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Abstract

This work is concerned with the impact of gas network disruptions on dual-firing power generation. The question is addressed through the following optimization problem. Markets drive the price of gas, oil and electricity. The log-prices evolve as correlated mean-reverting processes in discrete time. A generating unit has dual-firing capabilities, here in the sense that it can convert either gas or oil to electricity. Oil and gas are subject to different constraints and uncertainties. Gas is obtained in real-time through the gas network. Due to gas supply disruptions, gas access is not guaranteed. Oil is stored locally and available in real-time. The oil storage capacity is limited onsite. Oil can be reordered to replenish the oil tank, with a lead time between the order time and the delivery time. Oil is paid for at the order time price. In this paper, we formulate the stochastic optimization problem for a risk-neutral operator, and study the sensitivity of the value of the dual-firing generating unit to the gas network availability parameters.

Keywords Dual-firing, Power generation, Energy asset management, Natural gas-electric coordination, Resilience, Markov decision processes, Optimal control, Dynamic programming, Stochastic optimization

MSC (2010): 90B05, 90B25, 90C15, 90C39, 90C40

1 Introduction

This paper is concerned with the impact of gas network disruptions on dual-firing power generation. The question is addressed through the following optimization problem. There are three markets to describe the price of gas, oil and electricity. The log-prices evolve as correlated mean-reverting processes in discrete time. A dual-firing power generation unit can convert fuel to electricity. Usually the prices and heat rates are such that it is more profitable to use gas rather than oil. However the fuels are subject to different constraints and sources of uncertainty. Gas is physically obtained in real-time through a gas pipeline. Due to gas supply disruptions, physical gas access is not guaranteed. The availability of the gas network is described by a two-state Markov chain. The parameters of the transition matrix determine the frequency and mean duration of the gas network disruptions. Oil is stored locally and available in real-time. However, the oil storage capacity is limited onsite. Oil can be reordered to replenish the oil tank, but there is a lead time between the order time and the delivery time. Oil is paid for at the order time price.

In this paper, the questions that are raised concern the determination of an optimal fuel utilization and oil replenishment policy, the value of the dual-firing generator asset, and the sensitivity of the value of the generator asset to the parameters of the random process describing the availability of the gas network. Gains in value over a single-fuel generator may be used to justify investments in dual-firing capabilities and oil storage capacity. Intuitively, we expect that the value added by dual-firing increases as the gas network becomes less reliable. To calculate the gain in value however, one must first determine an optimal operations policy. In this paper, we formulate the problem as a Markov decision process in a continuous state space with unbounded rewards. We examine its relationship with an equivalent multistage stochastic programming formulation. As we cannot solve these problems exactly, we resort to lower and upper bound calculations. We then study the variation of these bounds with respect to perturbations of the gas network availability process. The results of this work can be used in several ways: to study the investment return in dual-firing capabilities and oil storage capacity; to value gas provision contracts with low reliability; to value the benefits of gas network reliability improvements to power generation operators.

1.1 Context and Related Work

Natural gas fired electric power generators make an important part of the U.S. electricity generation mix [6]. However, gas-fired generators can be affected by gas supply disruptions. For instance, in some severe weather situations such as cold weather, high demand for gas and high demand for electricity put the gas system under stress, which can go as far as interrupting pipeline gas supply and force gas-fired power generation outages. Power generation outages have a significant effect on work and life and can disrupt the functioning of many critical sites such as hospitals, airports and manufacturing facilities.

To mitigate the risk of gas access disruptions, power generator units can be equipped with dual-firing capabilities, to be able to use another fuel when gas is unavailable [30]. Dual-firing capabilities brings resiliency to the electric system, assuming the alternate fuel can be used during temporary interruptions of gas supply [23]. Typical alternate fuels are petroleum and coal. Both can be stored onsite to mitigate the dependence upon their own delivery infrastructure (storing natural gas onsite would necessitate cryogenic facilities which may be expensive to operate and difficult to stabilize during a power outage). In this paper we focus on oil as the alternate fuel and assume the existence of onsite oil storage capacity. Power generation from petroleum is marginal in the U.S. due to high costs and emissions. Nevertheless oil-fired generators are used during peak load times and their high cost is a factor contributing to driving up the spot price of electricity.

The literature on the valuation of gas-fired electric power generation is extensive, especially in the applied energy finance literature. Gas generators are often viewed as a collection of call options on the "spark spread", which is the difference between the price of 1 MWh of electric energy versus the price of gas multiplied by the quantity needed to produce the electric MWh. See [21, 3, 4]. When the spark spread is positive, gas is converted into electric power. Flexibility in gas procurement contracts is often valued as a swing option, which gives the holder the obligation to withdraw a prescribed minimal cumulated amount of gas and the option to withdraw a maximal cumulated amount, over a contractual time window, subject to minimal and maximal amounts at each exercise. At each exercise, the holder receives the difference between the gas spot price and the contractual price. See [13, 24].

Several mathematical models and methods have been proposed to study the dependency between power generation and the gas network. For instance, [22] employ a two-stage nonlinear optimization model for coordinating operations of the power system and natural gas network. [15] use simulation to analyze the extent to which the electrical network can be affected by the gas network. [18] study the influence of interdependency of natural gas network and electricity infrastructure on power system security. [9] consider generation expansion problems under stochastic electricity and natural gas prices. However, few structural results have been developed.

The purpose of this paper is to assess the benefit of having oil as an alternative fuel to natural gas to operate a combustion turbine. To do this, we use stylized models to describe uncertainties. We make the assumption that the gas network access is a Markov process that can be described by a finite Markov chain. We describe the price of electricity, price of oil and price of gas as a mean-reverting process (in log scale). These are reasonable approximations. For instance, [5, 20, 14, 7, 2] use mean-reverting processes to describe electricity spot prices, [25] introduced a mean-reverting model to describe commodity price dynamics, [10] tested the term structure of futures prices of oil and natural gas from 1994 to 1999 and found mean reversion in these two commodities.

1.2 Contributions and Organization

The contributions of this paper can be summarized as follows. We incorporate the uncertainty of the gas network into a dual-firing power generation problem. We formulate the control problem as a Markov decision process where the electricity price, gas price and oil price follow a mean-reverting process (in log scale) and the availability of gas is described as a Markov chain. We analyze the structure of an equivalent multistage stochastic optimization formulation. We develop lower and upper bounds on the value function. We carry out the perturbation analysis of the lower and upper bounds over the parameters of the gas availability process.

Technical contributions. Technically, our paper works as follows. We first formulate a Markov decision process with continuous unbounded state variables and discrete bounded decision variables, and then reformulate it as a multistage mixed-integer stochastic optimization problem. The first result is Proposition 1 which shows that the problem is in fact totally unimodular (TU), and can thus be approached using tools from convex stochastic optimization [28]. Totally unimodular multistage stochastic mixed-integer optimization is discussed in [29]. The authors use finite random variables and rely on an extensive-form formulation. Unlike [29], we use a value function to shield us from the exponential growth of the problem dimension with the number of stages. This also allows us to handle continuous random variables. Our formulation circumvents difficulties that arise with different sets of assumptions (see e.g. [17, 12, 16]) because in our problem the integrality of decisions can be propagated forward in time, the stochastic process component that affects the feasible sets is integral, and our representation of the expected value function at the next state provably preserves the TU structure.

The second result, stated in Propositions 2 and 3, describes a lower bound and its sensitivity to perturbations of the problem data. The method consists in using a suboptimal solution that decouples the problem into independent parts, for which closed-form computations with continuous random variables are possible. The third result, stated in Proposition 4, describes the sensitivity of an upper bound obtained by assuming perfect foresight of the continuous random variables, while keeping the uncertainty for the discrete random variables. This technique works because the subproblems with perfect foresight are finite Markov decision processes, which can be solved exactly by dynamic programming techniques. The last result, Proposition 5, shows that the lower and upper bounds give the exact solution to a degenerate version of the problem.

Organization. The remainder of the paper is organized as follows. Section 2 describes the system model in the framework of Markov decision processes. Section 3 furnishes an equivalent formulation in the framework of multistage stochastic optimization, from which structural results are obtained. Section 4 furnishes a lower bound on the value function and studies its sensitivity to perturbations of the gas availability process. Section 5 furnishes an upper bound and studies its sensitivity to perturbations of the gas availability process. Section 6 illustrates the results numerically, and Section 7 concludes.

