \operatorname{tr}(\hat{X}_N) \leq \frac{n\mu}{\sigma}, \end{aligned}$$

where  $\hat{S}_B = (Q_B^*)^T S Q_B^*$ ,  $\hat{X}_N = (Q_N^*)^T X Q_N^*$ , and the latter inequalities follow from [\(9\)](#). Using [Lemma 1](#), we then have

$$\begin{aligned} \lambda_{[n-|B|+1]}(S) + \dots + \lambda_{[n]}(S) &\leq \operatorname{tr}(\hat{S}_B) \leq \frac{n\mu}{\sigma}, \\ \lambda_{[n-|N|+1]}(X) + \dots + \lambda_{[n]}(X) &\leq \operatorname{tr}(\hat{X}_N) \leq \frac{n\mu}{\sigma}, \end{aligned}$$

which implies

$$(32) \quad \begin{aligned} \lambda_{[n-i+1]}(S) &\leq \frac{n\mu}{\sigma}, \quad i = 1, \dots, |B|, \\ \lambda_{[n-i+1]}(X) &\leq \frac{n\mu}{\sigma}, \quad i = 1, \dots, |N|, \end{aligned}$$

by the positive definiteness of  $X$  and  $S$ . Recall from [\(30\)](#) that  $n\mu = X \bullet S \leq n\tau_2 \lambda_{\min}(XS)$ , which yields

$$(33) \quad \frac{\lambda_{\min}(XS)}{\mu} \geq \frac{1}{\tau_2} \geq \frac{1}{\tau}.$$



where  $\tau_2 = \tau\tau_1$  and  $\tau_1 \leq 1$ . Then, [Lemma 17](#) and [\(33\)](#) can be applied to [\(32\)](#) to derive lower bounds for the eigenvalues of  $X$  and  $S$ :

$$\begin{aligned}\lambda_{[i]}(X) &\geq \frac{\lambda_{\min}(XS)}{\lambda_{[n-i+1]}(S)} \geq \frac{\sigma \lambda_{\min}(XS)}{n\mu} \geq \frac{\sigma}{n\tau}, \quad i = 1, \dots, |B|, \\ \lambda_{[i]}(S) &\geq \frac{\lambda_{\min}(XS)}{\lambda_{[n-i+1]}(X)} \geq \frac{\sigma \lambda_{\min}(XS)}{n\mu} \geq \frac{\sigma}{n\tau}, \quad i = 1, \dots, |N|.\end{aligned}$$

The  $T$  part follows from [Lemma 8](#), where we have a sequence of bounded solutions  $(X, y, S)$  belonging to  $\mathcal{N}(\tau)$ . Analogous to the proof of [Theorem 10](#), it can be observed, using the basis transformation  $Q^*$ , that

$$(34) \quad \begin{aligned}\|(Q_{T \cup N}^*)^T X Q_{T \cup N}^*\| &= \left\| \begin{bmatrix} \hat{X}_T & \hat{X}_{TN} \\ \hat{X}_{NT} & \hat{X}_N \end{bmatrix} \right\| \leq \|X - X^*\| \leq c(n\mu)^\gamma, \\ \|(Q_{B \cup T}^*)^T S Q_{B \cup T}^*\| &= \left\| \begin{bmatrix} \hat{S}_B & \hat{S}_{BT} \\ \hat{S}_{TB} & \hat{S}_T \end{bmatrix} \right\| \leq \|S - S^*\| \leq c(n\mu)^\gamma,\end{aligned}$$

where  $\gamma = 2^{-d}$  and  $Q_{T \cup N}^*$  and  $Q_{B \cup T}^*$  are treated in the same way as in [Lemma 9](#). Therefore, from [\(34\)](#) it follows that

$$\begin{aligned}\lambda_{[n-|N|-|T|+1]}(X) + \dots + \lambda_{[n]}(X) &\leq c\sqrt{n}(n\mu)^\gamma, \\ \lambda_{[n-|B|-|T|+1]}(S) + \dots + \lambda_{[n]}(S) &\leq c\sqrt{n}(n\mu)^\gamma,\end{aligned}$$

and consequently,

$$\begin{aligned}\lambda_{[n-i+1]}(X) &\leq c\sqrt{n}(n\mu)^\gamma, \quad i = 1, \dots, |N| + |T|, \\ \lambda_{[n-i+1]}(S) &\leq c\sqrt{n}(n\mu)^\gamma, \quad i = 1, \dots, |B| + |T|.\end{aligned}$$

