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Abstract Discrete Multi-Scenario Truss Sizing Optimization problems (MSTSO) are challenging to solve due to their combinatorial, nonlinear, and non-convex nature. We show two important characteristics of the feasible set of MSTSO problems. First, given a feasible truss structure for the continuous MSTSO problem and the ray corresponding to that feasible structure, we prove the feasibility of the solutions along that up to the boundary of the feasible set. Second, we prove that a truss structure that withstands a set of external force scenarios, also withstands any convex combination of those external force scenarios. We use these two characteristics to extend the Neighborhood Search Mixed Integer Linear Optimization (NS-MILO) algorithm to solve large-scale discrete MSTSO problems. We demonstrate through extensive computational experiments that the NS-MILO algorithm with a reasonable time budget provides high-quality solutions for large-scale MSTSO problems.

Keywords Truss Sizing Optimization · Multi-Scenario Truss Design Optimization · Mixed Integer Linear Optimization · Euler buckling constraints

1 Introduction

Truss structures are widely used in a variety of applications. A truss design problem is concerned with the optimal selection the topology, and sizing of the bars. In Truss Sizing Optimization (TSO) problems, the truss topology is given and the cross sectional areas of bars are to be decided (Dorn et al., 1964). In real world applications, a truss structure should withstand different circumstances i.e., different loading conditions. Multi-Scenario Truss Sizing Optimization (MSTSO) problems consider a TSO problem under multiple external load scenarios. It is

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worth mentioning that multi-load TSO problems are different from MSTSO problems. Multi-load TSO problems refers to the problems where multiple external forces are exerted on the structure at once while in multi-load scenario TSO problems, the structure must withstand different load conditions (Bendsøe et al., 1995). In other words, in a multi-load scenario TSO problems, a structure is under different load scenarios. In TSO problems, cross-sectional areas can be selected from a continuous or a discrete set.

Mellaert et al. (2016) proposed a Mixed Integer Linear Optimization (MILO) model for single scenario TSO problems considering displacement constraints, member constraints, and joint constraints. Shahabsafa et al. (2018) proposed a Mixed Integer Linear Optimization (MILO) model for single scenario TSO problems with weight minimization objective function. They considered additional physical constraints such as Hooke's law and Euler buckling constraints. In Truss Topology Design (TTD) problems, potential bars can take zero cross-sectional areas. Mela (2014) suggested a MILO model for Multiple Loading Scenarios of Truss Topology Design and Sizing Optimization (MTTDSO) problems considering physical constraints. Achtziger (1998) studied the MTTDSO problem with the objective of minimizing the maximum compliance of load scenarios.

There is another class of problems that can be reformulated as MSTSO namely when the structure is under uncertain loading scenarios (Lg et al., 2009). Stochastic optimization is a mathematical approach to deal with uncertainty by considering discrete random variables in the problem. In special cases, a stochastic problem can be transformed into a multiple load scenarios problem (Csbfalvi, 2018). Alvarez and Carrasco (2005) demonstrated that the minimum expected compliance under stochastic loading scenarios coincides with the dual of a special convex minimax problem. They proved that the expected compliance minimization is equivalent to multiload problem with specific finite set of multiple load scenarios. Makrodimopoulos et al. (2010) proposed a new compliance-based objective for the truss design problem under multiple load scenarios. Their model minimizes the summation of the maximum strain energy of the structure. They proved that, in kinematically stable and determinate structures, the proposed problem is equivalent to a weight minimization problem.

It is worth mentioning uncertainty in loading scenarios can be modeled using robust optimization instead of stochastic optimization (Calafiore and Dabbene, 2008; Zhao and Wang, 2014). In these cases the problem cannot typically be reformulated as MSTSO problem. Kanno and Takewaki (2006) applied the quadratic embedding method of uncertainty and the S-procedure and reformulated the problem as a nonlinear semidefinite optimization problem. Dunning et al. (2011) considered uncertainties in loading magnitude and direction in the compliance minimization problem. They applied an analytical approach for normally distributed external forces, and converted the robust problem to a multiple load scenarios problem. Liu and Gea (2018) proposed a new continuous formulation for TTDSO problems where bars are under uncertain forces. The ellipsoid-bounded uncertainty comes from both loading direction and amplitude. The objective is to minimize the maximum compliance.

In this article, we focus on truss sizing optimization problems. We extend the TSO model proposed by Shahabsafa et al. (2018) to formulate minimum-weight MSTSO problems considering force balance equations, Hook's law, yield stress bounds, displacement bounds, and Euler buckling constraints. Then, we present two important characteristics of the feasible set of MSTSO problems. Given a feasible truss structure for the continuous MSTSO problem, we prove that any multiple of the vector of cross-sectional areas that satisfies the upper bounds is also feasible for the problem when the multiplier is grater than one. We also prove that a truss structure feasible for all load scenarios is also feasible for any convex combination of those external forces. MSTSO problems become unsolvable as the size of the truss grows. We extend the Neighborhood Search Mixed Integer Linear Optimization (NS-MILO) method, that is developed by Shahabsafa et al. (2018), to solve discrete MSTSO problems, and through extensive computational results demonstrate that the extended NS-MILO algorithm is able to solve large-scale MSTSO problems.