2 Model Description

This section describes our gas-electric-oil system as a Markov decision process in discrete-time over a finite horizon.

State variables. The state at time $t = 0, 1, \ldots, T$ is denoted S_t. The state is described by five state variables:

$$S_t = (l_t, p_t^e, p_t^g, p_t^o, b_t). \tag{1}$$

The variable l_t is the storage level of oil in the oil tank at the beginning of period t, p_t^e is the price of electricity, p_t^g is the price of gas, p_t^o is the price of oil, and b_t indicates whether the gas network is under disruption and unavailable ($b_t = 0$) or functions normally ($b_t = 1$). The storage level l_t is limited by K, which is the capacity of the oil tank. The prices are positive. The variables $\log(p_t^e)$, $\log(p_t^g)$, $\log(p_t^o)$ are called the log-prices.

Decisions variables. The decision at time $t = 0, \ldots, T-1$ is denoted A_t. The decision is described by the following decision variables:

$$A_t = (u_t^g, u_t^o, q_t) \tag{2}$$

The variables u_t^g, u_t^o are 0-1 indicators determining the fuel utilization during the current period. If $u_t^g = 1$, the unit converts gas to power. If $u_t^o = 1$, the unit converts oil to power. Those are mutually exclusive: $u_t^g + u_t^o \in \{0, 1\}$. The variable q_t is the quantity of oil that is ordered at the beginning period t. This quantity is delivered to the oil tank at the end of period t and is ready to be consumed at the beginning of the next period. The decision u_t^g is constrained by the availability of the gas network:

$$u_t^g \leq b_t. \tag{3}$$

The decisions u_t^o and q_t are subject to oil-tank capacity constraints:

$$\begin{aligned} & l_t + q_t \leq K && \text{if} \quad u_t^o = 0, \\ l_t - O_c \geq 0 \quad \text{and} \quad & l_t - O_c + q_t \leq K && \text{if} \quad u_t^o = 1. \end{aligned} \tag{4}$$

The first constraint in (4) applies when oil is not consumed during the current period. It says that the oil level post-delivery cannot exceed the oil storage capacity K. The second constraint applies when oil is used as fuel during the current period. It says that the oil level must be enough to satisfy the consumption, and then that the oil level post-consumption and post-delivery cannot exceed the capacity. The quantity O_c describes the quantity oil consumed to produce a quantity C of electricity (more on this in the definition of the reward function below).

We assume that the starting level l_0 and the quantity q_t are restricted to nonnegative multiples of O_c, and that K is a multiple of O_c (by rounding K down if necessary). This implies that $l_t \in \{0, O_c, 2O_c, \ldots, K\}$ and $q_t \in \{0, O_c, 2O_c, \ldots, K, K + O_c\}$, that is, the decision space is finite. In this context we also assume that $K \geq O_c$, otherwise oil will never be used.

Rewards. The reward $R_t = R(S_t, A_t)$ for $t = 0, \ldots, T-1$ gives the expected reward of being in state S_t and making decision A_t. The reward function is described as follows:

$$\begin{aligned} R(S_t, A_t) &= -p_t^o q_t && \text{if} \quad u_t^g = 0 \text{ and } u_t^o = 0, \\ R(S_t, A_t) &= C p_t^e - G_c p_t^g - p_t^o q_t && \text{if} \quad u_t^g = 1, \\ R(S_t, A_t) &= C p_t^e - p_t^o q_t && \text{if} \quad u_t^o = 1. \end{aligned} \tag{5}$$

The term $-p_t^o q_t$ is the cost of ordering a quantity q_t of oil at the spot price of oil $p_t^o = \exp(\log(p_t^o))$. The term $C p_t^e$ is the revenue from producing a quantity C of electricity at the spot price of electricity p_t^e. We assume that the generating unit is all-or-nothing and that the time the unit runs per period is one hour. In this case C is also the power capacity of the turbine. The term $G_c p_t^g$ is the cost of the gas needed to produce the quantity C of electricity. The cost depends on the quantity G_c of gas consumed to produce C and on the spot price of natural gas p_t^g. No term involves the price of oil when electricity is produced from oil because oil is withdrawn from the oil tank at that time.

We also define a terminal reward $R_T(S_T)$ at the terminal time $t = T$,

$$R_T(S_T) = p_T^o l_T. \tag{6}$$

The term $p_T^o l_T$ represents the revenue from the sale of the oil that remains in the tank at the end of the horizon, valued at the spot price of oil.

State Transitions. The log-prices follow a mean-reverting process in discrete-time. Let Δ denote the period duration. Let κ^e, κ^g, κ^o be positive parameters for the mean-reversion rates. We assume that Δ is made small enough such that the κ parameters are in $(0, 1/\Delta)$. Let $\log(\zeta^e)$, $\log(\zeta^g)$, $\log(\zeta^o)$ be parameters for the mean levels. Let $\sigma^e, \sigma^g, \sigma^o$ be positive parameters for the volatilities. Let $W_{t+1} = (W^e_{t+1}, W^g_{t+1}, W^o_{t+1})$ be a Gaussian multivariate random vector with zero mean, unit variances, and correlation matrix Σ. The vectors W_{t+1} at different time steps are mutually independent. We define

$$
\begin{aligned}
\log p^e_{t+1} - \log p^e_t &= \kappa^e(\log(\zeta^e) - \log p^e_t)\Delta + \sigma^e\sqrt{\Delta}W^e_{t+1} \\
\log p^g_{t+1} - \log p^g_t &= \kappa^g(\log(\zeta^g) - \log p^g_t)\Delta + \sigma^g\sqrt{\Delta}W^g_{t+1} \\
\log p^o_{t+1} - \log p^o_t &= \kappa^o(\log(\zeta^o) - \log p^o_t)\Delta + \sigma^o\sqrt{\Delta}W^o_{t+1}\ .
\end{aligned} \tag{7}
$$

The gas network state b_t follows a Markov chain with two states labeled 0,1. The transition matrix P of the gas network Markov chain is given by

$$
P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}, \qquad \begin{aligned} p_{01} &= 1 - p_{00} \in (0,1) \\ p_{10} &= 1 - p_{11} \in (0,1). \end{aligned} \tag{8}
$$

The oil level state variable l_t evolves as follows:

$$
\begin{aligned}
l_{t+1} &= l_t + q_t & \text{if} \quad u^o_t = 0, \\
l_{t+1} &= l_t - O_c + q_t & \text{if} \quad u^o_t = 1.
\end{aligned} \tag{9}
$$

Assuming $l_0 \in [0, K]$, the constraints guarantee that l_{t+1} remains in $[0, K]$.

Objective. We maximize the expected discounted cumulated reward over the finite horizon. For this objective, the minimal standard set of policies that contains an optimal policy is the set of nonstationary deterministic Markov policies. Let π be an admissible policy from that set, i.e. the decisions also satisfy the constraints. Viewing S_t, A_t as stochastic processes, let $\mathbb{P}^\pi_{s_0}$ and $\mathbb{E}^\pi_{s_0}$ denote the probability measure and expectation operator induced by the stochastic state transitions, the choice of A_t according to policy π, and the initial state s_0. Let $\gamma \in (0, 1]$ be the discount factor. The objective is then written as

$$
V_0(s_0) = \max_\pi \mathbb{E}^\pi_{s_0} \left\{ \sum_{t=0}^{T-1} \gamma^t R(S_t, A_t) + \gamma^T R_T(S_T) \right\}. \tag{10}
$$

Optimality Conditions. Following Bellman's optimality principle, the optimality conditions can be expressed recursively in terms of the value functions V_t:

$$
\begin{aligned}
&V_T(s) = R_T(s), \\
&V_t(s) = \sup_a \{R(s,a) + \gamma \mathbb{E}\{V_{t+1}(S_{t+1}) \mid S_t = s, A_t = a\}\}
\end{aligned} \tag{11}
$$

for $t = T-1, \ldots, 1, 0$.

