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On parametric second-order conic optimization

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This paper is dedicated to Marco Lopez on the occasion of his 70th birthday.

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Abstract In this paper, using an optimal partition approach, we study the parametric analysis of a second-order conic optimization problem, where the objective function is perturbed along a fixed direction. We introduce the notions of nonlinearity interval and transition point of the optimal partition, and we prove that the set of transition points is finite. Additionally, on the basis of Painlevé-Kuratowski set convergence, we provide sufficient conditions for the existence of a nonlinearity interval, and we show that the continuity of the primal or dual optimal set mapping might fail on a nonlinearity interval. We then propose, under the strict complementarity condition, an iterative procedure to compute a nonlinearity interval of the optimal partition. Furthermore, under primal and dual nondegeneracy conditions, we show that a transition point can be numerically identified from the higher-order derivatives of the Lagrange multipliers associated with a nonlinear reformulation of the parametric second-order conic optimization problem. Our theoretical results are supported by numerical experiments.

Keywords Parametric second-order conic optimization · Optimal partition · Nonlinearity interval · Transition point

Mathematics Subject Classification (2010) 90C31 · 90C22 · 90C51

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1 Introduction

In this paper, we investigate the *optimal partition* approach for the parametric analysis of second-order conic optimization (SOCO) problems. Let $\mathcal{L}_+^{\bar{n}} := \mathbb{L}_+^{n_1} \times \mathbb{L}_+^{n_2} \times \ldots \times \mathbb{L}_+^{n_p}$ be the Cartesian product of p second-order cones [2], where $\bar{n} = \sum_{i=1}^p n_i$,

$$\mathbb{L}_+^{n_i} := \Big\{ x^i := (x_1^i, \ldots, x_{n_i}^i)^T \in \mathbb{R}^{n_i} \mid x_1^i \geq \|x_{2:n_i}^i\| \Big\},$$

and $\|.\|$ denotes the ℓ_2 norm. Parametric primal-dual SOCO problems are defined as

$$\begin{aligned}
&(\mathrm{P}_\epsilon) \qquad \inf_x \Big\{ (c + \epsilon \bar{c})^T x \mid Ax = b,\ x \in \mathcal{L}_+^{\bar{n}} \Big\},\\
&(\mathrm{D}_\epsilon) \qquad \sup_{(y;s)} \Big\{ b^T y \mid A^T y + s = c + \epsilon \bar{c},\ s \in \mathcal{L}_+^{\bar{n}} \Big\},
\end{aligned}$$

in which $x := (x^1; \ldots; x^p)$, $s := (s^1; \ldots; s^p)$, $b \in \mathbb{R}^m$, $A := (A^1, \ldots, A^p) \in \mathbb{R}^{m \times \bar{n}}$, where $A^i \in \mathbb{R}^{m \times n_i}$, $c := (c^1; \ldots; c^p) \in \mathbb{R}^{\bar{n}}$ and $\bar{c} := (\bar{c}^1; \ldots; \bar{c}^p) \in \mathbb{R}^{\bar{n}}$, where $c^i, \bar{c}^i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, p$, and $(.\,; \ldots; .)$ denotes the concatenation of the column vectors. Note that $\epsilon \in \mathbb{R}$ is the perturbation parameter, and $\bar{c}$ is a fixed direction.

The optimal value of (P_ϵ) is denoted by $\psi(\epsilon) : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$, and it is called the *optimal value function*[1]. Let $\mathcal{E} \subseteq \mathbb{R}$ be the domain of the optimal value function, i.e., the set of all ϵ such that $\psi(\epsilon) > -\infty$. The interior point condition guarantees that $\mathcal{E}$ is nonempty and non-singleton.

Assumption 1 *The interior point condition holds for both* (P_ϵ) *and* (D_ϵ) *at* $\epsilon = 0$*, i.e., there exists a feasible solution* $\big(x^\circ(0); y^\circ(0); s^\circ(0)\big)$ *such that*

$$x^{\circ i}(0), s^{\circ i}(0) \in \mathrm{int}(\mathbb{L}_+^{n_i}), \qquad i = 1, \ldots, p,$$

where $\mathrm{int}(\mathbb{L}_+^{n_i}) := \Big\{ x^i \in \mathbb{R}^{n_i} \mid x_1^i > \|x_{2:n_i}^i\| \Big\}$.

Assumption 1 implies the existence of an interior solution $\big(x^\circ(\epsilon); y^\circ(\epsilon); s^\circ(\epsilon)\big)$ at every $\epsilon \in \mathrm{int}(\mathcal{E})$, see [17, Lemma 3.1], where int(.) denotes the interior of a convex set. Furthermore, under Assumption 1, $\psi(\epsilon)$ is proper and concave [8, Lemma 2.2] and continuous [11, Corollary 2.109] on $\mathcal{E}$, and thus $\mathcal{E}$ is a closed, possibly unbounded, interval [8, Lemma 2.2].

Given a fixed ϵ, it is well-known that primal-dual interior point methods (IPMs) [28] can efficiently solve (P_ϵ) and (D_ϵ) in polynomial time. We refer the reader to [2] and the references cited therein for a review of IPMs for SOCO, and to [4] for implementation of IPMs for SOCO.

[1] In this context, ∞ simply means that the primal problem is infeasible.

1.1 Related works

Sensitivity and stability analysis has been extensively studied for nonlinear optimization problems. Classical results about semicontinuity of the optimal set and the optimal value function date back to 1960's using the set-valued mapping theory [7,23]. Zlobec et al. [6,39] identified the region of stability for perturbed convex optimization problems. The sensitivity of KKT solutions was studied by Fiacco [15] and Fiacco and McCormick [14] under linear independence constraint qualification, second-order sufficient condition, and strict complementarity condition. Robinson [31] released the strict complementarity condition but imposed a stronger second-order sufficient condition. Sensitivity analysis of nonlinear semidefinite optimization (SDO) and nonlinear SOCO problems has been widely studied in the past twenty years [10,11,35]. Under Slater and nondegeneracy conditions, by applying the Implicit Function Theorem [12], Shapiro [35] established the differentiability of the optimal solution for a nonlinear SDO problem. Bonnans and Ramírez [10] characterized strongly regular KKT solutions for nonlinear SOCO problems. Directional stability and Lipschitz continuity of optimal solutions have been extensively studied in [11] for nonlinear optimization problems in abstract setting. We refer the reader to [16] for a survey of classical results.