The article is organized as follows. In Section 2, we formally introduce the multi-scenario continuous and discrete truss sizing optimization problem. In Section 3, we present two important properties of the feasible set of multi-scenario truss sizing optimization problems...

2 Problem description

In this article, we are considering TSO problems with multiple load scenarios. We reformulate MSTSO problems by extending the TSO model introduced by Shahabsafa et al. (2018).

Let m and n denote the number of bars and degrees of freedom of the ground structure, respectively. The dimension of the space is denoted by d ($d = 2, 3$). We assume that each node is either fixed in all directions or pinned. Let $\mathcal{I} = \{1, \ldots, m\}$ be the index set of the bars of the ground structure and let $\mathcal{H} = \{1, \ldots, t\}$ represent the index set of the external force scenarios, where t denotes the number of the external force scenarios.

Vector $x \in \mathbb{R}^m_{++}$ denotes the cross-sectional areas of the bars, where $\mathbb{R}^m_{++}$ is the m-dimensional space of strictly positive vectors. Let $f^h \in \mathbb{R}^n$, for $h \in \mathcal{H}$, denote the vector of the (multi-load) external force in scenario h. Note that external forces are exerted on the nodes of the truss structure. Let $q^h \in \mathbb{R}^m$, for $h \in \mathcal{H}$, represent the vector of the internal forces of scenario h. The relationship between the external forces and internal forces are governed by the following equations:

$$Rq^h = f^h, \quad h \in \mathcal{H}.$$

The stress on bar $i \in \mathcal{I}$ in scenario $h \in \mathcal{H}$ is denoted by σ_i^h, and is defined as

$$\sigma_i^h = \frac{q_i^h}{x_i}.$$

Let $R \in \mathbb{R}^{n \times m}$ be the topology matrix of the truss structure. Let $u^h \in \mathbb{R}^n$ and $\Delta l^h \in \mathbb{R}^m$, for $h \in \mathcal{H}$, be the nodal displacements and the elongations of the bars in scenario h, respectively. The relationship between the nodal displacements and the elongations of the bars is as follows:

$$\Delta l^h = R^T u^h, \quad h \in \mathcal{H}. \tag{1}$$

Let E_i and l_i, for $i \in \mathcal{I}$, be the Young's modulus and the length of bar $i \in \mathcal{I}$, respectively. The Hooke's law governs the relationship between the stress and elongation of the bars; namely

$$\sigma_i^h = \frac{E_i}{l_i} \Delta l_i^h, \quad i \in \mathcal{I}, h \in \mathcal{H}. \tag{2}$$

We assume that the cross-sectional areas of the bars are circular. Let $\gamma_i = \pi E_i / 4l_i^2$ for all $i \in \mathcal{I}$. The Euler buckling constraints are written as

$$\sigma_i^h + \gamma_i x_i \geq 0, \quad i \in \mathcal{I}, h \in \mathcal{H}.$$

Let $\sigma^{\min}, \sigma^{\max} \in \mathbb{R}^m$ be the lower and upper bounds on the stress of the bars. Note that $\sigma^{\min} < 0 < \sigma^{\max}$. Let $x^{\min}$ and $x^{\max}$ be the lower and upper bounds on the bars' cross-sectional areas. Note that $u^{\min} < 0 < u^{\max}$. Also, let $u^{\min}$ and $u^{\max}$ denote the lower and upper bounds of the nodal displacements, respectively. Note that $u^{\min}, \sigma^{\max} \in \mathbb{R}^m$

Let ρ be the density of the material used to construct the truss structure. The objective is to minimize the total weight of the structure which is equal to $\rho\, l^T x$. The continuous MSTSO problem can be formulated as follows:

$$\begin{array}{llllll}
\min & \rho\, l^T x, & & & & \\
\text{s.t.} & R q^h & = & f^h, & & h \in \mathcal{H}, \\
& R^T u^h & = & \Delta l^h, & & h \in \mathcal{H}, \\
& \frac{E_i}{l_i} \Delta l_i^h - \sigma_i^h & = & 0, & & i \in \mathcal{I}, h \in \mathcal{H}, \\
& q_i^h - x_i \sigma_i^h & = & 0, & & i \in \mathcal{I}, h \in \mathcal{H}, \\
& \sigma_i^h + \gamma_i x_i & \geq & 0, & & i \in \mathcal{I}, h \in \mathcal{H}, \\
& u^{\min} & \leq & u^h & \leq u^{\max}, & h \in \mathcal{H}, \\
& \sigma^{\min} & \leq & \sigma^h & \leq \sigma^{\max}, & h \in \mathcal{H}, \\
& x^{\min} & \leq & x & \leq x^{\max}. &
\end{array} \tag{3}$$