3 Multistage Stochastic Optimization Formulation

Let $q_t^o = q_t/O_c$, $l_t^o = l_t/O_c$, $K^o = K/O_c$ be quantities rescaled to be integer-valued. Suppose momentarily that p_t^e, p_t^g, p_t^o, b_t are all fixed, and given l_0^o, consider the problem

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{t=0}^{T-1} \gamma^t [C p_t^e (u_t^g + u_t^o) - G_c p_t^g u_t^g - O_c p_t^o q_t^o] + \gamma^T p_T^o O_c l_T^o \\ \text{subject to} \quad & u_t^g \leq b_t, \quad u_t^o \leq l_t^o, \quad u_t^g + u_t^o \leq 1, \\ & l_{t+1}^o = l_t^o - u_t^o + q_t^o, \quad l_{t+1}^o \leq K^o, \\ & (u_t^g, u_t^o, q_t^o, l_{t+1}^o) \in \mathbb{Z}_+^4 \qquad \text{for } t = 0, \ldots, T-1, \end{aligned} \tag{12}$$

where $\mathbb{Z}_+$ is the set of nonnegative integers. It can be checked that this formulation is equivalent to the problem stated in Section 2 posed over a fixed realization of the prices and gas network states. By eliminating $q_t^o = l_{t+1}^o + u_t^o - l_t^o$, but enforcing its nonnegativity, we get the following description of the feasible set,

$$\begin{aligned} \mathcal{U}_t(l_t^o; b_t) := \{(u_t^g, u_t^o, l_{t+1}^o) \in \mathbb{Z}_+^3 : \quad & u_t^g \leq b_t, \ u_t^o \leq l_t^o, \quad u_t^g + u_t^o \leq 1, \\ & l_{t+1}^o + u_t^o \geq l_t^o, \quad l_{t+1}^o \leq K^o \ \}. \end{aligned} \tag{13}$$

Letting $(p_t^e, p_t^g, p_t^o, b_t)$ be a random process again and adapting the decisions to the generated filtration leads to a multistage stochastic programming formulation of the original problem. We state it below in nested form, using the notation V_t for the value functions, since except for the change of variables, they coincide with the value functions V_t in (11):

$$\begin{aligned} V_T(l_T^o; p_T^e, p_T^g, p_T^o, b_T) &= p_T^o O_c l_T^o, \\ V_t(l_t^o; p_t^e, p_t^g, p_t^o, b_t) &= \max_{(u_t^g,\, u_t^o,\, l_{t+1}^o) \in \mathcal{U}_t(l_t^o;\, b_t)} \{(C p_t^e - G_c p_t^g) u_t^g + (C p_t^e - O_c p_t^o) u_t^o \\ &\qquad - O_c p_t^o (l_{t+1}^o - l_t^o) + \gamma \, \mathcal{V}_t(l_{t+1}^o; p_t^e, p_t^g, p_t^o, b_t)\}, \qquad (14) \\ \mathcal{V}_t(l_{t+1}^o; p_t^e, p_t^g, p_t^o, b_t) &:= \mathbb{E}\{V_{t+1}(l_{t+1}^o; p_{t+1}^e, p_{t+1}^g, p_{t+1}^o, b_{t+1}) \mid p_t^e, p_t^g, p_t^o, b_t\}. \qquad (15) \end{aligned}$$

Proposition 1. *Let l_t^o be integer, but suppose that the integrality constraints in $\mathcal{U}_t(l_t^o;\, b_t)$ are relaxed. Suppose that $\mathcal{V}_t(l_{t+1}^o; p_t^e, p_t^g, p_t^o, b_t)$ is extended to a continuous domain for l_{t+1}^o by piecewise-linear interpolation. Then, it holds that the integrality of the optimal decisions $(u_t^g,\ u_t^o,\ l_{t+1}^o)$ is preserved, and that there is no gain or loss in optimality. Furthermore, the piecewise-linear interpolant for V_t is actually concave in l_t^o, which implies that the value function V_t has nonincreasing differences in l_t^o, for almost all $(p_t^e, p_t^g, p_t^o, b_t)$.*

Having nonincreasing differences in l_t^o means that as the tank fills up, the extra value from having an extra unit of stored oil diminishes or at best stays the same.

Proof. Suppose that in (14), the function $\mathcal{V}_t(l^o_{t+1}; p^e_t, p^g_t, p^o_t, b_t)$ is replaced by a continuous piecewise linear interpolant $\phi_t(l^o_{t+1})$, chosen to coincide with $\mathcal{V}_t(l^o_{t+1};\ p^e_t,\ p^g_t,\ p^o_t,\ b_t)$ at integer l^o_{t+1}. Note that $\mathcal{V}_t$ and thus ϕ_t are nondecreasing in l^o_{t+1}, and actually increasing, since increasing l^o_{t+1} relaxes the oil inventory constraints, while the increment can be used either to offset the cost of an optimal oil purchase, or otherwise to increase the terminal reward.

Suppose inductively that ϕ_t is also concave in l^o_{t+1}. Concavity holds at $t = T-1$, since V_T for all p^o_T and thus $\mathcal{V}_{T-1}$ are affine in l^o_T. Then $\phi_t(l^o_{t+1})$ admits the following representation (with state-dependent coefficients),

$$\begin{aligned}
\phi_t(l^o_{t+1}) &= \max\nolimits_{y \in \mathcal{Y}(l^o_{t+1})}\{c_0 + \textstyle\sum_{i=1}^{K^o} c_i y_i\},\\
\mathcal{Y}(l^o_{t+1}) &= \{y \in \mathbb{R}^{K^o} : 0 \le y_i \le 1,\ \textstyle\sum_{i=1}^{K^o} y_i = l^o_{t+1}\},\\
c_0 &:= \mathcal{V}_t(0; p^e_t, p^g_t, p^o_t, b_t),\\
c_i &:= \mathcal{V}_t(i; p^e_t, p^g_t, p^o_t, b_t) - \mathcal{V}_t(i-1; p^e_t, p^g_t, p^o_t, b_t),
\end{aligned} \tag{16}$$

where $c_1 \ge c_2 \ge \cdots \ge c_{K^o}$ by concavity of $\mathcal{V}_t$. We make the change of variable $z_i = \sum_{j=1}^{i} y_j$ and set by convention $z_0 = 0$. Therefore,

$$\begin{aligned}
y_i &= z_i - z_{i-1},\\
\textstyle\sum_{i=1}^{K^o} c_i y_i &= \textstyle\sum_{i=1}^{K^o} c_i(z_i - z_{i-1}) = c_{K^o} z_{K^o} + \sum_{i=1}^{K^o-1}(c_i - c_{i+1}) z_i.
\end{aligned}$$

Setting $h_i = c_{i+1} - c_i$ we get

$$\begin{aligned}
\phi_t(l^o_{t+1}) &= \max\nolimits_{z \in \mathcal{Z}(l^o_{t+1})}\{c_0 + c_{K^o} l^o_{t+1} - \textstyle\sum_{i=1}^{K^o-1} h_i z_i\} \qquad (17)\\
\mathcal{Z}(l^o_{t+1}) &= \{z \in \mathbb{R}^{K^o} : 0 \le z_i - z_{i-1} \le 1,\ z_{K^o} = l^o_{t+1}\} \quad (\text{with } z_0 = 0)\\
h_i &:= \mathcal{V}_t(i-1; p^e_t, p^g_t, p^o_t, b_t) - 2\mathcal{V}_t(i; p^e_t, p^g_t, p^o_t, b_t) + \mathcal{V}_t(i+1; p^e_t, p^g_t, p^o_t, b_t).
\end{aligned}$$

Going back to (14), we make a last change of variable: we introduce $l^u_t = l^o_t - u^o_t$ (oil inventory at the end of the period just before replenishment). Thus $u^o_t = l^o_t - l^u_t$ where l^o_t is fixed. Finally, we relax the integrality requirements of the set $\mathcal{U}_t(l^o_t, b_t)$. The input l^o_t is still assumed to be integer. This leads to the following continuous relaxation of the maximization problem in (14):

$$\begin{aligned}
\text{maximize}\quad & (Cp^e_t - G_c p^g_t)u^g_t - (Cp^e_t - O_c p^o_t)l^u_t + Cp^e_t l^o_t - O_c p^o_t l^o_{t+1}\\
& + \gamma\left[c_0 + c_{K^o} l^o_{t+1} - \textstyle\sum_{i=1}^{K^o-1} h_i z_i\right]\\
\text{subject to}\quad & 0 \le u^g_t \le b_t, \quad 0 \le l^u_t \le l^o_t,\\
& l^u_t - u^g_t \ge l^o_t - 1, \quad l^o_{t+1} - l^u_t \ge 0, \quad 0 \le l^o_{t+1} \le K^o,\\
& 0 \le l^o_{t+1} - z_{K^o-1} \le 1, \quad 0 \le z_1 \le 1,\\
& 0 \le z_i - z_{i-1} \le 1 \quad \text{for } i = 2, \ldots, K^o - 1.
\end{aligned} \tag{18}$$

The constraints have the form $\mathbf{A}x \leq \mathbf{b}$. Recall that the vertex optimal solutions to a linear problem are integral if and only if $\mathbf{b}$ is integral and $\mathbf{A}$ is totally unimodular. A sufficient condition for $\mathbf{A}$ to be totally unimodular is that its elements are in $\{0, -1, +1\}$, each row has at most 2 nonzero elements, and if a row has two nonzero elements they must have opposite sign. Those conditions are verified with (18). The right-hand side $\mathbf{b}$ is integer since l_t^o is fixed and integer. Hence the optimal vertex solutions to (18) are integer, and the relaxation is tight. Reverting back to the original variables, those are automatically integer as well. We have thus established that if l_t^0 is integer and the linear interpolant of $\mathcal{V}_t(\cdot; p_t^e, p_t^g, p_t^o, b_t)$ is concave, then the set $\mathcal{U}(l_t^o, b_t)$ can be replaced by its convex relaxation.