Sensitivity analysis based on the optimal partition is well-understood for linear optimization (LO) [1,21,24] and linearly constrained quadratic optimization (LCQO) problems [8]. The optimal partition approach was initially studied by Adler and Monteiro [1], Jansen et al. [24], and Greenberg [21] for LO. The approach was extended for LCQO problems by Berkelaar et al. [8]. For LO and LCQO, the domain $\mathcal{E}$ is partitioned into so-called invariancy sets. An invariancy set is a subset of $\mathcal{E}$, either a singleton or an open subinterval, on which the optimal partition is constant w.r.t. ϵ. For (P_ϵ) and (D_ϵ), Goldfarb and Scheinberg [17] investigated the differentiability of $\psi(\epsilon)$ and provided auxiliary problems to compute the boundary points of an invariancy set. Yildirim [38] extended the approach in [17] for linear conic optimization problems. Unlike the classical sensitivity analysis results for nonlinear optimization problems [11], which hinge on strong second-order sufficient conditions, and are mostly exploring a small neighborhood of a locally optimal solution, the optimal partition approach fully describes the behavior of the optimal set mapping and the optimal value function, by using the optimal partition of the problem on the domain $\mathcal{E}$. This is remarkable, since the concept of optimal partition is uniquely defined for any instance of linear conic optimization with strong duality, regardless of strict complementarity and nondegeneracy conditions, see [10,27,26,37]. Recently, Mohammad-Nezhad and Terlaky [25] introduced the concepts of a nonlinearity interval and transition point for the optimal partition of parametric SDO problems. Subsequently, Hauenstein et al. [22] proposed a numerical procedure to partition $\mathrm{int}(\mathcal{E})$ into the finite union of invariancy intervals, nonlinearity intervals, and transition points.

1.2 Contribution

To date, only a few studies have been devoted to parametric analysis of linear conic optimization problems. In particular, the optimal partition and parametric analysis of SOCO problems have not been fully investigated in the literature. Motivated by the study of parametric SDO problem in [17,22,25] and the identification of the optimal partition in [37], we study the optimal partition approach for (P_ϵ) and (D_ϵ) and highlight its similarities and differences to/from a parametric LO problem. We introduce the concepts of nonlinearity interval and transition point for the optimal partition of (P_ϵ) and (D_ϵ). A nonlinearity interval is a non-singleton maximal subinterval of $\mathrm{int}(\mathcal{E})$ on which the optimal partitions are identical, but both the primal and dual optimal sets change with ϵ. Consequently, $\psi(\epsilon)$ is nonlinear on a nonlinearity interval. A transition point is a singleton invariancy set which does not belong to a nonlinearity interval. Our ultimate goal is to elaborate on the optimal partition approach in [17,25] and develop sensitivity analysis methodologies for SOCO problems.

Roughly speaking, our main contributions are

- Characterization of nonlinearity intervals and transition points for the optimal partition of parametric SOCO;
- A numerical procedure for the computation of a nonlinearity interval;
- Sufficient conditions for the identification of a transition point.

More specifically, using the algebraic definition of a transition point, we prove that the set of transition points is finite, see Lemma 3. Furthermore, we provide sufficient conditions for the existence of a nonlinearity interval on the basis of Painlevé-Kuratowski set convergence, see Lemma 4, and we show that continuity might fail on a nonlinearity interval, see problem (10). Under the existence of a strictly complementarity solution at a given $\bar{\epsilon}$, we formulate nonlinear auxiliary problems to compute a subinterval of a nonlinearity interval. We then use the auxiliary problems alongside a procedure from real algebraic geometry to compute the boundary points of the nonlinearity interval surrounding $\bar{\epsilon}$, see Algorithm 1. Finally, under primal and dual nondegeneracy conditions, we show that the derivative information from a nonlinear reformulation of (D_ϵ) can be invoked to identify a transition point of the optimal partition, see Theorem 4.

1.3 Organization of the paper

The rest of this paper is organized as follows. Preliminaries are provided in Section 2. In Sections 2.1 and 2.2, we provide background information about the optimal partition and nondegeneracy conditions; In Section 2.3, we review set-valued analysis and the continuity of the feasible set and optimal set mappings for (P_ϵ) and (D_ϵ). In Section 3, we investigate the sensitivity of the optimal partition and optimal solutions w.r.t. ϵ. In Section 3.1, we formally define the concepts of nonlinearity interval and transition point of the optimal partition; In Section 3.2, we provide sufficient conditions for the existence of a nonlinearity interval. Furthermore, under strict complementarity condition, we present a numerical procedure for the computation of a nonlinearity interval; In Section 3.3, under primal and dual nondegeneracy conditions, we show how to identify a transition point using higher-order derivatives of the Lagrange multipliers from a nonlinear reformulation of (D_ϵ). In Section 4, we provide numerical results to demonstrate the convergence of the solutions generated by the numerical procedure and the magnitude of the derivatives. Our concluding remarks and directions for future research are summarized in Section 5.

Notation We adopt the notation in accordance with [27], where $\mathbb{R}_+x$ is defined as

$$\mathbb{R}_+x := \left\{\check{x} \mid \check{x} = \zeta x,\ \zeta \in \mathbb{R}_+\right\},$$

and R^i is the $n_i \times n_i$ diagonal matrix given by

$$R^i := \mathrm{diag}(1,\ -1, \dots, -1). \tag{1}$$

A primal-dual optimal solution of (P_ϵ) and (D_ϵ) is denoted by $\big(x(\epsilon); y(\epsilon); s(\epsilon)\big)$, while $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ stands for a maximally complementary solution. The relative interior, boundary, and the closure of a convex set are denoted by ri(.), bd(.), and cl(.), respectively. Letting N^i be an $m \times n_i$ matrix for $i \in I \subseteq \{1, \dots, p\}$, $(N^i)_I$ is an $m \times \sum_{i \in I} n_i$ matrix formed by the matrices N^i for $i \in I$ put side by side. Finally, dist(.,.) denotes the distance function defined by the ℓ_2 norm.