In model (3), it is assumed that cross-sectional areas of the bars are continuous decision variables. However, in practice, bars take values only from a predefined discrete set because of manufacturing and economic restrictions. Without loss of generality, we may assume that the number of candidate sizes is the same for all the bars. Furthermore, for ease of presentation, we assume that all the bars have the same set $\mathcal{S}$ of potential cross-sectional areas. Let $\mathcal{S}$ be the set of possible non-zero cross-sectional areas of the bars defined as:

$$\mathcal{S} = \{s_1, s_2, \ldots, s_v\},$$

where $0 < s_1 < s_2 < \ldots < s_v$ and v is the cardinality of set $\mathcal{S}$. Let $\mathcal{K} = \{1, \ldots, v\}$ denote the set of indices corresponding to the discrete set $\mathcal{S}$. The cross-sectional area of bar i, for $i \in \mathcal{I}$, takes values from the set $\mathcal{S}$ in the discrete MSTSO problem. We extend the *incremental* model proposed by Shahabsafa et al. (2018, Section 2.2) for discrete single-scenario TSO problems, to discrete MSTSO problems. Let $\bar{\mathcal{K}} = \{1, \ldots, v-1\}$ and $\delta_k = s_{k+1} - s_k$ for $k \in \bar{\mathcal{K}}$. The incremental formulation of TSO problem with the discrete set of cross-sectional areas is as follows:

$$\begin{array}{ll}
x_i = \sum\limits_{k \in \bar{\mathcal{K}}} \delta_k z_{ik}, & i \in \mathcal{I}, \\
z_{ik} \leq z_{i,k-1}, & i \in \mathcal{I},\ k \in \bar{\mathcal{K}} \setminus \{1\}, \\
z_{ik} \in \{0, 1\}, & i \in \mathcal{I},\ k \in \bar{\mathcal{K}}.
\end{array} \tag{4}$$

In the incremental formulation (4), if $z_{i1} = 0$, then $x_i = s_1$. If $z_{ik} = 1$ and $z_{i,k+1} = 0$, then $x_i = s_{i,k+1}$. If $z_{iv} = 1$, then $x_i = s_v$.

Let σ_{ik}^h, for $i \in \mathcal{I}$, $k \in \mathcal{K}$ and $h \in \mathcal{H}$, be the stress on bar i in scenario h if $x_i = s_k$, that is

$$\sigma_{ik}^h = \begin{cases} \lambda_i \Delta l_i^h, & \text{if } x_i = s_k; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\sigma_i^h = \sum_{k\in\mathcal{K}} \sigma_{ik}^h$. The following set of constraints enforce yield stress and Euler buckling constraints:

$$\begin{aligned}
&\max\left(-\gamma_i s_1, \sigma_i^{\min}\right)(1-z_{i1}) &&\le \sigma_{i1}^h \le \sigma_i^{\max}(1-z_{i1}), && i\in\mathcal{I}, h\in\mathcal{H},\\
&\max\left(-\gamma_i s_k, \sigma_i^{\min}\right)\left(z_{i,k-1}-z_{ik}\right) &&\le \sigma_{ik}^h \le \sigma_i^{\max}\left(z_{i,k-1}-z_{ik}\right), && i\in\mathcal{I}, k\in\bar{\mathcal{K}}\setminus\{1\}, h\in\mathcal{H},\\
&\max\left(-\gamma_i s_v, \sigma_i^{\min}\right) z_{i,v-1} &&\le \sigma_{iv}^h \le \sigma_i^{\max} z_{i,v-1}, && i\in\mathcal{I}, h\in\mathcal{H}.
\end{aligned}$$

The discrete MSTSO problem can then be formulated as:

$$\begin{array}{llllll}
\min & \rho\, l^T x, & & & & \\
\text{s.t.} & R q^h & = & f^h, & & h\in\mathcal{H},\\
& R^T u^h & = & \Delta l^h, & & h\in\mathcal{H},\\
& x_i - s_1 - \sum_{k\in\bar{\mathcal{K}}} \delta_k z_{ik} & = & 0, & & i\in\mathcal{I},\\
& \frac{E_i}{l_i}\Delta l_i^h - \sum_{k\in\mathcal{K}} \sigma_{ik}^h & = & 0, & & i\in\mathcal{I}, h\in\mathcal{H},\\
& q_i^h - \sum_{k\in\mathcal{K}} s_k \sigma_{ik}^h & = & 0, & & i\in\mathcal{I}, h\in\mathcal{H},\\
& z_{ik} & \le & z_{i,k-1}, & & i\in\mathcal{I},\ k\in\bar{\mathcal{K}}\setminus\{1\},\\
& u^{\min} & \le & u^h & \le u^{\max}, & h\in\mathcal{H},\\
& \max\left(-\gamma_i s_1, \sigma_i^{\min}\right)(1-z_{i1}) & \le & \sigma_{i1}^h & \le \sigma_i^{\max}(1-z_{i1}), & i\in\mathcal{I}, h\in\mathcal{H},\\
& \max\left(-\gamma_i s_k, \sigma_i^{\min}\right)\left(z_{i,k-1}-z_{ik}\right) & \le & \sigma_{ik}^h & \le \sigma_i^{\max}\left(z_{i,k-1}-z_{ik}\right), & i\in\mathcal{I}, k\in\bar{\mathcal{K}}\setminus\{1\}, h\in\mathcal{H},\\
& \max\left(-\gamma_i s_v, \sigma_i^{\min}\right) z_{i,v-1} & \le & \sigma_{iv}^h & \le \sigma_i^{\max} z_{i,v-1}, & i\in\mathcal{I}, h\in\mathcal{H},\\
& z_{ik}\in\{0,1\}, & & & & i\in\mathcal{I},\ k\in\bar{\mathcal{K}}.
\end{array} \tag{5}$$

In Section 3, we present two important characteristics of the feasible set of the continuous MSTSO problem (3), one of which holds for the discrete MSTSO (5) as well.