It remains to complete the induction argument. We start with the concavity of the linear interpolant. Let $x = (u_t^g, l_t^u, l_{t+1}^o, z_1, \ldots, z_{K^o-1})$, and let $\mathcal{X}(l_t^o, b_t)$ denote the feasible set for x in (18). Let $\varphi^*(l_t^o)$ denote the optimal value of (18) from the maximization over $x \in \mathcal{X}(l_t^o, b_t)$. Now, let $\mathcal{C} = \{(l_t^o, x) : x \in \mathcal{X}(l_t^o, b_t),\ 0 \leq l_t^o \leq K^o\}$. As a polytope, $\mathcal{C}$ is convex. The objective function of (18) is linear in (l_t^o, x) and thus jointly concave on $\mathcal{C}$, see e.g. [8]. From these conditions, it follows that $\varphi^*(l_t^o)$ is concave in l_t^o. On the discrete values of l_t^o we have $\varphi^*(l_t^o) = V_t(l_t^o; p_t^e, p_t^g, p_t^o, b_t)$. Taking the expectation over $(p_t^e, p_t^g, p_t^o, b_t)$ preserves the concavity properties in l_t^o, thus $\phi_{t-1}(l_t^o) = \mathbb{E}\{\varphi^*(l_t^o) | p_{t-1}^e, p_{t-1}^g, p_{t-1}^o, b_{t-1}\}$ is concave in l_t^o. Note that on the discrete values of l_t^o, $\phi_{t-1}(l_t^o) = \mathcal{V}_{t-1}(l_t^o; p_{t-1}^e, p_{t-1}^g, p_{t-1}^o, b_{t-1})$, so we have shown that the linear interpolant of $\mathcal{V}_{t-1}$ is concave in l_t^o, as required.

Finally, if l_t^o is integer, it has also been shown that l_{t+1}^o is integer almost surely. Thus, the value functions $V_t(l_t^o; p_t^e, p_t^g, p_t^o, b_t)$ for integer l_t^o only query the functions $V_{t+1}(l_{t+1}^o;\ p_{t+1}^e,\ p_{t+1}^g,\ p_{t+1}^o,\ b_{t+1})$ at integer l_{t+1}^o. This ensures one can always replace $\mathcal{V}_{t+1}$ by a piecewise continuous function with breakpoints at integer l_{t+1}^o without loss of optimality. $\square$

An interesting technical detail in the proof is the description (17) of the expected value function at the next state. Usually, polyhedral approximations of concave (respectively, convex) value functions are described as the minimum (respectively, maximum) of linear functions, which can be converted to a family of inequality contraints (linear cuts), e.g. as in stochastic dual dynamic programming (SDDP) methods [27]. The description (17) is based instead on a maximum, while continuing to use primal variables. This representation turns out to be instrumental for establishing by (18) the integrality of the solution of the convex relaxation.

When referring back to (18), note that the coefficients c_0, c_{K^o} and h_i depend on p_t^e, p_t^g, p_t^o, b_t. Proposition 1 addresses concavity properties of the value function with respect to l_t^o, but does not address the structure of the value function with respect to the price state variables.

Finally, we note that the mathematical result of Proposition 1 has a technological interpretation. The ability to *simultaneously* use different fuels is called *co-firing*. Co-firing is technologically more challenging than dual-firing. Mathematically, co-firing removes the integrality constraint on the variables u_t^e, u_t^o. The result of Proposition 1 shows that co-firing (of oil and gas) has *zero value* over dual-firing. This is of course under the assumption that the unit can operate under any single of the two fuels.

4 Lower Bounds

Lower bounds on the value function (11) can be obtained by calculating the value of a given policy. One possibility that leads to closed-form calculations is to run on gas when gas is available ($u_t^g = 1$ if $b_t = 1$ and $Cp_t^e - G_c p_t^g > 0$), and run on oil when gas is unavailable. To be able to run on oil without tracking the storage level l_t, we can select a suboptimal oil replenishment policy. For instance, we replenish with $q_t = O_c$ each time oil is used. In this case it can make sense to use oil when $b_t = 0$ and $Cp_t^e - O_c p_t^o > 0$, having the guarantee that $l_t \geq O_c$.

The value function is then bounded by the sum of two independent value functions:

$$V(s_0) \geq V^{\text{gas}}(s_0) + V^{\text{oil}}(s_0), \tag{19}$$

where $V^{\text{gas}}(s_0)$ is the value of converting gas to power, which depends on the electricity-gas spread ("spark spread" [3]) and on the gas network availability probabilities, and $V^{\text{oil}}(s_0)$ is the value of converting oil to power, which depends on the electricity-oil spread and on the gas network unavailability probabilities.

The two value functions can be summed up in the right-hand side of (19) because their contributions to the expected reward at time t conditionally to b_t are mutually exclusive, the suboptimal fuel utilization policy does not compare the fuel prices, and the two policies relative to V^{gas} and V^{oil} affect non-interacting components of the state. Actually, in the special case where the recourse to oil is nonexistent (for instance if $K = 0$), $V^{\text{gas}}(s_0)$ would coincide with the exact optimal value $V(s_0)$ of the problem.

To calculate the bounds, we first write (7) in matrix form. Let $\xi_t = [\log p_t^e \;\; \log p_t^g \;\; \log p_t^o]^\top$. We write $z_{t+1} \sim \mathcal{N}(\mu_z, \Sigma_z)$ to indicate that z_{t+1} follows a multivariate Gaussian of mean μ_z and covariance matrix Σ_z. We have

$$\xi_{t+1} = D\xi_t + z_{t+1}, \qquad D = \begin{bmatrix} (1-\kappa^e\Delta) & 0 & 0 \\ 0 & (1-\kappa^g\Delta) & 0 \\ 0 & 0 & (1-\kappa^o\Delta) \end{bmatrix},$$

$$z_{t+1} \sim \mathcal{N}(\mu_z, \Sigma_z),$$

$$\mu_z = \Delta \begin{bmatrix} \kappa^e \log(\zeta^e) \\ \kappa^g \log(\zeta^g) \\ \kappa^o \log(\zeta^o) \end{bmatrix}, \qquad \Sigma_z = \Delta \begin{bmatrix} (\sigma^e)^2 & \sigma^e\sigma^g\Sigma_{12} & \sigma^e\sigma^o\Sigma_{13} \\ \sigma^e\sigma^g\Sigma_{12} & (\sigma^g)^2 & \sigma^g\sigma^o\Sigma_{23} \\ \sigma^e\sigma^o\Sigma_{13} & \sigma^g\sigma^o\Sigma_{23} & (\sigma^o)^2 \end{bmatrix}. \tag{20}$$

From there, ξ_t given ξ_0 follows $\mathcal{N}(\bar{x}_t, X_t)$ where $\bar{x}_t = \begin{bmatrix} \bar{x}_{t,e} \\ \bar{x}_{t,g} \\ \bar{x}_{t,o} \end{bmatrix}$, $X_t = \begin{bmatrix} X_{t,ee} & X_{t,eg} & X_{t,eo} \\ X_{t,eg} & X_{t,gg} & X_{t,go} \\ X_{t,eo} & X_{t,go} & X_{t,oo} \end{bmatrix}$ are defined by the recursion

$$\bar{x}_{t+1} = D\bar{x}_t + \mu_z, \quad X_{t+1} = DX_tD^\top + \Sigma_z, \quad \text{with} \quad \bar{x}_0 = \xi_0, \quad X_0 = 0. \tag{21}$$

One has $\bar{x}_t = \sum_{k=0}^{t-1} D^k\mu_z + D^t\xi_0$ and $X_t = \sum_{k=0}^{t-1} D^k\Sigma_z D^k$ for $t \geq 1$.