2 Preliminaries

2.1 Optimal partition and optimal solutions

The primal and dual optimal set mappings are defined as

$$\mathcal{P}^*(\epsilon) := \{x \mid (c + \epsilon\bar{c})^T x = \psi(\epsilon),\ Ax = b,\ x \in \mathcal{L}_+^{\bar{n}}\},$$
$$\mathcal{D}^*(\epsilon) := \{(y; s) \mid b^T y = \psi(\epsilon),\ A^T y + s = c + \epsilon\bar{c},\ s \in \mathcal{L}_+^{\bar{n}}\}.$$

Assumption 1 ensures that at every $\epsilon \in \mathrm{int}(\mathcal{E})$ strong duality holds[2], and that both $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ are compact, see [11, Theorem 5.81]. Under the strong duality assumption, $\mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)$ is the set of solutions of

$$F\big((x; y; s)\big) := \begin{pmatrix} Ax - b \\ A^T y + s - c - \epsilon\bar{c} \\ x \circ s \end{pmatrix} = 0, \qquad x, s \in \mathcal{L}_+^{\bar{n}}, \tag{2}$$

where $\circ : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{n}}$ is a bilinear map [2,13] defined as

$$x^i \circ s^i = L(x^i) s^i, \quad i = 1, \dots, p, \tag{3}$$

where

$$L(x^i) := \begin{pmatrix} x_1^i & (x_{2:n_i}^i)^T \\ x_{2:n_i}^i & x_1^i I_{n_i - 1} \end{pmatrix}$$

is a symmetric matrix, I_{n_i-1} is the identity matrix of size $n_i - 1$, and $x \circ s := (x^1 \circ s^1; \dots; x^p \circ s^p) = 0$ denotes the complementarity condition. The Jacobian of the equation system in (2) is given by

$$\nabla F((x; y; s)) := \begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ L(s) & 0 & L(x) \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} L(x) &:= \mathrm{diag}\big(L(x^1), \dots, L(x^p)\big), \\ L(s) &:= \mathrm{diag}\big(L(s^1), \dots, L(s^p)\big). \end{aligned} \tag{5}$$

Among all primal and dual optimal solutions, we are interested in strictly complementary and maximally complementary solutions.

[2] In this paper, strong duality means that the duality gap is zero at optimality, and the optimal sets $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ are nonempty.

Definition 1 Let a primal-dual optimal solution $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big) \in \mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)$ be given for a fixed ϵ. Then $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ is called *maximally complementary* if

$$x^*(\epsilon) \in \mathrm{ri}\,\big(\mathcal{P}^*(\epsilon)\big) \quad \text{and} \quad \big(y^*(\epsilon); s^*(\epsilon)\big) \in \mathrm{ri}\,\big(\mathcal{D}^*(\epsilon)\big).$$

A maximally complementary solution $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ is called *strictly complementary* if $x^*(\epsilon) + s^*(\epsilon) \in \mathrm{int}(\mathcal{L}^{\bar{n}}_+)$.

Remark 1 Throughout this paper, the strict complementarity condition is said to hold at ϵ if there exists a strictly complementary solution $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$.

Not every SOCO problem has a strictly complementary solution [2]. However, under Assumption 1, a maximally complementary solution always exists for every $\epsilon \in \mathrm{int}(\mathcal{E})$.

The notion of the optimal partition was originally defined for LO, where the index set of the variables is partitioned into two disjoint complementary sets [18,24]. Associated with any instance of SOCO with strong duality, the optimal partition is uniquely defined by using solutions from the relative interior of the optimal set [10,37]. Mathematically, given a fixed ϵ, the *optimal partition* of SOCO is defined as $\big(\mathcal{B}(\epsilon), \mathcal{N}(\epsilon), \mathcal{R}(\epsilon), \mathcal{T}(\epsilon)\big)$, where

$$\begin{aligned}
\mathcal{B}(\epsilon) &:= \{i \mid x^i_1(\epsilon) > \|x^i_{2:n_i}(\epsilon)\|, \text{ for some } x(\epsilon) \in \mathcal{P}^*(\epsilon)\},\\
\mathcal{N}(\epsilon) &:= \{i \mid s^i_1(\epsilon) > \|s^i_{2:n_i}(\epsilon)\|, \text{ for some } \big(y(\epsilon); s(\epsilon)\big) \in \mathcal{D}^*(\epsilon)\},\\
\mathcal{R}(\epsilon) &:= \{i \mid x^i_1(\epsilon) = \|x^i_{2:n_i}(\epsilon)\| > 0,\ s^i_1(\epsilon) = \|s^i_{2:n_i}(\epsilon)\| > 0,\\
&\qquad \text{for some } \big(x(\epsilon); y(\epsilon); s(\epsilon)\big) \in \mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)\},\\
\mathcal{T}(\epsilon) &:= \{\mathcal{T}_1(\epsilon), \mathcal{T}_2(\epsilon), \mathcal{T}_3(\epsilon)\},\\
\mathcal{T}_1(\epsilon) &:= \{i \mid x^i(\epsilon) = s^i(\epsilon) = 0, \text{ for all } \big(x(\epsilon); y(\epsilon); s(\epsilon)\big) \in \mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)\},\\
\mathcal{T}_2(\epsilon) &:= \{i \mid s^i(\epsilon) = 0, \text{ for all } \big(y(\epsilon); s(\epsilon)\big) \in \mathcal{D}^*(\epsilon),\ x^i_1(\epsilon) = \|x^i_{2:n_i}(\epsilon)\| > 0,\\
&\qquad \text{for some } x(\epsilon) \in \mathcal{P}^*(\epsilon)\},\\
\mathcal{T}_3(\epsilon) &:= \{i \mid x^i(\epsilon) = 0, \text{ for all } x(\epsilon) \in \mathcal{P}^*(\epsilon),\ s^i_1(\epsilon) = \|s^i_{2:n_i}(\epsilon)\| > 0,\\
&\qquad \text{for some } \big(y(\epsilon); s(\epsilon)\big) \in \mathcal{D}^*(\epsilon)\}.
\end{aligned}$$

The convexity of the optimal set implies that the subsets $\mathcal{B}(\epsilon)$, $\mathcal{N}(\epsilon)$, $\mathcal{R}(\epsilon)$, and $\mathcal{T}(\epsilon)$ are mutually disjoint and their union is the index set $\{1, \ldots, p\}$. Additionally, it follows from the complementarity condition that for all $\big(x(\epsilon); y(\epsilon); s(\epsilon)\big) \in \mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)$ we have $x^i(\epsilon) = 0$ for all $i \in \mathcal{N}(\epsilon)$ and $s^i(\epsilon) = 0$ for all $i \in \mathcal{B}(\epsilon)$.