3 Theoretical properties of MSTSO problems

In this section, we present two important characteristics of the feasible set of the MSTSO problems. These properties are then used in extending the NS-MILO algorithm (Shahabsafa et al., 2018) to solve discrete MSTSO problems 4.

Let $\mathcal{F}$ be the feasible set of the continuous MSTSO problem (3) and let $(x, u, \Delta l, \sigma, q) \in \mathcal{F}$ be a feasible solution for problem (3) where

$$\begin{aligned}
u &:= (u^1, \ldots, u^t), & u^h &\in \mathbb{R}^n, & h&\in\mathcal{H},\\
\Delta l &:= (\Delta l^1, \ldots, \Delta l^t), & \Delta l^h &\in \mathbb{R}^m, & h&\in\mathcal{H},\\
\sigma &:= (\sigma^1, \ldots, \sigma^t), & \sigma^h &\in \mathbb{R}^m, & h&\in\mathcal{H},\\
q &:= (q^1, \ldots, q^t), & q^h &\in \mathbb{R}^m, & h&\in\mathcal{H}.
\end{aligned} \tag{6}$$

Let $\mathcal{X}$ denote the m-dimensional subspace of the bars' cross-sectional areas. Additionally, let $\mathcal{F}_\mathcal{X}$ denote the orthogonal projection of the feasible set $\mathcal{F}$ on subspace $\mathcal{X}$, i.e., $\mathcal{F}_\mathcal{X} = \{x \in \mathbb{R}^m_{++} : \exists\, u, \Delta l, \sigma, q \text{ s.t. } (x, u, \Delta l, \sigma, q) \in \mathcal{F}\}$. The truss structure corresponding to cross-sectional area $x \in \mathbb{R}^m$ is referred as *structure* x.

3.1 Feasibility along rays

In this section, we prove that if structure x is feasible for the continuous MSTSO problem (3) and $\alpha > 1$, then ray αx is feasible for problem (3), if αx is within the bounds of the cross-sectional areas of the feasible set.

Theorem 1 *Let $x \in \mathcal{F}_\mathcal{X}$ and $\alpha > 1$. If $\alpha x \leq x^{\max}$, then $\alpha x \in \mathcal{F}_\mathcal{X}$.*

Proof As $x \in \mathcal{F}_\mathcal{X}$, there exists $(u, \Delta l, \sigma, q)$ such that $(x, u, \Delta l, \sigma, q) \in \mathcal{F}$, as defined in (6). Let $\bar{x} = \alpha x$, and let $(\bar{u}, \overline{\Delta l}, \bar{\sigma}, \bar{q})$ be defined as follows:

$$\begin{aligned} \bar{u}^h &= \frac{1}{\alpha}\, u^h, && \forall h \in \mathcal{H}, \\ \overline{\Delta l}^h &= \frac{1}{\alpha}\, \Delta l^h, && \forall h \in \mathcal{H}, \\ \bar{\sigma}^h &= \frac{1}{\alpha}\, \sigma^h, && \forall h \in \mathcal{H}, \\ \bar{q}^h &= q^h, && \forall h \in \mathcal{H}. \end{aligned} \tag{7}$$

From (1), (2) and (7), we have

$$\begin{aligned} \overline{\Delta l}^h &= \frac{1}{\alpha} R^T u^h = R^T \bar{u}^h, && h \in \mathcal{H}, \\ \bar{\sigma}^h_i &= \frac{1}{\alpha}\frac{E_i}{l_i} {\Delta l_i}^h = \frac{E_i}{l_i}\overline{\Delta l}^h_i, && i \in \mathcal{I}, h \in \mathcal{H}. \end{aligned}$$

If $\sigma^h_i \geq 0$ for $h \in \mathcal{H}$, then we have $\bar{\sigma}^h_i + \gamma_i \bar{x}_i \geq 0$. Also, if $\sigma^h_i < 0, h \in \mathcal{H}$, then we have

$$\bar{\sigma}^h_i + \gamma_i \bar{x}_i = \frac{1}{\alpha}\sigma^h_i + \gamma_i \alpha x_i \geq \sigma^h_i + \gamma_i x_i \geq 0$$

Furthermore, $x^{\min} \leq x \leq \alpha x = \bar{x} \leq x^{\max}$, where the last inequality holds by the assumption. Note that if $u^h \geq 0, h \in \mathcal{H}$, then

$$u^{\min} \leq 0 \leq \frac{u^h}{\alpha} = \bar{u}^h \leq u^h \leq u^{\max}$$

Also, if $u^h < 0, h \in \mathcal{H}$, then

$$u^{\min} \leq u^h \leq \frac{u^h}{\alpha} = \bar{u}^h \leq 0 \leq u^{\max}$$

Similarly, one can show $\sigma^{\min} \leq \bar{\sigma}^h \leq \sigma^{\max}$. Thus, we conclude that $(\bar{x}, \bar{u}, \overline{\Delta l}, \bar{\sigma}, \bar{q})$ is feasible for problem (3). □

Note that Theorem 1 does not hold for the discrete MSTSO problem 5, since the feasible set of problem (5) is discrete and finite.