Using the bounds in [Lemma 17](#) and [\(33\)](#) we can derive

$$\begin{aligned}\lambda_{[i]}(X) &\geq \frac{\lambda_{\min}(XS)}{\lambda_{[n-i+1]}(S)} \geq \frac{\lambda_{\min}(XS)}{c\sqrt{n}(n\mu)^\gamma} \geq \frac{\mu}{c\sqrt{n\tau}(n\mu)^\gamma}, \quad i = 1, \dots, |N| + |T|, \\ \lambda_{[i]}(S) &\geq \frac{\lambda_{\min}(XS)}{\lambda_{[n-i+1]}(X)} \geq \frac{\lambda_{\min}(XS)}{c\sqrt{n}(n\mu)^\gamma} \geq \frac{\mu}{c\sqrt{n\tau}(n\mu)^\gamma}, \quad i = 1, \dots, |B| + |T|.\end{aligned}$$

In the sequel, we can identify the partition  $B$ ,  $N$  and  $T$  if

$$(35) \quad \frac{n\mu}{\sigma} < \frac{\mu}{c\sqrt{n\tau}(n\mu)^\gamma} < c\sqrt{n}(n\mu)^\gamma < \frac{\sigma}{n\tau},$$

which can be written equivalently as

$$\mu < \min \left\{ \frac{1}{n} \left( \frac{\sigma}{cn^{\frac{3}{2}}\tau} \right)^{\frac{1}{\gamma}}, (c^2 n^{2\gamma+1} \tau)^{\frac{1}{1-2\gamma}} \right\}.$$

This completes the proof.  $\square$

**COROLLARY 19.** *Let  $(X^0, y^0, S^0) \in \mathcal{N}_\kappa(\tau)$  and  $\mu^0 = \frac{X^0 \bullet S^0}{n}$ . Then, the Dikin-type primal-dual affine scaling method with steplength  $\alpha = \frac{1}{\tau\sqrt{n}}$  and the neighborhood [\(29\)](#) (see [Section 6.6](#) in [\[14\]](#)) needs at most*

$$\left\lceil \tau n \log \left( n\mu^0 \left( \min \left\{ \frac{1}{n} \left( \frac{\sigma}{cn^{\frac{3}{2}}\tau} \right)^{\frac{1}{\gamma}}, (c^2 n^{2\gamma+1} \tau)^{\frac{1}{1-2\gamma}} \right\} \right)^{-1} \right) \right\rceil$$

*iterations to get an  $(X, y, S) \in \mathcal{N}_\kappa(\tau)$  which allows to identify the optimal partition.*

*Proof.* The proof easily follows from the iteration complexity result for the Dikin-type primal-dual affine scaling method with steplength  $\alpha = \frac{1}{\tau\sqrt{n}}$  [\[14\]](#). Then, the complementarity gap drops below a certain threshold  $\epsilon_c$  after

$$\left\lceil \tau n \log \left( \frac{n\mu^0}{\epsilon_c} \right) \right\rceil$$

iterations. The result follows if we replace  $\epsilon_c$  by the right hand side of (31).  $\square$

**6. Concluding remarks.** In this paper, we considered the identification of the optimal partition for SDO where strict complementarity may fail. The magnitude of the eigenvalues belonging to each set  $B$ ,  $N$  and  $T$  is quantified using a condition number and the degree of singularity of the problem. We then use the estimates of the optimal partition to get an  $\epsilon$ -feasible maximally complementary solution by solving two least square problems. Further, we set bounds on each element of the partition  $B$ ,  $N$  and  $T$  for approximate solutions in a certain neighborhood.