Let

$$h_t^{eg} = Cp_t^e - G_c p_t^g. \tag{22}$$

To get the best lower bound, the optimal decision given $b_t = 1$ is $u_t^g = 1$ if $h_t^{eg} > 0$, and $u_t^g = u_t^o = 0$ otherwise. The probability that $b_t = 1$ is described by the second element of the row vector

$$[\mathbb{P}(b_t = 0), \mathbb{P}(b_t = 1)] := g_t = g_0 P^t := [g_{t0}, g_{t1}]. \tag{23}$$

It follows that

$$V^{\text{gas}}(s_0) = g_{01}[h_0^{eg}]^+ + \textstyle\sum_{t=1}^{T-1} \gamma^t g_{t1}\mathbb{E}\{[h_t^{eg}]^+\}, \tag{24}$$

$$[\log(Cp_t^e), \log(G_c p_t^g)]^\top \sim \mathcal{N}(\bar{y}_t, Y_t),$$

$$\bar{y}_t = \begin{bmatrix} \log C + \bar{x}_{t,e} \\ \log G_c + \bar{x}_{t,g} \end{bmatrix}, \quad Y_t = \begin{bmatrix} X_{t,ee} & X_{t,eg} \\ X_{t,eg} & X_{t,gg} \end{bmatrix}.$$

It remains to evaluate $\mathbb{E}\{[h_t^{eg}]^+\}$. Let $\Phi(\cdot)$ be the cumulative distribution function (cdf) of the standard normal distribution. By techniques similar to those used to establish Margrabe's formula [21], or by observing that $\mathbb{E}\{[h_t^{eg}]^+\} = \mathbb{E}\{Cp_t^e\} - \mathbb{E}\{\min\{Cp_t^e, G_c p_t^g\}\}$ and then using [19], one finds

$$\begin{aligned} \mathbb{E}\{[h_t^{eg}]^+\} = {} & C\exp(\bar{x}_{t,e} + \tfrac{1}{2}X_{t,ee})\Phi\left(\tfrac{(\log(C/G_c)+\bar{x}_{t,e}-\bar{x}_{t,g})+X_{t,ee}-X_{t,eg}}{\sqrt{X_{t,ee}+X_{t,gg}-2X_{t,eg}}}\right) \\ & - G_c\exp(\bar{x}_{t,g} + \tfrac{1}{2}X_{t,gg})\Phi\left(\tfrac{(\log(C/G_c)+\bar{x}_{t,e}-\bar{x}_{t,g})-X_{t,gg}+X_{t,eg}}{\sqrt{X_{t,ee}+X_{t,gg}-2X_{t,eg}}}\right). \end{aligned} \tag{25}$$

For consistency, note that $\mathbb{E}\{[h_0^{eg}]^+\} = h_0^{eg}$ by letting X_0 tend to 0.

The calculations for $V^{\text{oil}}(s_0)$ are similar, except that $V^{\text{oil}}(s_0)$ is based on two components, $V^{\text{oil},1}(s_0)$ and $V^{\text{oil},2}(s_0)$. The first component ensures we can always run on oil at times $t \geq 1$. Recall that $K \geq O_c$ by assumption. If $l_0 < O_c$ at time 0, we order O_c. It will be left to the second component to order additional oil if we consume oil at time 0, and to manage the balance of orders with consumption onwards. Then, $l_t = \max\{l_0, O_c\}$ for all $t \geq 1$. At time T the oil level l_T is liquidated via the terminal reward. Overall we have

$$V^{\text{oil},1}(s_0) = 1_{\{l_0 < O_c\}}(-O_c p_0^o) + \gamma^T \max\{l_0, O_c\}\mathbb{E}\{p_T^o\}, \tag{26}$$

$$\mathbb{E}\{p_T^o\} = \exp(\bar{x}_{T,o} + X_{T,oo}/2). \tag{27}$$

The second component is the return from the conversion from oil to power. Recall that $g_{t0} = 1 - g_{t1}$ is the probability that the gas network is unavailable at time t. Let $h_t^{eo} = Cp_t^e - O_c p_t^o$ be the electric-oil spread, noting that $-O_c p_t^o$ is actually the cost of replenishing for the next time oil is used. Setting time 0 apart to check the feasibility of the oil-electric conversion at the initial time, we get

$$V^{\text{oil},2}(s_0) = g_{00} 1_{\{l_0 \geq O_c\}}[h_0^{eo}]^+ + \textstyle\sum_{t=1}^{T-1} \gamma^t g_{t0} \mathbb{E}\{[h_t^{eo}]^+\}, \tag{28}$$

$$\begin{aligned} \mathbb{E}\{[h_t^{eo}]^+\} &= C \exp(\bar{x}_{t,e} + \tfrac{1}{2} X_{t,ee}) \Phi\left(\tfrac{(\log(C/O_c) + \bar{x}_{t,e} - \bar{x}_{t,o}) + X_{t,ee} - X_{t,eo}}{\sqrt{X_{t,ee} + X_{t,oo} - 2X_{t,eo}}}\right) \\ &\quad - O_c \exp(\bar{x}_{t,o} + \tfrac{1}{2} X_{t,oo}) \Phi\left(\tfrac{(\log(C/O_c) + \bar{x}_{t,e} - \bar{x}_{t,o}) - X_{t,oo} + X_{t,eo}}{\sqrt{X_{t,ee} + X_{t,oo} - 2X_{t,eo}}}\right). \end{aligned} \tag{29}$$

The quality of $V^{\text{oil},1} + V^{\text{oil},2}$ is negatively impacted by the suboptimality of the oil replenishment strategy. Trying to improve over the replenishment strategy would lead to a formulation similar to the original problem we set out to solve, with one fewer state variable (the price of gas). We can nevertheless improve the quality of the bound by considering other suboptimal policies that are easy to value. For instance, the expected return of doing nothing with oil is $\gamma^T l_0 \mathbb{E}\{p_T^o\} \geq 0$, from the terminal reward. In light of this remark, we can use

$$V^{\text{oil}}(s_0) = \max\{V^{\text{oil},1}(s_0) + V^{\text{oil},2}(s_0), \gamma^T l_0 \mathbb{E}\{p_T^o\}\}. \tag{30}$$

To summarize, we have shown that $V(s_0) \geq V^{LB}(s_0) := V^{\text{gas}}(s_0) + V^{\text{oil}}(s_0)$ with $V^{\text{gas}}(s_0)$ given by (24) and $V^{\text{oil}}(s_0)$ given by (30). These bounds offer a first look into the sensitivity of the problem value to the parameters describing the reliability of the gas network:

Proposition 2. *The lower bound $V^{LB}(s_0)$ varies linearly in the gas network availability probabilities* (23)*, except for a kink at the point where $V^{oil,1} + V^{oil,2} = \gamma^T l_0 \mathbb{E}\{p_T^o\}$.*

Proof. Suppose that $V^{\text{oil}}(s_0) = V^{\text{oil},1}(s_0) + V^{\text{oil},2}(s_0) > \gamma^T l_0 \mathbb{E}\{p_T^o\}$, see (30), which means that using oil backup is justified economically. Then we have, from (24), (28) and $g_{t0} = 1 - g_{t1}$,

$$\frac{\partial V^{\mathrm{LB}}(s_0)}{\partial g_{t1}} = \begin{cases} [h_0^{eg}]^+ - 1_{\{l_0 \geq O_c\}}[h_0^{eo}]^+ & \text{for } t = 0, \\ \gamma^t(\mathbb{E}\{[h_t^{eg}]^+\} - \mathbb{E}\{[h_t^{eo}]^+\}) & \text{for } t = 1, \ldots, T-1. \end{cases} \tag{31}$$

Alternatively, suppose that $V^{\mathrm{oil}}(s_0) = \gamma^T l_0 \mathbb{E}\{p_T^o\} < V^{\mathrm{oil},1}(s_0) + V^{\mathrm{oil},2}(s_0)$. To know if oil backup is justified economically, we would need to consider a better oil replenishment strategy. With our current lower bound, we can only assert that

$$\frac{\partial V^{\mathrm{LB}}(s_0)}{\partial g_{t1}} = \begin{cases} [h_0^{eg}]^+ & \text{for } t = 0, \\ \gamma^t \mathbb{E}\{[h_t^{eg}]^+\} & \text{for } t = 1, \ldots, T-1. \end{cases} \tag{32}$$

At the point where $V^{\mathrm{oil},1}(s_0) + V^{\mathrm{oil},2}(s_0) = \gamma^T l_0 \mathbb{E}\{p_T^o\}$, $V^{\mathrm{LB}}(s_0)$ is in general not differentiable.