3.2 Convex hull of the external force scenarios

Convex combinations of external forces are studied in the shakedown analysis and optimal shakedown design of elsto-plastic trusses under multi-parameter static loading, see, e.g., Giambanco and Palizzolo (1995), Kaliszky and Lógó (2002), and Atkočiūnas et al. (2008). However, to the best of our knowledge, this concept is not studied in the optimal design of the trusses that are limited to have elastic behavior while considering force balance equations, Hooke's law, Euler buckling constraints, yield stress, and displacement bounds.

We prove that a truss structure that withstands a set of external force scenarios, also withstands a force scenario that can be written as a convex combination of the existing external force scenarios. To do so, we prove that if we have a multi-scenario truss design problem and add a scenario which is a convex combination of the current external forces, then the feasible set of the problem does not change. Below, $f \coloneqq (f^1, f^2, \ldots, f^t)$.

Theorem 2 *Consider MSTSO problem (3). Let f^{t+1} be a convex combination of the external force scenarios f, and $\tilde{\mathcal{F}}$ be the feasible set of (3) when the external force f^{t+1} added as a new scenario i.e. $f \to \tilde{f} \coloneqq (f^1, \ldots, f^t, f^{t+1})$ and $\mathcal{H} \to \tilde{\mathcal{H}} \coloneqq \mathcal{H} \cup \{t+1\}$. Then, $\tilde{\mathcal{F}}_{\mathcal{X}} = \mathcal{F}_{\mathcal{X}}$.*

Proof Clearly $\tilde{\mathcal{F}}_{\mathcal{X}} \subseteq \mathcal{F}_{\mathcal{X}}$, since $\tilde{\mathcal{H}} \supset \mathcal{H}$. Thus, we need to prove that $\mathcal{F}_{\mathcal{X}} \subseteq \tilde{\mathcal{F}}_{\mathcal{X}}$. If $\mathcal{F} = \emptyset$, then $\tilde{\mathcal{F}} = \emptyset$, and the theorem holds. Assume that $\tilde{\mathcal{F}} \neq \emptyset$. We have assumed $\sum_{h \in \mathcal{H}} \lambda_h f^h = f^{t+1}$, for some $\lambda_h \geq 0, h \in \mathcal{H}$ such that $\sum_{h \in \mathcal{H}} \lambda_h = 1$. Let $x \in \mathcal{F}_{\mathcal{X}}$ be given. Then, there exists $(u, \Delta l, \sigma, q)$ such that $(x, u, \Delta l, \sigma, q) \in \mathcal{F}$. Let $(\tilde{u}, \tilde{\Delta l}, \tilde{q}, \tilde{\sigma})$ be defined as

$$\begin{aligned} \tilde{q} &= \sum_{h \in \mathcal{H}} \lambda_h q^h, \\ \tilde{u} &= \sum_{h \in \mathcal{H}} \lambda_h u^h, \\ \tilde{\sigma} &= \sum_{h \in \mathcal{H}} \lambda_h \sigma^h. \end{aligned} \tag{8}$$

Then we have that

$$\begin{aligned} R\tilde{q} &= R\left(\sum_{i \in S} \lambda_h q^h\right), \\ &= \sum_{h \in \mathcal{H}} \lambda_h \left(Rq^h\right), \\ &= \sum_{h \in \mathcal{H}} \lambda_h f^h, \\ &= \tilde{f}. \end{aligned} \tag{9}$$

Similarly, all the linear constraints in (3) are satisfied. Also notice that the non-linear constraints $q_i^{t+1} - x_i \sigma_i^{t+1} = 0$ holds as for fixed x, the constraint becomes linear. Thus, we can conclude that $(x, u^{t+1}, q^{t+1}, \sigma^{t+1})$, as defined in (8), satisfies all the constraints of problems (3) for the external force scenario f^{t+1}. □

Remark 1 Theorem 2 holds for the discrete MSTSO problem (5).

From Theorem 2 and Remark 1, we conclude that adding a new external force which is a convex combination of the current external forces does not change the feasible set of cross-sectional areas of problems (3) and (5) (proof is for $\mathcal{F}_{\mathcal{X}}$), and thus, the set of optimal cross-sectional areas and the objective values of (3) and (5) will not change their optimal solutions. We can use this result to reduce the number of scenarios in discrete and continuous MSTSO problems by eliminating the scenarios that are in the convex hull of the rest of the scenarios.

To find out whether an external force scenario f^{t+1} is a convex combination of other external force scenarios f one solves the following linear feasibility problem:

$$\begin{aligned} & f^{t+1} = \sum_{h \in \mathcal{H}} \lambda_h f^h, \\ & \sum_{h \in \mathcal{H}} \lambda_h = 1, \\ & 0 \leq \lambda_h \leq 1, h \in \mathcal{H}. \end{aligned} \tag{10}$$

If (10) has a solution, then external force f^{t+1} is a convex combination of other external force scenarios f and can be eliminated from the set of external force scenarios without impacting the problem's optimal structures. To identify all the external force scenarios that can be eliminated, one starts by solving the feasibility problem (10) for the external force scenario f^1 and eliminate f^1 if the corresponding feasibility problem has a solution, and iteratively do the same for all the other scenarios.