The definition of maximally and strictly complementary solutions can be rephrased by using the optimal partition of the problem: $\big(x(\epsilon); y(\epsilon); s(\epsilon)\big) \in \mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)$ is maximally complementary if and only if $x^i(\epsilon) \in \mathrm{int}(\mathbb{L}^{n_i}_+)$ for all $i \in \mathcal{B}(\epsilon)$, $s^i(\epsilon) \in \mathrm{int}(\mathbb{L}^{n_i}_+)$ for all $i \in \mathcal{N}(\epsilon)$, $x^i_1(\epsilon) > 0$ for all $i \in \mathcal{R}(\epsilon) \cup \mathcal{T}_2(\epsilon)$, and $s^i_1(\epsilon) > 0$ for all $i \in \mathcal{R}(\epsilon) \cup \mathcal{T}_3(\epsilon)$. A maximally complementary solution $(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon))$ is strictly complementary if and only if $\mathcal{T}(\epsilon) = \emptyset$.

2.2 Nondegeneracy conditions

The concepts of primal and dual nondegeneracy were introduced in [30] for linear conic optimization and in [2,3] for SOCO and SDO. Here, we tailor and adapt the nondegeneracy conditions only for a maximally complementary solution.

Assume that $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ is a maximally complementary solution of (P_ϵ) and (D_ϵ) at a given ϵ. Then $x^*(\epsilon)$ is called *primal nondegenerate* if

$$\Big(\big(A^i \bar{P}^{*i}(\epsilon)\big)_{\mathcal{R}(\epsilon)\cup\mathcal{T}_2(\epsilon)},\ (A^i)_{\mathcal{B}(\epsilon)}\Big) \tag{6}$$

has full row rank, where the columns of $\bar{P}^{*i}(\epsilon) \in \mathbb{R}^{n_i \times n_i - 1}$ are normalized eigenvectors of the positive eigenvalues of $L\big((x^{*i}(\epsilon)\big)$. Furthermore, $\big(y^*(\epsilon); s^*(\epsilon)\big)$ is called *dual nondegenerate* if

$$\Big(\big(A^i R^i s^{*i}(\epsilon)\big)_{\mathcal{R}(\epsilon)\cup\mathcal{T}_3(\epsilon)},\ (A^i)_{\mathcal{B}(\epsilon)\cup\mathcal{T}_1(\epsilon)\cup\mathcal{T}_2(\epsilon)}\Big) \tag{7}$$

has full column rank, where R^i is defined in (1). Given a fixed ϵ, if there exists a primal (dual) nondegenerate optimal solution, then the dual (primal) optimal set mapping is single-valued at ϵ. Furthermore, if there exists a strictly complementary solution at ϵ, then the reverse direction is true as well. The proof can be found in [2].

Remark 2 In this paper, the primal and dual nondegeneracy conditions are said to hold at ϵ if there exists a nondegenerate maximally complementary solution at ϵ.

Remark 3 To test the primal nondegeneracy of an optimal solution $x(\epsilon)$ (not necessarily maximally complementary), the index sets $\mathcal{R}(\epsilon)\cup\mathcal{T}_2(\epsilon)$ and $\mathcal{B}(\epsilon)$ in (6) are replaced by indices of second-order cones which have $x(\epsilon)$ on their relative boundaries and in their interiors, respectively. Analogously, to check dual nondegeneracy, $\mathcal{R}(\epsilon)\cup\mathcal{T}_3(\epsilon)$ and $\mathcal{B}(\epsilon)\cup\mathcal{T}_1(\epsilon)\cup\mathcal{T}_2(\epsilon)$ are replaced in (7) by the indices of second-order cones which have $s(\epsilon)$ on their relative boundaries and at their origins, respectively.

We invoke the primal and dual nondegeneracy conditions in Section 3.3 for the identification of a transition point.

2.3 Set-valued analysis

In this section, we briefly review the continuity of set-valued mappings from [32]. Let $\mathbb{N}$ be the set of natural numbers, $\mathcal{J}$ be the collection of subsets $J \subset \mathbb{N}$ with $\mathbb{N}\setminus J$ being finite, $\mathcal{J}_\infty$ denote the collection of all infinite subsets of $\mathbb{N}$, and $\{\mathcal{C}_k\}_{k=1}^\infty$ be a sequence of subsets of $\mathbb{R}^{\bar{n}}$. The *outer limit* of $\{\mathcal{C}_k\}_{k=1}^\infty$ is defined as

$$\limsup_{k\to\infty} \mathcal{C}_k := \big\{x \mid \exists\ J \in \mathcal{J}_\infty \text{ and } x_k \in \mathcal{C}_k \text{ for } k \in J \text{ s.t. } \lim_{k\in J} x_k = x\big\},$$

where $\lim_{k\in J} x_k$ denotes the limit of a convergent sequence x_k as $k \to \infty$ and $k \in J$. On the other hand, the *inner limit* of $\{\mathcal{C}_k\}_{k=1}^\infty$ is given by

$$\liminf_{k\to\infty} \mathcal{C}_k := \big\{x \mid \exists\ J \in \mathcal{J} \text{ and } x_k \in \mathcal{C}_k \text{ for } k \in J \text{ s.t. } \lim_{k\in J} x_k = x\big\}.$$

If the inner and outer limits coincide, then the limit of $\{\mathcal{C}_k\}_{k=1}^\infty$ exists and converges to $\mathcal{C}$ *in the sense of Painlevé-Kuratowski*, i.e.,

$$\lim_{k\to\infty} \mathcal{C}_k := \limsup_{k\to\infty} \mathcal{C}_k = \liminf_{k\to\infty} \mathcal{C}_k = \mathcal{C}.$$

When $\mathcal{C}_k \neq \emptyset$, $\limsup_{k\to\infty} \mathcal{C}_k$ denotes the collection of all accumulation points of $\{x_k\}_{k=1}^\infty$ such that $x_k \in \mathcal{C}_k$, while $\liminf_{k\to\infty} \mathcal{C}_k$ represents the collection of all limit points of $\{x_k\}_{k=1}^\infty$. Recall that both the lim sup and lim inf of a sequence of sets are closed [32, Section 3.1].