The rounding procedure obtains an approximate maximally complementary solution from a central solution sufficiently close to the optimal set. Let  $(X, y, S) \in \mathcal{N}_\kappa(\tau)$ , where  $X := Q\Lambda(X)Q^T$  and  $S := P\Lambda(S)P^T$ . To extend the rounding procedure to solutions in the neighborhood of the central path, we need to fix the orthogonal basis either at  $Q$  or  $P$ , because  $X$  and  $S$  do not commute in general. To do so, we can solve the primal least squares problem (19), where  $X(\mu)$  and  $Q(\mu)$  are replaced by  $X$  and  $Q$ , respectively, in the definition of  $\bar{A}^i$ ,  $\bar{C}$ , and the right hand side in (19). We then solve the following least squares problem to compute a dual solution:

$$\begin{aligned} \min_{(E', y', S')} \quad & \|E'\|^2 + \|y - y'\|^2 + \|S' - Q^T S Q\|^2 \\ \text{s.t.} \quad & \sum_{i=1}^m y'_i \bar{A}^i + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S' \end{bmatrix} - E' = \begin{bmatrix} \bar{C}_B & \bar{C}_{BT} & \bar{C}_{BN} \\ \bar{C}_{TB} & \bar{C}_T & \bar{C}_{TN} \\ \bar{C}_{NB} & \bar{C}_{NT} & \bar{C}_N \end{bmatrix}, \end{aligned}$$

where

$$E' := \begin{bmatrix} E'_B & E'_{BT} & E'_{BN} \\ E'_{TB} & E'_T & E'_{TN} \\ E'_{NB} & E'_{NT} & E'_N \end{bmatrix}.$$

Let  $X'$  be the updated primal solution after applying the search direction from (19). This way, we get an approximate complementary solution  $(X', (y')^*, (S')^*)$  if the complementary gap  $X \bullet S$  is sufficiently small.

Optionally, we may fix the basis at  $P$  and solve (25) to arrive at  $(y', S')$ , where  $(y(\mu), S(\mu))$  and  $Q(\mu)$  are replaced by  $(y, S)$  and  $P$ , respectively, in the definition of  $\bar{A}^i$  and the right hand side in (25). Subsequently we solve

$$\begin{aligned} \min_{(e', X')} \quad & \|e'\|^2 + \|X' - P^T X P\|^2 \\ \text{s.t.} \quad & \bar{A}_B^i \bullet X' - e'_i = b_i, \quad i = 1, \dots, m, \end{aligned}$$

which, together with  $(y', S')$  gives an approximate maximally complementary solution  $((X')^*, y', S')$ . The bounds for primal-dual infeasibility for both approaches can be computed in a similar manner as in Section 4.

### Appendix A. The lower bound on $\sigma$ .

LEMMA A.1. *The condition number  $\sigma$  is bounded below by*

$$(A.1) \quad \min \left\{ \frac{1}{R_{\mathcal{P}^*} \sum_{i=1}^m \rho(A_i)}, \frac{1}{R_{\mathcal{D}^*}} \right\},$$

where

$$\begin{aligned} \log(R_{\mathcal{P}^*}) &= L(4(m + 2|B| + |N| + n(n + 1)))^{2m + |B| + |N| + n^2}, \\ \log(R_{\mathcal{D}^*}) &= L(4(m + |B| + 2|N| + n(n + 1)))^{2m + |B| + |N| + \frac{3n^2 + n}{2}}, \end{aligned}$$

$L$  denotes the binary input length, and  $\rho(\cdot)$  is the spectral radius.

*Proof.* We apply the definition of the analytic center of the optimal set to find a solution in the relative interior of the optimal set, and we then derive a lower bound for its minimum eigenvalue. It should be noted that Ramana [25] used this definition to compute a lower bound for the volume of a ball inscribed in the feasible set of a so called strict semidefinite feasibility problem.

Let  $X^* = Q^* \Lambda(X^*) (Q^*)^T$  and  $S^* = Q^* \Lambda(S^*) (Q^*)^T$  be a primal-dual optimal solution. The complementarity condition  $X^* S^* = 0$  implies that  $\lambda_j(X^*) = 0$  for  $j \in N \cup T$  and  $\lambda_j(S^*) = 0$  for  $j \in B \cup T$ . Suppose that the size of the partition  $B$ ,  $N$ , and  $T$  is already known. Furthermore, assume w.o.l.g., that the first  $|B|$  eigenvectors span the range space of  $X^*$  and the last  $|N|$  eigenvectors span the range space of  $S^*$ . For the sake of simplicity, let  $\lambda^x$  and  $\lambda^s$  denote the vector of eigenvalues of  $X$  and  $S$ , respectively. Then, the relative interior of the optimal set can be written as

$$\begin{aligned}
& \sum_{j=1}^n \lambda_j^x q_j^T A_i q_j = b_i, \quad \forall i = 1, \dots, m, \\
& C - \sum_{i=1}^m y_i A_i = \sum_{j=1}^n \lambda_j^s q_j q_j^T, \\
& \lambda_j^x > 0, \quad j = 1, \dots, |B|, \\
& \lambda_j^x = 0, \quad j > |B|, \\
& \lambda_j^s = 0, \quad j = 1, \dots, |B| + |T|, \\
& \lambda_j^s > 0, \quad j > |B| + |T|.
\end{aligned} \tag{A.2}$$