4 Computational experiments for multi-scenario TSO problems

In this section, we present our computational experiments for MSTSO problems. In Section 4.1, we demonstrate how adding external force scenarios adversely impacts the solution time. In Section 4.2 we extend the NS-MILO algorithm, that was developed by Shahabsafa et al. (2018), to solve large-scale multi-scenario truss sizing optimization problems, and then in Section 4.3, we demonstrate through extensive computational experiments that the extended NS-MILO algorithm is able to solve large-scale MSTSO problems.

4.1 The effect of adding external force scenarios

4.2 Extension of the NS-MILO algorithm

The NS-MILO algorithm is a neighborhood-search based algorithm, proposed by Shahabsafa et al. (2018), to solve large-scale truss sizing optimization problems. In the NS-MILO algorithm, MILO_k subproblems ($k = 2, 3, 5$) are iteratively solved, where in a MILO_k subproblem, the size of the discrete set for each bar is at most equal to $k \leq v$. The NS-MILO algorithm starts by solving the continuous MSTSO problem and using Theorem 1 to generates a MILO_2 subproblem based on the solution of the continuous optimization problem (see Algorithm 1 for details). Then, it continues iteratively solving MILO_2 subproblems until an initial integer feasible solution is found. After that, it solves a series of MILO_3 subproblems in a given time budget, and ultimately, a series of MILO_5 subproblems are solved. For additional details of NS-MILO algorithm we refer the reader to Shahabsafa et al. (2018).

In order to use the NS-MILO approach for MSTSO problems, we make the following modifications:

(i) The time budgets of the MILO_2 subproblems are equal to tm seconds, where t and m denote the number of scenarios and the number of bars, respectively. Additionally, the time budgets of other MILO_k subproblems are equal to $\beta dkm(1 + t^2)$ seconds, where d is the dimension of the truss, and β is the coefficient representing the complexity of the structure. We use $\beta = 1$ for wing structures and $\beta = 0.5$ for the cantilever structures presented next.

(ii) Let (P_m) be the MSTSO problem (3), except that we have set $x_i^{\min} = s_4$, i.e., the lower bounds of the cross-sectional areas are set to the 4^{th} value of the discrete set. We assumed that $v \geq 4$. We have increased $x_i^{\min}$ so as to move away from the boundaries of the feasible set of the continuous problem and help MILO_2 subproblems to find an initial integer feasible solution.

The NS-MILO algorithm for the MSTSO problems is presented in Algorithm 1.

Algorithm 1 The NS-MILO algorithm for MSTSO problems

$\boldsymbol{x}^{\mathbf{0}} :=$ Solve $(\bar{P})$
$\alpha := 1$
repeat
  $(\hat{\boldsymbol{x}}, \hat{\eta}) :=$ Solve($\text{MILO}_2(\boldsymbol{\alpha x^0})$)
  $\alpha := \alpha + 0.1$
until $\text{MILO}_2(\boldsymbol{\alpha x_0})$ is feasible
repeat
  $\eta_{\text{curr}} := \hat{\eta}$
  $(\hat{\boldsymbol{x}}, \hat{\eta}) :=$ FindSol($\text{MILO}_3(\hat{\boldsymbol{x}})$)
until $\hat{\eta} = \eta_{\text{curr}}$
repeat
  $\eta_{\text{curr}} := \hat{\eta}$
  $(\hat{\boldsymbol{x}}, \hat{\eta}) :=$ FindSol($\text{MILO}_5(\hat{\boldsymbol{x}})$)
until $\hat{\eta} = \eta_{\text{curr}}$
return $\hat{\boldsymbol{x}}$

4.3 Computational experiments of the NS-MILO algorithm

In Section 4.3.1, we present the results on the well-known 72-bar truss and compare the solution obtained by the NS-MILO approach with that of other methods used in the literature to solve the problem. Additionally, to demonstrate how the NS-MILO approach scales as the size of a MSTSO problem grows, we present computational results for the multi-scenario cantilever trusses and wing trusses in Sections 4.3.2 and 4.3.3, respectively.

4.3.1 72-bar truss

A solution to the 72-bar instance that satisfies Euler buckling constraints has not been reported in the literature. Thus to benchmark the NS-MILO approach for this instance, we show in Table 1 the results of the 72-bar instance without Euler buckling constraints. As we can see, the weight of the solution of the continuous model (**??**) matches that of the solution by Haftka and Gürdal (2012) with 0.1% precision. Additionally, the solution of the discrete model obtained by the NS-MILO approach in 0.29 hour is the same as the ones by **?**, Sadollah et al. (2015), and Ho-Huu et al. (2016). Note that we let the full-MILO approach run for 120 hrs for the 72-bar instance without Euler buckling constraints, and the optimality gap is still 32.61% when the full-MILO approach stops.