A *set-valued* mapping $\Phi(\epsilon) : \mathbb{R} \rightrightarrows \mathbb{R}^{\bar{n}}$ assigns a subset of $\mathbb{R}^{\bar{n}}$ to each element of $\epsilon \in \mathbb{R}$. The domain of the set-valued mapping $\Phi(\epsilon)$ is defined as

$$\operatorname{dom}(\Phi) := \big\{\epsilon \in \mathbb{R} \mid \Phi(\epsilon) \neq \emptyset\big\},$$

and its range is given by

$$\operatorname{range}(\Phi) := \big\{x \in \mathbb{R}^{\bar{n}} \mid x \in \Phi(\epsilon), \text{ for some } \epsilon \in \mathbb{R}\big\} = \bigcup_{\epsilon \in \mathbb{R}} \Phi(\epsilon).$$

Various forms of continuity exist for a set-valued mapping. In this paper, continuity of a set-valued mapping is formed on the basis of Painlevé-Kuratowski set convergence, see [32, Section 3.2], which is equivalent to the notion of continuity of a *point-to-set map* in [23].

Let Γ be a subset of $\mathbb{R}$ containing $\bar{\epsilon}$, and define

$$\limsup_{\epsilon \to \bar{\epsilon}} \Phi(\epsilon) := \big\{x \mid \exists \{\epsilon_k\}_{k=1}^{\infty} \subseteq \Gamma \text{ with } \epsilon_k \to \bar{\epsilon}, \exists\, x_k \to x \text{ with } x_k \in \Phi(\epsilon_k)\big\},$$

$$\liminf_{\epsilon \to \bar{\epsilon}} \Phi(\epsilon) := \big\{x \mid \forall \{\epsilon_k\}_{k=1}^{\infty} \subseteq \Gamma \text{ with } \epsilon_k \to \bar{\epsilon}, \exists\, x_k \to x \text{ with } x_k \in \Phi(\epsilon_k)\big\}.$$

Then a set-valued mapping $\Phi(\epsilon)$ is called *outer semicontinuous* at $\bar{\epsilon}$ relative to Γ if

$$\limsup_{\epsilon \to \bar{\epsilon}} \Phi(\epsilon) \subseteq \Phi(\bar{\epsilon})$$

and *inner semicontinuous* at $\bar{\epsilon}$ relative to Γ if

$$\liminf_{\epsilon \to \bar{\epsilon}} \Phi(\epsilon) \supseteq \Phi(\bar{\epsilon})$$

holds. The set-valued mapping $\Phi(\epsilon)$ is *Painlevé-Kuratowski continuous* at $\bar{\epsilon}$ relative to Γ if it is both outer and inner semicontinuous at $\bar{\epsilon}$ relative to Γ.

By Assumption 1, we can show that $\mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)$ is outer semicontinuous relative to $\operatorname{int}(\mathcal{E})$. The result follows from [23, Theorem 8].

Lemma 1 *The set-valued mappings $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ are outer semicontinuous relative to* $\operatorname{int}(\mathcal{E})$.

The optimal set mapping may fail to be inner semicontinuous relative to $\operatorname{int}(\mathcal{E})$, e.g., when either the primal or the dual nondegeneracy condition fails at ϵ, while the strict complementarity condition holds. Nevertheless, sufficient conditions can be given for the continuity of $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$, regardless of the nondegeneracy conditions. First, it is easy to show, under Assumption 1, that the optimal set mapping is uniformly bounded near any $\epsilon \in \operatorname{int}(\mathcal{E})$ [34, Lemma 3.11], i.e., there exists $\varsigma > 0$ and a compact set $\mathcal{C} \subset \mathbb{R}^{\bar{n}} \times \mathbb{R}^m \times \mathbb{R}^{\bar{n}}$ such that

$$\bigcup_{\epsilon' \in (\epsilon - \varsigma, \epsilon + \varsigma)} \mathcal{P}^*(\epsilon') \times \mathcal{D}^*(\epsilon') \subseteq \mathcal{C}.$$

Then the continuity follows from the uniqueness condition.

Lemma 2 *If $\mathcal{P}^*(\epsilon)$ is single-valued at $\epsilon \in \operatorname{int}(\mathcal{E})$, then $\mathcal{P}^*(\epsilon)$ is continuous at ϵ relative to $\operatorname{int}(\mathcal{E})$. An analogous result holds for $\mathcal{D}^*(\epsilon)$.*

Proof The proof is immediate from the outer semicontinuity of $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ relative to $\operatorname{int}(\mathcal{E})$, the uniformly boundedness of $\mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)$ near any $\epsilon \in \operatorname{int}(\mathcal{E})$, and [23, Corollary 8.1]. □

Even though the primal or dual optimal set mapping is not necessarily inner semicontinuous relative to $\operatorname{int}(\mathcal{E})$, the set of points at which $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ fail to be continuous relative to $\operatorname{int}(\mathcal{E})$ is proven to be the union of countably many nowhere dense subsets of $\text{int}(\mathcal{E})$, i.e., it is of *first category* in $\text{int}(\mathcal{E})$. This is the consequence of Lemma 1 and Theorem 5.55 in [33]. Then the following result is in order.

Theorem 2 (Theorem 1.3 in [29]) *The set of points at which $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ are continuous is dense in* $\text{int}(\mathcal{E})$.

Remark 4 From this point on, unless stated otherwise, by the inner/outer semicontinuity of $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ at a given $\epsilon \in \text{int}(\mathcal{E})$ we mean inner/outer semicontinuity at ϵ relative to $\text{int}(\mathcal{E})$.

The continuity results are used in Sections 3.2 and 3.3 for the identification of a nonlinearity interval and a transition point.