Any solution  $((\lambda^x)^*, (\lambda^s)^*, Q^*, y^*)$  of (A.2) gives a maximally complementary primal-dual pair of optimal solutions. Recall from (6) that  $\sigma_B \geq \lambda_{\min}(X^*)$  and  $\sigma_N \geq \lambda_{\min}(S^*)$ , where  $(X^*, S^*) \in \text{ri}(\mathcal{P}^* \times \mathcal{D}^*)$ . The analytic center of the primal optimal set can be computed by solving

$$\begin{aligned}
& \max_{\lambda^x, \lambda^s, Q, y} \log \left( \prod_{j=1}^{|B|} \lambda_j^x \right) \\
& \text{s.t.} \quad \sum_{j=1}^{|B|} \lambda_j^x q_j^T A_i q_j = b_i, \quad i = 1, \dots, m, \\
& C - \sum_{i=1}^m y_i A_i = \sum_{j=|B|+|T|+1}^n \lambda_j^s q_j q_j^T, \\
& \|q_j\|^2 = 1, \quad j = 1, \dots, n, \\
& q_j^T q_k = 0, \quad j, k = 1, \dots, n, \quad j > k, \\
& \lambda_j^x > 0, \quad j = 1, \dots, |B|, \\
& \lambda_j^s > 0, \quad j > |B| + |T|.
\end{aligned} \tag{A.3}$$

This problem can be written equivalently as

$$\begin{aligned}
& \max_{\lambda^s, Q, y} g(q_1, \dots, q_{|B|}) \\
& \text{s.t.} \quad C - \sum_{i=1}^m y_i A_i = \sum_{j=|B|+|T|+1}^n \lambda_j^s q_j q_j^T, \\
& \|q_j\|^2 = 1, \quad j = 1, \dots, n, \\
& q_j^T q_k = 0, \quad j, k = 1, \dots, n, \quad j > k, \\
& \lambda_j^s > 0, \quad j > |B| + |T|, \\
& g(q_1, \dots, q_{|B|}) = \max_{\lambda^x} \left\{ \log \left( \prod_{j=1}^{|B|} \lambda_j^x \right) \mid \sum_{j=1}^{|B|} \lambda_j^x q_j^T A_i q_j = b_i, \quad i = 1, \dots, m, \right. \\
& \quad \left. \lambda_j^x > 0, \quad j = 1, \dots, |B| \right\},
\end{aligned}$$

where  $g(q_1, \dots, q_{|B|})$  is determined by a convex optimization problem with a strictly concave objective function, i.e., when  $q_j$  for  $j = 1, \dots, |B|$  are fixed. The necessary and sufficient optimality conditions for the

inner maximization problem, which should be feasible for some  $Q$ , are expressed as

$$\begin{cases} (\lambda_j^x)^{-1} - \sum_{i=1}^m u_i q_j^T A_i q_j = 0, & j = 1, \dots, |B|, \\ \sum_{j=1}^{|B|} \lambda_j^x q_j^T A_i q_j = b_i, & i = 1, \dots, m, \\ \lambda_j^x > 0, & j = 1, \dots, |B|, \end{cases}$$

where  $u \in \mathbb{R}^m$ . All this implies that some  $\lambda_j^x$  can be represented as

$$\lambda_j^x = \left( \sum_{i=1}^m u_i q_j^T A_i q_j \right)^{-1} > 0, \quad j = 1, \dots, |B|,$$

i.e., for the given  $q_j$  for  $j = 1, \dots, |B|$ ,  $\sum_{j=1}^{|B|} \lambda_j^x q_j q_j^T$  is in the relative interior of the primal optimal set. Using  $\|q_j\| = 1$  for  $j = 1, \dots, B$ ,  $|q_j^T A_i q_j| \leq q_j^T |A_i| q_j$  and  $q_j^T |A_i| q_j \leq \lambda_{\max}(|A_i|)$  (the linear operator  $|\cdot| : \mathbb{S}^n \rightarrow \mathbb{S}^n$  is applied to the eigenvalues of a symmetric matrix, i.e., each eigenvalue of  $|A_i|$  corresponds to the absolute value of an eigenvalue of  $A_i$ ), we can conclude for  $j = 1, \dots, |B|$  that