Table 1: Solution (in^2) and the weights (lbm) for the 72-bar truss problem without Euler buckling constraints.

	Continuous		Discrete							
Vars.	Haftka and Gürdal (2012)	Model (**??**)	Wu and Chow (1995)	Kaveh and Talatahari (2009)	Sadollah et al. (**?**)	Kaveh and Ghazan (**?**)	Sadollah et al. (2015)	Ho-Huu et al. (2016)	Full-MILO	NS-MILO
1-4	0.157	0.156	0.196	0.196	1.800	0.196	0.196	0.196	0.196	0.196
5-12	0.536	0.546	0.602	0.563	0.602	0.563	0.563	0.563	0.563	0.563
13-16	0.410	0.410	0.307	0.442	0.111	0.391	0.391	0.391	0.442	0.391
17-18	0.569	0.570	0.766	0.563	0.111	0.563	0.563	0.563	0.602	0.563
19-22	0.507	0.524	0.391	0.563	1.266	0.563	0.563	0.563	0.785	0.563
23-30	0.520	0.517	0.391	0.563	0.563	0.563	0.563	0.563	0.563	0.563
31-34	0.100	0.100	0.141	0.111	0.111	0.111	0.111	0.111	0.111	0.111
35-36	0.100	0.100	0.111	0.250	0.111	0.111	0.111	0.111	0.111	0.111
37-40	1.280	1.268	1.800	1.228	0.442	1.228	1.228	1.228	1.000	1.228
41-48	0.515	0.512	0.602	0.563	0.442	0.442	0.442	0.563	0.563	0.563
49-52	0.100	0.100	0.141	0.111	0.111	0.111	0.111	0.111	0.111	0.111
53-54	0.100	0.100	0.307	0.111	0.111	0.111	0.111	0.111	0.111	0.111
55-58	1.897	1.886	1.563	1.800	0.196	1.990	1.990	1.990	1.990	1.990
59-66	0.516	0.512	0.766	0.442	0.563	0.563	0.563	0.442	0.442	0.442
67-70	0.100	0.100	0.141	0.141	0.442	0.111	0.111	0.111	0.111	0.111
71-72	0.100	0.100	0.111	0.111	0.602	0.111	0.111	0.111	0.111	0.111
W (lbm)	379.66	379.61	427.20	393.38	390.73	389.33	389.33	389.33	392.96	389.33

Furthermore, the solution provided by the NS-MILO approach for the 72-bar truss with Euler buckling constraints is presented in Table 2. As we can see, the solution provided by the NS-MILO approach is better than that of the full-MILO approach. Note that the NS-MILO approach provided the solution in 27 s, while the full-MILO approach produced the solution in 24 hrs.

4.3.2 Multi-scenario 2D and 3D cantilever trusses

Results of the 2D cantilever trusses with two scenarios are presented in Table 3. As we can see, the solutions obtained by the NS-MILO approach are equal to the ones obtained by the full-MILO approach for the instances with 20 and 60 bars. In all the trusses with more than 60 bars, the solution provided by the NS-MILO approach is better than the one provided by the full-MILO approach. For instance, the solution provided by the NS-MILO approach for the 2D cantilever truss with 300 bars is %15 lighter than the solution obtained by the full-MILO approach. Furthermore, the NS-MILO approach was able to get the solutions significantly faster than the full-MILO approach in all the 2D cantilever trusses.

Table 2: Cross-sectional areas (in^2) and the weights (lbm) for the 72-bar truss problem solutions with Euler buckling constraints.

Vars.	Continuous	Discrete	
		Full-MILO	NS-MILO
1–4	1.470	1.457	1.457
5–12	2.283	2.380	2.380
13–16	1.649	1.990	1.620
17–18	2.774	2.630	2.880
19–22	1.498	1.563	1.563
23–30	1.776	1.800	1.800
31–34	0.100	0.196	0.196
35–36	0.330	0.785	0.442
37–40	1.518	1.563	1.620
41–48	1.933	1.990	1.990
49–52	0.482	0.391	0.442
53–54	0.825	0.602	0.766
55–58	2.084	1.990	1.800
59–66	1.906	1.990	1.990
67–70	0.321	0.391	0.442
71–72	0.100	0.111	0.111
W	1264.750	1316.148	1302.500

Table 3: Weights (kg) and the solution times (s) for the two-scenario 2D cantilever instances using the NS-MILO approach.

n_b	m	t	Full-MILO		NS-MILO			w_f/w_n
			w_f	t_f	n_s	w_n	t_n	
4	20	2	10.31	281.31	9	10.31	3.66	1.00
12	60	2	37.15	86,400.03	9	37.15	1,553.20	1.00
20	100	2	74.20	86,400.07	14	73.95	6,221.25	1.00
28	140	2	119.08	86,400.02	23	119.01	7,309.91	1.00
36	180	2	259.68	86,400.08	10	191.80	7,755.97	1.35
44	220	2	321.42	86,400.11	16	288.22	13,616.79	1.12
52	260	2	518.21	86,400.13	22	397.79	13,684.09	1.30
60	300	2	558.01	86,400.02	19	471.61	23,177.21	1.18