3 Sensitivity of the optimal partition

In [25], the notion of a nonlinearity interval and a transition point was formally introduced for the optimal partition of a parametric SDO problem. In this section, we introduce those notions for the optimal partition of a parametric SOCO problem, which is defined on the basis of a different algebraic structure, see Section 2.1. From now on, the optimal partition of (P_ϵ) and (D_ϵ) at a given ϵ is denoted by $\pi(\epsilon) := \big(\mathcal{B}(\epsilon), \mathcal{N}(\epsilon), \mathcal{R}(\epsilon), \mathcal{T}(\epsilon)\big)$.

3.1 Invariancy sets, nonlinearity intervals, and transition points

For parametric LO and LCQO problems, the interval $\mathcal{E}$ can be entirely partitioned into invariancy sets, on which the optimal partition remains unchanged w.r.t. ϵ [8, 24]. An invariancy set can be analogously defined for a parametric SOCO problem. This definition is in accordance with [38, Section 4] for a linear conic optimization problem.

Definition 2 Let $\mathcal{E}_{\text{inv}}$ be a subset of $\text{int}(\mathcal{E})$, and for $\epsilon \in \mathcal{E}_{\text{inv}}$ let $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ be a maximally complementary solution. Then $\mathcal{E}_{\text{inv}}$ is called an *invariancy set* if $\pi(\epsilon') = \pi(\epsilon'')$ for all $\epsilon', \epsilon'' \in \mathcal{E}_{\text{inv}}$, and the extreme rays $\mathbb{R}_+ x^{*i}(\epsilon)$ for $i \in \mathcal{R}(\epsilon) \cup \mathcal{T}_2(\epsilon)$ and $\mathbb{R}_+ s^{*i}(\epsilon)$ for $i \in \mathcal{R}(\epsilon) \cup \mathcal{T}_3(\epsilon)$ are invariant w.r.t. $\epsilon \in \mathcal{E}_{\text{inv}}$. If it is not a singleton, then $\mathcal{E}_{\text{inv}}$ is called an *invariancy interval*.

Analogous to a parametric SDO problem, there exists a unique primal optimal set associated with an invariancy set, see e.g., [25, Lemma 3.3], which in turn implies the openness of an invariancy interval. The boundary points of an invariancy set can be obtained by solving a pair of auxiliary SOCO problems, see [38, Section 4].

It is easy to see that a singleton invariancy set $\{\bar{\epsilon}\}$ exists, i.e., when either the optimal partition $\pi(\epsilon)$, or the extreme rays $\mathbb{R}_+ x^{*i}(\epsilon)$ and $\mathbb{R}_+ s^{*i}(\epsilon)$ for some $i \in \mathcal{R}(\epsilon) \cup \mathcal{T}_2(\epsilon) \cup \mathcal{T}_3(\epsilon)$, or both changes in every neighborhood of $\bar{\epsilon}$. However, unlike parametric LO and LCQO problems, infinitely many singleton invariancy sets may exist for a parametric SOCO problem, as demonstrated in the following parametric problem[3]:

$$\begin{array}{ll} \min & -\epsilon x_2^1 - (1-\epsilon)x_3^1 \\ \text{s.t.} & x_1^1 = 1, \\ & x_3^1 - x_1^2 = 0, \\ & x_2^1 - x_2^2 = 1, \\ & x_1^1 \geq \sqrt{(x_2^1)^2 + (x_3^1)^2}, \\ & x_1^2 \geq |x_2^2|, \end{array} \tag{8}$$

where $\mathcal{E} = \mathbb{R}$. One can check that the optimal partition on $\mathbb{R}$ is given by

$$\big(\mathcal{B}(\epsilon), \mathcal{N}(\epsilon), \mathcal{R}(\epsilon), \mathcal{T}(\epsilon)\big) = \begin{cases} \big(\emptyset, \emptyset, \{1,2\}, (\emptyset, \emptyset, \emptyset)\big), & \epsilon \in (-\infty, 0), \\ \big(\emptyset, \emptyset, \{1\}, (\emptyset, \{2\}, \emptyset)\big), & \epsilon = 0, \\ \big(\{2\}, \emptyset, \{1\}, (\emptyset, \emptyset, \emptyset)\big), & \epsilon \in (0,1), \\ \big(\emptyset, \emptyset, \{1\}, (\{2\}, \emptyset, \emptyset)\big), & \epsilon = 1, \\ \big(\emptyset, \{2\}, \{1\}, (\emptyset, \emptyset, \emptyset)\big), & \epsilon \in (1, \infty), \end{cases}$$

where

$$x^{*1}(\epsilon) = \begin{pmatrix} 1 \\ \frac{\epsilon}{\sqrt{(1-\epsilon)^2+\epsilon^2}} \\ \frac{1-\epsilon}{\sqrt{(1-\epsilon)^2+\epsilon^2}} \end{pmatrix}, \ s^{*1}(\epsilon) = \begin{pmatrix} \sqrt{(1-\epsilon)^2+\epsilon^2} \\ -\epsilon \\ \epsilon - 1 \end{pmatrix}, \qquad \forall \epsilon \in (0,1).$$

The optimal partitions on $(-\infty, 0)$ and $(1, \infty)$ are invariant w.r.t. ϵ. However, while the index sets of $\pi(\epsilon)$ are fixed on the interval $(0,1)$, the extreme rays $\mathbb{R}_+ x^{*1}(\epsilon)$ and $\mathbb{R}_+ s^{*1}(\epsilon)$ vary continuously with ϵ. In this case, $(0,1)$ is called a nonlinearity interval, and $\{0,1\}$ denotes the set of transition points.

Now, we formally define nonlinearity intervals and transition points of the optimal partition.

Definition 3 Let $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ be a maximally complementary solution. A *nonlinearity interval* is a non-singleton open maximal subinterval $\mathcal{E}_{\text{non}}$ of $\operatorname{int}(\mathcal{E})$ such that $\pi(\epsilon') = \pi(\epsilon'')$ for any two $\epsilon', \epsilon'' \in \mathcal{E}_{\text{non}}$, while $\mathbb{R}_+ x^{*i}(\epsilon)$ and $\mathbb{R}_+ s^{*i}(\epsilon)$ vary with ϵ for some $i \in \mathcal{R}(\epsilon) \cup \mathcal{T}_2(\epsilon) \cup \mathcal{T}_3(\epsilon)$.