□

The parameters of the gas network transition matrix P in (8) can be related to the probabilities g_{t1}. The dependence is nonlinear. To see this, consider the stationary probabilities of the gas network Markov chain, written g_∞ (row vector). Their derivatives can be described as $\dot{g}_\infty = g_\infty \dot{P}(I - P)^\#$, see [11], where $(I - P)^\# = (I - P + 1g_\infty)^{-1} - 1g_\infty$, see [26], is the generalized group inverse of $I - P$, and the dot operation denotes differentiation with respect to a parameter of interest (which can be p_{00} or p_{11}). The steady-state probabilities are most sensitive to perturbations when p_{00} and p_{11} are both close to 1, i.e., rare but long gas network disruptions.

Proposition 3 below gives the detailed results. We consider perturbations such that the perturbed matrix is still a transition matrix. This means that e.g. an increment of p_{00} is balanced by a decrement of p_{01} of equal magnitude, and the resulting probabilities should remain in $(0, 1)$.

Proposition 3. *When $V^{oil}(s_0) \neq \gamma^T l_0 \mathbb{E}\{p_T^o\}$, the lower bound $V^{LB}(s_0)$ is differentiable in the parameters p_{00}, p_{11} of the gas network transition matrix. Its gradient is described by*

$$\frac{\partial V^{LB}(s_0)}{\partial p_{ii}} = \sum_{t=1}^{T-1} \gamma^t [g_0 \sum_{k=0}^{t-1} P^k \tfrac{\partial P}{\partial p_{ii}} P^{t-1-k}] \begin{bmatrix} \varepsilon^{oil} \mathbb{E}\{[h_t^{eo}]^+\} \\ \mathbb{E}\{[h_t^{eg}]^+\} \end{bmatrix} \quad \textit{for } i = 0, 1, \tag{33}$$

$$\varepsilon^{oil} = 1_{\{V^{oil}(s_0) > \gamma^T l_0 \mathbb{E}\{p_T^o\}\}}, \quad \frac{\partial P}{\partial p_{00}} := \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial P}{\partial p_{11}} := \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}. \tag{34}$$

Proof. From $g_t = g_0 P^t$ we have $\dot{g}_t = g_0 \sum_{k=0}^{t-1} P^k \dot{P} P^{t-1-k}$. We introduce $\varepsilon^{\mathrm{oil}}$ to unify the results of Proposition 2. Then, the differentiation of $V^{LB}(s_0)$ with respect to p_{00}, p_{11} at a point where V^{LB} is differentiable ($V^{\mathrm{oil},1}(s_0) + V^{\mathrm{oil},2}(s_0) \neq \gamma^T l_0 \mathbb{E}\{p_T^o\}$) gives the expression of the proposition. □

5 Upper Bounds

Upper bounds on the value function (11) can be obtained by relaxing constraints and/or allowing decisions with foresight. Bounds based on perfect information are easy to write down but are often loose. One improvement is to assume perfect foresight of future prices, while keeping the gas network process b_t random. This is formalized as follows: we have

$$\begin{aligned} V(s_0) &= \max_{\pi} \mathbb{E}^{\pi}_{s_0}\{\textstyle\sum_{t=0}^{T-1} \gamma^t R(S_t, A_t) + \gamma^T R_T(S_T)\} \\ &\leq V^{\mathrm{UB}}(s_0) = \mathbb{E}_{\xi}\{V^{\xi}(s_0)\}, \end{aligned} \tag{35}$$

$$V^{\xi}(s_0) := \max_{\tilde{\pi}} \mathbb{E}^{\tilde{\pi}}_{\tilde{s}_0}\{\textstyle\sum_{t=0}^{T-1} \gamma^t R^{\xi}_t(\tilde{S}_t, A_t) + \gamma^T R^{\xi}_T(\tilde{S}_T)\}. \tag{36}$$

The expectation $\mathbb{E}_{\xi}$ is over the price process, given s_0. Since ξ is treated as a fixed input of V^{ξ}, in the dynamic programming problem (36), ξ_t is removed from the state space. The reduced state is denoted $\tilde{S}_t = (l_t, b_t)$. The reduced state space is finite with cardinality $2(1 + K/O_c)$. The notation R^{ξ}_t emphasizes that the reward function is nonstationary, as it depends on ξ_t which is time-dependent. The reward is defined as $R^{\xi}_t(\tilde{S}_t, A_t) = R(S^{\xi}_t, A_t)$ where $S^{\xi}_t = (\xi_t, \tilde{S}_t) = (p^e_t, p^g_t, p^o_t, b_t, l_t)$. Similarly, $R^{\xi}_T(\tilde{S}_t) = R_T(S^{\xi}_t)$. The policy $\tilde{\pi}$, which is specific to ξ, maps reduced states to decisions A_t. The finite-horizon problem (36) can be solved exactly in T iterations by value iteration, using the auxiliary value functions $V^{\xi}_T(l_T, b_T) = l_T p^o_T$ and

$$V^{\xi}_t(l_t, b_t) = \max_{A_t}\{R^{\xi}_t(l_t, b_t, A_t) + \gamma[p_{b_t,0} V^{\xi}_{t+1}(\ell_{t+1}, 0) + p_{b_t,1} V^{\xi}_{t+1}(\ell_{t+1}, 1)]\} \tag{37}$$

for $t = T-1, \ldots, 0$, where we use the post-decision state $\ell_{t+1} = l_t - O_c u^o_t + q_t$, given (l_t, b_t) and $A_t = (u^g_t, u^o_t, q_t)$. One then sets $V^{\xi}(s_0) = V^{\xi}_0(l_0, b_0)$.

The expectation over ξ in (35) can be estimated by sample average approximation methods [28] and will produce a statistical upper bound. Typically, using a large number M of independent samples $\xi^{(m)}$, one calculates

$$\bar{V}_M = \tfrac{1}{M}\textstyle\sum_{m=1}^{M} V^{\xi^{(m)}}(s_0), \quad \bar{\sigma}_M = \tfrac{1}{M}\sqrt{\textstyle\sum_{m=1}^{M}[V^{\xi^{(m)}}(s_0) - \bar{V}_M]^2}, \tag{38}$$

and then adopt a statistical upper bound such as $\hat{V}^{\mathrm{UB}}(s_0) = \bar{V}_M + 1.96\bar{\sigma}_M$ which holds with approximate confidence 97.5%.

Proposition 4 below describes the sensitivity of the upper bound (35) to perturbations of the gas network transition matrix.

Proposition 4. *Let $\ell_{t+1,j}$ denote the optimal ℓ_{t+1} when being in state (l_t, j) at time t, for $j = 0, 1$, as determined by* (37). *Define the matrix $\left[\frac{\partial P}{\partial p_{ii}}\right]$ as in* (34).