Results of the 3D cantilever trusses with two and three scenarios are presented in Table 4. As we can see, the solution provided by the NS-MILO approach for the 3D cantilever truss with 20 bars is the same as th one provided by the full-MILO approach. In all the 3D cantilever trusses with more than 20 bars, the solution obtained by the NS-MILO approach is better than the one provided by the full-MILO approach, and the gap between the weight of the solution provided by the NS-MILO and full-MILO approaches increases rapidly as the size of the problem grows. For instance, the weights of the solutions obtained by the NS-MILO approach for the two-scenario and three-scenario 3D cantilever truss with 300 bars are only %15.8 and %16.3 of the one generated by the full-MILO approach, respectively. In other words, the solution obtained by the full-MILO approach is more than six times heavier than the solution obtained by the NS-MILO approach for the 3D cantilever truss with 300 bars. Except for the three-scenario cantilever with 220 bars, the NS-MILO approach required significantly less time than the 24 hrs allocated to the full-MILO approach.

Table 4: Weights (kg) and the solution times (s) for the multi-scenario 3D cantilever instances using the NS-MILO approach.

n_b	m	t	Full-MILO		NS-MILO			w_f/w_n
			w_f	t_f	n_s	w_n	t_n	
1	20	2	13.47	86400.01	15	13.47	41.33	1.00
		3	14.28	86400.00	13	14.28	1317.91	1.00
3	60	2	41.07	86400.01	13	33.62	5424.24	1.22
		3	39.63	86400.01	8	37.62	7521.81	1.05
5	100	2	73.77	86400.05	13	55.74	5623.09	1.32
		3	105.42	86400.02	18	62.38	17388.69	1.69
7	140	2	199.50	86400.03	23	100.73	16165.49	1.98
		3	202.93	86400.02	24	108.72	37528.54	1.87
9	180	2	345.11	86400.02	16	139.78	19032.60	2.47
		3	375.75	86400.04	20	162.43	47978.13	2.31
11	220	2	549.47	86400.06	11	179.00	20693.89	3.07
		3	701.82	86400.05	20	192.46	121163.95	3.65
13	260	2	714.18	86400.06	9	225.08	17297.92	3.17
		3	978.16	86400.06	11	243.02	48180.25	4.03
15	300	2	1847.84	86400.01	10	293.33	26139.16	6.30
		3	1970.40	86400.08	11	322.91	48251.02	6.10

4.3.3 Multi-scenario wing trusses

Results of the 2-scenario and 3-scenario wing trusses are presented in Table 5. As we can see, the solutions provided by the NS-MILO approach are significantly better than the ones obtained by the full-MILO approach, and the difference between the weights of the solutions provided by the NS-MILO and full-MILO approaches increases as the size of the wing truss increases. For the 2-scenario and 3-scenario wing truss with 315 bars, solutions provided by the full-MILO approach are 9.61 and 8.91 times heavier than the ones obtained by the NS-MILO approach. The primary reason that full-MILO fails in solving large-scale multi-scenario wing trusses is that solving the continuous relaxation problems at the nodes of the branch and bound tree takes excessive time.

Observe that, within the 24-hour time budget, the number of nodes explored by Gurobi in the B&B algorithm in the full-MILO approach decreases rapidly as the size of the problem grows and as the number of the scenarios increases. For instance, for the 2-scenario wing truss with 315 bars, Gurobi, which is set to use 10 threads, has explored only 288 nodes, implying that on average the continuous relaxation at each node is solved in 3,000 s. For the 3-scenario wing trusses with 279 and 315 bars, Gurobi was not able to solve the continuous relaxation at the root node in 24 hrs. The solutions reported by Gurobi for these two problems are obtained by the heuristics that the solver utilizes to generate initial integer feasible solutions.

Table 5: Weights (kg) and the solution times (s) for the multi-scenario wing trusses using the NS-MILO approach.

m	t	Full-MILO			NS-MILO			w_f/w_n
		B&B nodes	w_f	t_f	n_s	w_n	t_n	
81	2	213,902	24,953.10	86,400.01	74	18,730.90	17,120.41	1.33
	3	22,952	45,989.98	86,400.04	70	19,334.45	21,479.01	2.38
117	2	39,808	47,131.95	86,400.05	46	14,246.51	36,734.13	3.30
	3	600	81,981.44	86,400.02	44	14,354.08	50,541.50	5.71
153	2	29,339	43,995.13	86,400.07	19	12,104.99	23,940.06	3.63
	3	20	74,600.89	86,400.32	41	12,306.48	83,625.31	6.06
207	2	6,510	62,157.36	86,400.06	31	10,399.04	47,733.30	5.98
	3	0	67,885.71	86,400.04	39	10,466.81	108,493.33	6.49
243	2	1,009	72,658.38	86,400.06	26	10,228.69	86,057.75	7.10
	3	15	72,658.38	86,400.07	42	10,033.00	180,992.04	7.24
279	2	181	86,170.94	86,400.03	22	9,661.80	67,236.63	8.92
	3	0	86,170.94	86,400.28	26	9,984.50	191,646.64	8.63
315	2	288	91,718.83	86,400.11	16	9,545.63	82,027.25	9.61
	3	0	91,718.83	86,400.17	31	9,742.68	466,123.89	9.41

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