Equivalently, the rank of $L\big(x^*(\epsilon)\big)$ and $L\big(s^*(\epsilon)\big)$ remain constant on a nonlinearity interval, where $L(x)$ and $L(s)$ are defined in (5). Obviously, if $\mathcal{R}(\epsilon) = \mathcal{T}_2(\epsilon) = \mathcal{T}_3(\epsilon) = \emptyset$ on $\operatorname{int}(\mathcal{E})$, then no nonlinearity interval exists.

Definition 4 A singleton invariancy set $\{\bar{\epsilon}\} \in \operatorname{int}(\mathcal{E})$ is called a *transition point* if for every $\varsigma > 0$ there exists an $\epsilon \in (\bar{\epsilon} - \varsigma, \bar{\epsilon} + \varsigma) \subseteq \operatorname{int}(\mathcal{E})$ such that $\pi(\bar{\epsilon}) \neq \pi(\epsilon)$.

It can be interpreted from Definitions 3 and 4 that a singleton invariancy set either belongs to a nonlinearity interval, or it is a transition point. Further, it immediately follows that the boundary points of invariancy or nonlinearity intervals belong to the set of transition points in $\operatorname{int}(\mathcal{E})$. On the other hand, an algebraic formulation of a transition point reveals that a transition point must be on the boundary of an invariancy or a nonlinearity interval, as stated in Lemma 3.

[3] See Example 3.1 in [25] for an instance of a parametric SDO problem with infinitely many singleton invariancy sets.

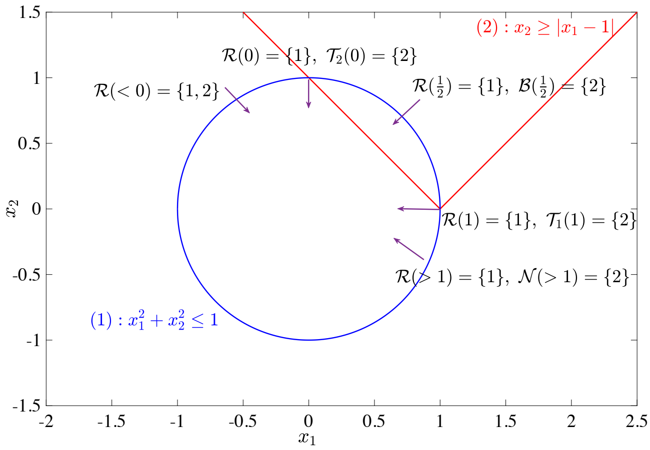

Fig. 1: The feasible region of problem (8).

Lemma 3 *The set of transition points in* $\operatorname{int}(\mathcal{E})$ *is finite.*

Proof Given a fixed optimal partition $\pi(\bar{\epsilon})$, the set of all ϵ with the optimal partition $\pi(\bar{\epsilon})$ can be formulated as

$$
\begin{aligned}
S_{\pi(\bar{\epsilon})} := \Big\{ \epsilon \in \operatorname{int}(\mathcal{E}) \mid \ \exists\ (x; y; s) \ \text{ s.t. } \ Ax &= b, \\
A^T y + s &= c + \epsilon \bar{c}, \\
x \circ s &= 0, \\
(x_1^i)^2 - \|x_{2:n_i}^i\|^2 &> 0, && i \in \mathcal{B}(\bar{\epsilon}), \\
(s_1^i)^2 - \|s_{2:n_i}^i\|^2 &> 0, && i \in \mathcal{N}(\bar{\epsilon}), \\
(x_1^i)^2 - \|x_{2:n_i}^i\|^2 &= 0, && i \in \mathcal{R}(\bar{\epsilon}) \cup \mathcal{T}_2(\bar{\epsilon}), \\
(s_1^i)^2 - \|s_{2:n_i}^i\|^2 &= 0, && i \in \mathcal{R}(\bar{\epsilon}) \cup \mathcal{T}_3(\bar{\epsilon}), \\
s^i &= 0, && i \in \mathcal{B}(\bar{\epsilon}) \cup \mathcal{T}_1(\bar{\epsilon}) \cup \mathcal{T}_2(\bar{\epsilon}), \\
x^i &= 0, && i \in \mathcal{N}(\bar{\epsilon}) \cup \mathcal{T}_1(\bar{\epsilon}) \cup \mathcal{T}_3(\bar{\epsilon}), \\
x_1^i &> 0, && i \in \mathcal{B}(\bar{\epsilon}) \cup \mathcal{R}(\bar{\epsilon}) \cup \mathcal{T}_2(\bar{\epsilon}), \\
s_1^i &> 0, && i \in \mathcal{N}(\bar{\epsilon}) \cup \mathcal{R}(\bar{\epsilon}) \cup \mathcal{T}_3(\bar{\epsilon}) \Big\}.
\end{aligned}
$$

Observe that $S_{\pi(\bar{\epsilon})}$ is a semialgebraic subset of $\mathbb{R}$, being the projection of a solution set formed by the intersection of polynomial equations and inequalities, see Theorem 2.2.1 in [9]. Note that $S_{\pi(\bar{\epsilon})}$ might be empty or disconnected in $\mathbb{R}$. Since a semialgebraic set has a finite number of connected components, see Theorem 2.4.5 in [9], and the boundary points of $S_{\pi(\bar{\epsilon})}$ are transition points in $\operatorname{int}(\mathcal{E})$, the set of transition points with a fixed optimal partition $\pi(\bar{\epsilon})$ is finite and thus algebraic. Then the result follows by noting that $\pi(.)$ can only take a finite number of possibilities. □

As a result of Lemma 3, one can partition $\operatorname{int}(\mathcal{E})$ into the finite union of invariancy intervals, nonlinearity intervals, and transition points. The connected components of a semialgebraic set in $\mathbb{R}$ are either points or intervals [9]. In Lemma 3, a singleton component of $S_{\pi(\bar{\epsilon})}$ is indeed a transition point, and the interior of a non-singleton component of $S_{\pi(\bar{\epsilon})}$ corresponds to either an invariancy or a nonlinearity interval with the optimal partition $\pi(\bar{\epsilon})$. Conversely, an invariancy interval with the optimal partition $\pi(\bar{\epsilon})$ corresponds to an open connected component of $S_{\pi(\bar{\epsilon})}$ in $\mathbb{R}$. It is unknown, however, if the component corresponding to a nonlinearity interval is open in $\mathbb{R}$. Equivalently, we do not know whether $\pi(\bar{\epsilon}) \neq \pi(\epsilon)$ holds at a transition point $\bar{\epsilon}$ for all ϵ belonging to the given component. Corollary 3.8 in [25] provides sufficient conditions for the openness of a non-singleton component.