$$\begin{aligned} \lambda_j^x &= \frac{1}{\sum_{i=1}^m u_i q_j^T A_i q_j} \geq \frac{1}{\sum_{i=1}^m |u_i| |q_j^T A_i q_j|} \geq \frac{1}{\sum_{i=1}^m |u_i| (q_j^T |A_i| q_j)} \\ &\geq \frac{1}{\sum_{i=1}^m |u_i| \lambda_{\max}(|A_i|)} \\ &\geq \frac{1}{\sum_{i=1}^m |u_i| \rho(A_i)}. \end{aligned}$$

In the sequel, it only remains to derive an upper bound on the magnitude of  $u$ . We can see from (A.3) and (A.4) that  $u$  satisfies

$$\begin{cases} \sum_{i=1}^m u_i \lambda_j^x q_j^T A_i q_j = 1, & j = 1, \dots, |B|, \\ \sum_{j=1}^{|B|} \lambda_j^x q_j^T A_i q_j = b_i, & i = 1, \dots, m, \\ C - \sum_{i=1}^m y_i A_i = \sum_{j=|B|+|T|+1}^n \lambda_j^s q_j q_j^T, \\ \|q_j\|^2 = 1, \quad q_j^T q_k = 0, & j, k = 1, \dots, n, \quad j > k, \\ \lambda_j^x > 0, & j = 1, \dots, |B|, \\ \lambda_j^s > 0, & j > |B| + |T|, \end{cases}$$

where the set of solutions is a compact set by the interior point assumption [3]. This is a system of equations which includes  $m + 2|B| + |N| + n(n+1)$  polynomials of degree 4 in  $2m + |B| + |N| + n^2$  variables. The size of the polynomials is dominated by  $L$  which is the upper bound on the binary size of the input data. Thus, by the first order theory of reals (Proposition 1.3 in [26]), there holds that  $\|u\| \leq R_{\mathcal{P}^*}$ , where

$$\log(R_{\mathcal{P}^*}) = L(4(m + 2|B| + |N| + n(n+1)))^{2m+|B|+|N|+n^2},$$

which completes the first part of the proof. The proof for dual side follows in a similar manner. The analytic

center of the dual optimal set is equivalent to

$$\begin{aligned}
& \max_{\lambda^x, Q, y} g(q_{|B|+|T|+1}, \dots, q_n) \\
& \text{s.t.} \quad \sum_{j=1}^{|B|} \lambda_j^x q_j^T A_i q_j = b_i, \quad i = 1, \dots, m, \\
& \quad \|q_j\|^2 = 1, \quad j = 1, \dots, n, \\
& \quad q_j^T q_k = 0, \quad j, k = 1, \dots, n, \quad j > k, \\
& \quad \lambda_j^x > 0, \quad j = 1, \dots, |B|, \\
& \quad g(q_{|B|+|T|+1}, \dots, q_n) = \max_{\lambda^s} \left\{ \log \left( \prod_{j=|B|+|T|+1}^n \lambda_j^s \right) \left| C - \sum_{i=1}^m y_i A_i = \sum_{j=|B|+|T|+1}^n \lambda_j^s q_j q_j^T, \right. \right. \\
& \quad \left. \left. \lambda_j^s > 0, \quad j > |B| + |T| \right\}.
\end{aligned}$$

The optimality conditions for the inner maximization problem are given by

$$\begin{cases}
(\lambda_j^s)^{-1} - q_j^T U q_j = 0, & j > |B| + |T|, \\
C - \sum_{i=1}^m y_i A_i = \sum_{j=|B|+|T|+1}^n \lambda_j^s q_j q_j^T, \\
\lambda_j^s > 0, & j > |B| + |T|,
\end{cases}$$

where  $U \in \mathbb{R}^{n \times n}$  is the matrix of Lagrange multipliers associated with the affine constraint in the inner maximization problem. Thus we get

$$\lambda_j^s = (q_j^T U q_j)^{-1} > 0, \quad j > |B| + |T|.$$

For the given  $q_j$  for  $j > |B| + |T|$ ,  $\sum_{j > |B| + |T|} \lambda_j^s q_j q_j^T$  is a solution in the relative interior of the dual optimal set. We can now observe that

$$\lambda_j^s = \frac{1}{q_j^T U q_j} \geq \frac{1}{q_j^T |U| q_j} \geq \frac{1}{\lambda_{\max}(|U|)} = \frac{1}{\rho(U)}.$$