Define $\frac{\partial V_0^\xi(l_0,b_0)}{\partial p_{ii}}$ *recursively using* $\frac{\partial V_{T-1}^\xi}{\partial p_{ii}} = 0$ *and*

$$\frac{\partial V_t^\xi(l_t,j)}{\partial p_{ii}} = \gamma\left(\left[\frac{\partial P}{\partial p_{ii}}\right]\left[\begin{array}{c} V_{t+1}^\xi(\ell_{t+1,j},0) \\ V_{t+1}^\xi(\ell_{t+1,j},1)\end{array}\right] + P\left[\begin{array}{c}\partial V_{t+1}^\xi(\ell_{t+1,j},0)/\partial p_{ii} \\ \partial V_{t+1}^\xi(\ell_{t+1,j},1)/\partial p_{ii}\end{array}\right]\right)_j, \tag{39}$$

where $(\cdot)_j$ *extracts the* $(j+1)$*-th element. Then it holds that*

$$\partial \mathbb{E}_\xi\{V^\xi(s_0)\}/\partial p_{ii} = \mathbb{E}_\xi\{\partial V_0^\xi(l_0,b_0)/\partial p_{ii}\}. \tag{40}$$

Proof. The maximization problem of (37) can be expressed in the form (14), except that the expectation in (15) is only over b_{t+1} given b_t. Thus, one can relax the integrality constraints and reason on the convex relaxation of the problem. The continuous relaxation of the feasible set $\mathcal{U}_{t+1}(l_t^o;b_t)$ is nonempty and compact. Recall that $l_t^o = l_t/O_c$. Slightly abusing notation we write (37) in terms of l_t^o and l_{t+1}^o:

$$\begin{aligned} V_t^\xi(l_t^o,b_t) &= \max_{(u_t^g,u_t^o,l_{t+1}^o)\in\mathcal{U}_{t+1}(l_t^o;b_t)} f_{l_t^o,b_t}^\xi(u_t^g,u_t^o,l_{t+1}^o) \\ f_{l_t^o,b_t}^\xi(u_t^g,u_t^o,l_{t+1}^o) &:= (Cp_t^e - G_c p_t^g)u_t^g + (Cp_t^e - O_c p_t^o)u_t^o - O_c p_t^o(l_{t+1}^o - l_t^o) \\ &\quad + \gamma[p_{b_t,0}V_{t+1}^\xi(l_{t+1}^o,0) + p_{b_t,1}V_{t+1}^\xi(l_{t+1}^o,1)]. \end{aligned}$$

At time T, $V_T^\xi(l_T^o,0) = V_T^\xi(l_T^o,1) = O_c p_T^o l_T^o$ is independent of b_T, p_{00} and p_{11}. At time $T-1$, the objective is differentiable in p_{00} and p_{11}; using $p_{b_t,1} = 1 - p_{b_t,0}$ we have

$$\begin{aligned} &\partial f_{l_{T-1}^o,0}^\xi/\partial p_{00} = \gamma[V_T^\xi(l_{T,0}^o,0) - V_T^\xi(l_{T,0}^o,1)] = 0, \\ &\partial f_{l_{T-1}^o,1}^\xi/\partial p_{00} = 0, \\ &\partial f_{l_{T-1}^o,0}^\xi/\partial p_{11} = 0, \\ &\partial f_{l_{T-1}^o,1}^\xi/\partial p_{11} = \gamma[-V_T^\xi(l_{T,1}^o,0) + V_T^\xi(l_{T,1}^o,1)] = 0. \end{aligned}$$

Hence $\partial V_{T-1}^\xi/\partial p_{00} = \partial V_{t+1}^\xi/\partial p_{11} = 0$. At time $T-2$ we have

$$\begin{aligned} &\partial f_{l_{T-2}^o,0}^\xi/\partial p_{00} = \gamma[V_{T-1}^\xi(l_{T-1,0}^o,0) - V_{T-1}^\xi(l_{T-1,0}^o,1)] \\ &\partial f_{l_{T-2}^o,1}^\xi/\partial p_{00} = 0, \\ &\partial f_{l_{T-2}^o,0}^\xi/\partial p_{11} = 0, \\ &\partial f_{l_{T-2}^o,1}^\xi/\partial p_{11} = -\gamma[V_{T-1}^\xi(l_{T-1,1}^o,0) - V_{T-1}^\xi(l_{T-1,1}^o,1)]. \end{aligned}$$

The feasible set $\mathcal{U}_{T-2}$ does not depend on (p_{00},p_{11}) and is always nonempty. Assuming the distribution of ξ is nondegenerate (i.e. no component is a constant or a function of other components), the set of values of ξ for which the maximizer exists and is unique has probability 1. Therefore, for almost all ξ, by Danskin's theorem (see [28] Thm. 7.21 or [1] Section 4.3.1) the maximum is differentiable at $(p_{00}, p_{11}) \in (0,1)^2$ with values $\partial V^{\xi}_{T-2}/\partial p_{ii} = \partial f^{\xi}_{l^o_{T-2}, b_{T-2}}/\partial p_{ii}$, keeping all decisions fixed to their optimal value. This means that $\partial u^e_t/\partial p_{00} = 0$, $\partial l^o_t/\partial p_{00} = 0$, etc., for all $t \geq T-2$. At times $t \leq T-3$ we have

$$\begin{aligned}
\partial f^{\xi}_{l^o_t,0}/\partial p_{00} &= \gamma\Big[V^{\xi}_{t+1}(l^o_{t+1,0}, 0) - V^{\xi}_{t+1}(l^o_{t+1,0}, 1) \\
&\qquad + p_{00}\tfrac{\partial V^{\xi}_{t+1}(l^o_{t+1,0},0)}{\partial p_{00}} + (1-p_{00})\tfrac{\partial V^{\xi}_{t+1}(l^o_{t+1,0},1)}{\partial p_{00}}\Big], \\
\partial f^{\xi}_{l^o_t,1}/\partial p_{00} &= \gamma\Big[(1-p_{11})\tfrac{\partial V^{\xi}_{t+1}(l^o_{t+1,1},0)}{\partial p_{00}} + p_{11}\tfrac{\partial V^{\xi}_{t+1}(l^o_{t+1,1},1)}{\partial p_{00}}\Big], \\
\partial f^{\xi}_{l^o_t,0}/\partial p_{11} &= \gamma\Big[p_{00}\tfrac{\partial V^{\xi}_{t+1}(l^o_{t+1,0},0)}{\partial p_{11}} + (1-p_{00})\tfrac{\partial V^{\xi}_{t+1}(l^o_{t+1,0},1)}{\partial p_{11}}\Big], \\
\partial f^{\xi}_{l^o_t,1}/\partial p_{11} &= \gamma\Big[-V^{\xi}_{t+1}(l^o_{t+1,1}, 0) + V^{\xi}_{t+1}(l^o_{t+1,1}, 1) \\
&\qquad + (1-p_{11})\tfrac{\partial V^{\xi}_{t+1}(l^o_{t+1,1},0)}{\partial p_{11}} + p_{11}\tfrac{\partial V^{\xi}_{t+1}(l^o_{t+1,1},1)}{\partial p_{11}}\Big],
\end{aligned} \tag{41}$$

and by a similar reasoning one argues that $\partial V^{\xi}_t(l^o_t, b_t)/\partial p_{ii} = \partial f^{\xi}_{l^o_t, b_t}/\partial p_{ii}$, keeping all decisions fixed to their optimal value. One proceeds recursively over t until $\partial V^{\xi}_0(l^o_0, b_0)/\partial p_{ii}$ is reached. Equation (39) expresses (41) in a more compact form.

Finally, we check the conditions for the interchange between expectation and differentiation with respect to (p_{00}, p_{11}) around $(\bar{p}_{00}, \bar{p}_{11}) \in (0,1)^2$, see [28] Thm 7.44 Assumptions A1, A2, A4:

[A1] $(\bar{p}_{00}, \bar{p}_{11})$ is such that V^{ξ}_0 is well-defined and finite. This holds true since

$$\begin{aligned}
V^{\xi}_0(l_0, b_0) &\leq \textstyle\sum_{t=0}^{T-1} \gamma^t C \mathbb{E}\{p^e_t\} + \gamma^T K \mathbb{E}\{p^o_T\} < \infty, \\
V^{\xi}_0(l_0, b_0) &\geq -\textstyle\sum_{t=0}^{T-1} \gamma^t K \mathbb{E}\{p^o_t\} > -\infty.
\end{aligned}$$

[A2] There exists a positive C^{ξ} such that $\mathbb{E}\{C^{\xi}\} < \infty$, and for all (p'_{00}, p'_{11}) in a neighborhood of $(\bar{p}_{00}, \bar{p}_{11})$, and for almost every ξ,

$$|V^{\xi}_0(l_0, b_0; p'_{00}, p'_{11}) - V^{\xi}_0(l_0, b_0; \bar{p}_{00}, \bar{p}_{11})| \leq C^{\xi} \|(p'_{00} - \bar{p}_{00}, p'_{11} - \bar{p}_{11})\|.$$

This holds true since the derivatives in (41) are bounded uniformly over the decisions, using

$$|V^{\xi}_{t+1}(\cdot, 1) - V^{\xi}_{t+1}(\cdot, 0)| \leq \textstyle\sum_{k=t+1}^{T-1} \gamma^{k-(t+1)} [G_c p^g_t - O_c p^o_t]^+ + K p^o_t$$

and using (41) recursively:

$$\left|\frac{\partial V^{\xi}_t(\cdot,1)}{\partial p_{ii}} - \frac{\partial V^{\xi}_t(\cdot,0)}{\partial p_{ii}}\right| \leq \gamma |V^{\xi}_{t+1}(\cdot, 1) - V^{\xi}_{t+1}(\cdot, 0)| + \gamma \left|\frac{\partial V^{\xi}_{t+1}(\cdot,1)}{\partial p_{ii}} - \frac{\partial V^{\xi}_{t+1}(\cdot,0)}{\partial p_{ii}}\right|.$$

The finiteness of $\mathbb{E}\{C^{\xi}\}$ results from $\mathbb{E}\{p^o_t\}$ and $\mathbb{E}\{[C p^e_t - G_c p^g_t]^+\}$ being finite.

[A4] For almost all ξ the function $V^{\xi}(l_t, b_t; p_{00}, p_{11})$ is differentiable at $(\bar{p}_{00}, \bar{p}_{11})$. This holds true from $\partial V_t^{\xi}(l_t^o, b_t;\ p_{00}, p_{11})/\partial p_{ii} = \partial f^{\xi}_{l_t^o, b_t}/\partial p_{ii}$ and (41).

□

We conclude this section with an observation that leads to a method to select M.

Proposition 5. *In the case $K = 0$, the upper bound $V^{UB}(s_0)$ in (35) is equal to the exact optimal value of the problem. In particular, it is equal to the lower bound $V^{LB}(s_0)$.*

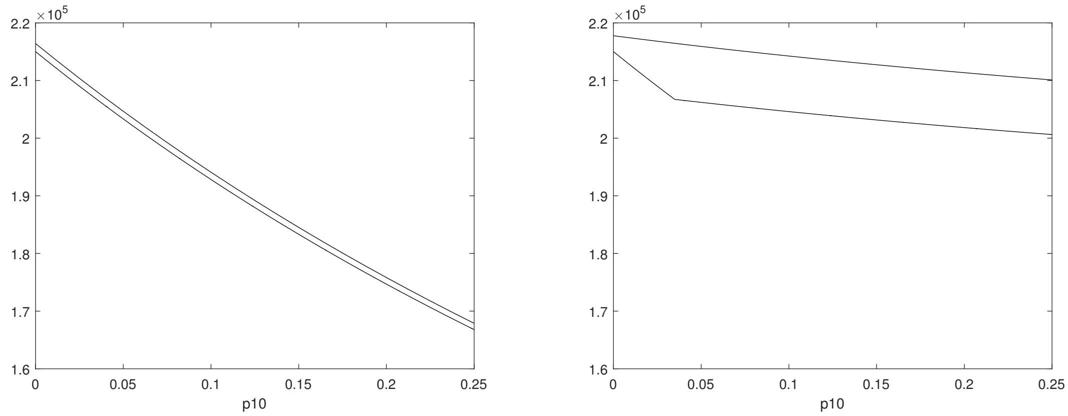
Proof. In the case $K = 0$ (no oil storage capacity), the only decision is whether $u_t^g = 0$ or $u_t^g = 1$ when $b_t = 1$. This decision does not influence the distribution of future states and is only based on the sign of the spark spread $(Cp_t^e - G_c p_t^g)$ at time t. Knowing the price of gas and electricity in advance will in general modify the conditional distributions for the oil price process, but since oil can never be used in the absence of oil storage capacity, this improved information has no impact on the optimal decision policy. Consequently there is no gain in optimality from perfect foresight assumed within V^{ξ}. □

Proposition 5 is interesting because it provides an accurate method to relate the statistical upper bound to the exact upper bound. In numerical studies, one can select M in (38) by first verifying that the statistical upper bound matches the exact lower bound in the case $K = 0$. When the match is exact to a desired tolerance level, one could also freeze the sample set, given the successful test on $K = 0$.

6 Numerical Test

We evaluate the lower and upper bound and their sensitivities to p_{00}, p_{11} for a particular instance. We set γ to 0.95. We assume that the power generator is operated as a peaker and runs either 0 or 1 hour per day. In our base case the problem is over $T = 30$ days, with the understanding that the period duration is $\Delta = 1$ hour, corresponding to the hour where the peaker may operate. The decision maker is risk neutral, and thus willing to be fully exposed to spot price variations and physical disruptions. The price of electricity is in \$/MWh, the price of gas is in \$/MMBtu, and the price of oil is in \$/barrel. In the oil price, we neglect the transportation costs and the emission costs. These location-specific components could be added to the oil price in a more detailed study. We assume that one barrel of oil produces 5.5 MMBtu of heat, and that the combustion turbine can convert 1MMBtu of heat to 0.1MWh of electricity. We neglect the effect of the ambient temperature on efficiency. The power capacity is $C = 100$ MW. The oil storage capacity is set to 3 days of 1-hour

Figure 1: Statistical upper bound $\hat{V}^{\mathrm{UB}}$ and lower bound V^{LB} on the test problem, as a function of the gas network failure probability $p_{10} = 1 - p_{11}$. Left: $K = 0$ (gas only). Right: $K = 3 \cdot O_c$ (dual-firing).



production by oil, thus $K = 3 \cdot 100 \cdot (1/0.1)/5.5 = 545.5$ barrels. With these values we also have $G_c = 100 \cdot (1/0.1) = 1000$ and $O_c = 100 \cdot (1/0.1)/5.5 = 181.8$. The price process parameters are as follows:

$$\begin{aligned} \log(\zeta^e) &= \log(100), & \kappa^e &= 0.5, & \sigma^e &= 1, \\ \log(\zeta^g) &= \log(5), & \kappa^g &= 0.3, & \sigma^g &= 0.3, \\ \log(\zeta^o) &= \log(50), & \kappa^o &= 0.1, & \sigma^o &= 0.05, \end{aligned} \qquad \Sigma = \begin{bmatrix} 1 & .2 & 0 \\ .2 & 1 & .2 \\ 0 & .2 & 1 \end{bmatrix},$$

where as seen from Σ we have assumed a positive correlation between gas and electric, a positive correlation between gas and oil, and no correlation between oil and electric. Usually oil moves slower than gas, and electric is the most volatile; this is reflected in the volatilities σ.

Following Proposition 5, we do a test run with $K = 0$ (no oil storage) to determine an appropriate value for M, the number of scenarios for the statistical upper bound. $M = 20\,000$ provides a reasonable match. The value of the lower and upper bounds appear in Figure 1 (Left) and are still distinguishable. The bounds are calculated for various values of the transition matrix P, namely, p_{10} varying from 0 to 0.25, and p_{01} set to 0.85.

Next, we compute the value of the lower bound and the upper bound when $K = 3 \cdot O_c = 545.5$. We report the values as a function of p_{10} as well. The results are depicted in Figure 1 (Right). On this example, the maximal gap between the two bounds is below 5% of the lower bound value. Finally, we check that the derivatives given in Propositions 3 and 4 are those that are observed on the curves of the figure. (This is indeed the case; those tests are not depicted.)

The comparison between the two cases ($K = 0$, $K = 3 \cdot O_c$) shows the extent to which dual-firing capabilities mitigates the loss of value from the unreliable gas network, as measured here by the transition probability p_{10}.

7 Concluding Remarks

In this paper, we formulate an optimal power production management problem for a dual-firing power generator. We discuss the impact of the gas network reliability on power production. We establish lower bounds and upper bounds that can be used to estimate the benefits of improving the reliability of the gas supply and the benefits of fuel flexibility. In our test the gap is relatively small, indicating that for the price model we considered, the lower bound is based on reasonable assumptions on optimal operations. Otherwise, the lower bound could be improved, for instance by calculating an approximate expected value function at the next state $\widetilde{\mathcal{V}}$ and then estimating on an independent test sample the value of the policy that uses the approximate value function. The upper bound could be improved, for instance by reintroducing uncertainty on some of the price components and conditioning the distributions on the perfect foresight information.

The stochastic access to the gas network can be interpreted as a random process that enables certain actions at times that are not controlled by the decision maker. The present work relates to the optimal control of such systems. Insights from this work could thus be found relevant to other domains with similar characteristics and reliability concerns.

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