According to the analytic center problem (A.3),  $U$  satisfies

$$\begin{cases}
\lambda_j^s q_j^T U q_j = 1, & j > |B| + |T|, \\
C - \sum_{i=1}^m y_i A_i = \sum_{j=|B|+|T|+1}^n \lambda_j^s q_j q_j^T, \\
\sum_{j=1}^{|B|} \lambda_j^x q_j^T A_i q_j = b_i, & i = 1, \dots, m, \\
\|q_j\|^2 = 1, \quad q_j^T q_k = 0, & j, k = 1, \dots, n, \quad j > k, \\
\lambda_j^s > 0, & j > |B| + |T|, \\
\lambda_j^x > 0, & j = 1, \dots, |B|,
\end{cases}$$

where the set of solutions is a compact set. This is a system of  $m + |B| + 2|N| + n(n+1)$  polynomials of degree 4 in  $2m + |B| + |N| + \frac{3n^2+n}{2}$  variables. Thus, by the first order theory of reals [26], there holds that  $\rho(U) \leq R_{\mathcal{D}^*}$ , where

$$\log(R_{\mathcal{D}^*}) = L(4(m + |B| + 2|N| + n(n+1)))^{2m+|B|+|N|+\frac{3n^2+n}{2}}.$$

This completes the proof.  $\square$

## Appendix B. Error bound for an LMI system.

LEMMA B.1 (Theorem 3.3 in [28]). Consider an LMI

$$\begin{cases} X \in D + \mathcal{L}, \\ X \succeq 0, \end{cases}$$

where  $D$  is a symmetric matrix and  $\mathcal{L}$  denotes a linear subspace of symmetric matrices. Let  $\bar{\mathcal{L}}$  be the smallest subspace containing  $D + \mathcal{L}$ , and  $\{X(\epsilon) \mid 0 < \epsilon \leq 1\}$  be a sequence of increasingly accurate solutions so that  $\|X(\epsilon)\|$  is bounded and

$$\text{dist}(X(\epsilon), D + \mathcal{L}) \leq \epsilon, \quad \lambda_{\min}(X(\epsilon)) \geq -\epsilon, \quad \forall \epsilon > 0,$$

where  $\text{dist}(X(\epsilon), D + \mathcal{L})$  denotes the distance of  $X(\epsilon)$  to its projection on the affine space  $D + \mathcal{L}$ . Then

$$\text{dist}(X(\epsilon), (D + \mathcal{L}) \cap \mathbb{S}_+^{n \times n}) \leq c\epsilon^\gamma,$$

where  $\mathbb{S}_+^{n \times n}$  denotes the cone of  $n \times n$  positive semidefinite matrices,  $c$  is a constant, and  $\gamma = 2^{-d(\bar{\mathcal{L}})}$  in which  $d(\bar{\mathcal{L}})$  denotes the degree of singularity of the linear subspace  $\bar{\mathcal{L}}$ .

LEMMA B.2 (Theorem 3.6 in [28]). For a linear subspace  $\bar{\mathcal{L}} \subset \mathbb{S}^{n \times n}$ , we have

$$d(\bar{\mathcal{L}}) \leq \min \{n - 1, \dim(\bar{\mathcal{L}}), \dim(\bar{\mathcal{L}}^\perp)\}.$$

*Example B.3.* We can show that the upper bound given in Lemma B.2 is indeed tight. To do so, consider the system of LMI

$$\begin{cases} X_{11} = 0, \\ X_{kk} = x_{1,k+1}, \quad k = 2, \dots, n-1, \\ X \succeq 0, \end{cases}$$

where the set of feasible solutions is given as

$$X^* = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & X_{nn}^* \end{bmatrix}, \quad X_{nn}^* \geq 0.$$

Using the facial reduction procedure in [28], we can see that the number of facial reduction steps is  $n - 1$  for all  $n \geq 2$ .

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