Note that in a parametric SOCO problem, if $\mathcal{R}(\epsilon) \cup \mathcal{T}_2(\epsilon) \cup \mathcal{T}_3(\epsilon) \neq \emptyset$, any two nonlinearity intervals or a nonlinearity interval and a transition point might have the same optimal partition. This is in contrast to parametric LO and LCQO problems where the invariancy sets are associated with distinct optimal partitions [8, 24]. For instance, consider the optimal partition of the following parametric SOCO problem:

$$\begin{aligned} \min \quad & (1-2\epsilon)x_2^1 - x_3^1 \\ \text{s.t.} \quad & x_1^1 = 1, \\ & x_1^2 = 2, \\ & x_2^2 - x_2^1 = 0, \\ & x_3^2 - 2x_3^1 = 0, \\ & x_1^1 \geq \sqrt{(x_2^1)^2 + (x_3^1)^2}, \\ & x_1^2 \geq \sqrt{(x_2^2)^2 + (x_3^2)^2}, \end{aligned} \tag{9}$$

where the optimal set is given by

$$x^*(\epsilon) = \begin{pmatrix} 1 \\ \frac{2\epsilon-1}{\sqrt{4\epsilon^2-4\epsilon+2}} \\ \frac{1}{\sqrt{4\epsilon^2-4\epsilon+2}} \\ 2 \\ \frac{2\epsilon-1}{\sqrt{4\epsilon^2-4\epsilon+2}} \\ \frac{2}{\sqrt{4\epsilon^2-4\epsilon+2}} \end{pmatrix}, \qquad s^*(\epsilon) = \begin{pmatrix} \sqrt{4\epsilon^2-4\epsilon+2} \\ 1-2\epsilon \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon \in (-\infty, \tfrac{1}{2}) \cup \ (\tfrac{1}{2}, \infty),$$

$$x^*(\tfrac{1}{2}) = \Big(1, 0, 1, 2, 0, 2\Big)^T, \quad s^*(\tfrac{1}{2}) = \Big(\gamma, 0, -\gamma, \tfrac{1-\gamma}{2}, 0, \tfrac{\gamma-1}{2}\Big)^T, \quad \gamma \in [0,1].$$

In this case, the two nonlinearity intervals $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$ with identical optimal partitions are separated by a transition point at $\epsilon = \frac{1}{2}$.

3.2 On the identification of a nonlinearity interval

Recall from Definition 3 that both the primal and dual optimal sets vary with ϵ on a nonlinearity interval. As problem (8) indicates[4], solely continuity of the optimal set mapping at $\bar{\epsilon}$ does not induce the existence of a nonlinearity interval. However, continuity becomes sufficient in the presence of the strict complementarity condition, see also [22, Lemma 3].

[4] For this problem, both the primal and dual optimal set mappings are continuous at the transition point $\epsilon = 0$.

Lemma 4 *Let $\{\bar{\epsilon}\}$ be a singleton invariancy set. If $\big(x^*(\bar{\epsilon}); y^*(\bar{\epsilon}); s^*(\bar{\epsilon})\big)$ is strictly complementary and both $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ are continuous at $\bar{\epsilon}$, then $\bar{\epsilon}$ belongs to a nonlinearity interval.*

Proof By the inner semicontinuity of $\mathcal{P}^*(\epsilon)$ at $\bar{\epsilon}$, for ϵ sufficiently close to $\bar{\epsilon}$ there exists a primal optimal solution $x(\epsilon)$ such that $x_1^i(\epsilon) - \|x_{2:n_i}^i(\epsilon)\| > 0$ for every $i \in \mathcal{B}(\bar{\epsilon})$, which implies $\mathcal{B}(\bar{\epsilon}) \subseteq \mathcal{B}(\epsilon)$, see [32, Theorem 3B.2(b)]. Analogously, the inner semicontinuity of $\mathcal{D}^*(\epsilon)$ at $\bar{\epsilon}$ implies the existence of ϵ such that $\mathcal{N}(\bar{\epsilon}) \subseteq \mathcal{N}(\epsilon)$. Finally, by the inner semicontinuity of both $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ at $\bar{\epsilon}$, we have $x_1^i(\epsilon) > 0$ and $s_1^i(\epsilon) > 0$ for $i \in \mathcal{R}(\bar{\epsilon})$, which by $x^i(\epsilon) \circ s^i(\epsilon) = 0$, give $i \in \mathcal{R}(\epsilon)$. All this implies that $x^*(\epsilon) + s^*(\epsilon) \in \operatorname{int}(\mathcal{L}_+^{\bar{n}})$ for all ϵ in a sufficiently small neighborhood of $\bar{\epsilon}$ and thus the index sets remain unchanged. □

Lemma 4 does not yield a complete characterization for the existence of a nonlinearity interval, in a sense that either strict complementarity or continuity might fail on a nonlinearity interval. All this makes the identification and computation of a nonlinearity interval a nontrivial task. For instance, $\mathcal{P}^*(\epsilon)$ fails to be continuous on a nonlinearity interval of the following parametric SOCO problem:

$$\begin{array}{ll}\min & (\tfrac{1}{2} - \tfrac{1}{2}\epsilon)x_2^1 + (\epsilon - \tfrac{1}{2})x_1^2 - \tfrac{1}{2}\epsilon x_2^2 + (\epsilon - \tfrac{1}{2})x_3^2 \\ \text{s.t.} & x_1^1 + x_1^2 = 4, \\ & x_3^1 + x_3^2 = 0, \\ & x_1^1 \geq \sqrt{(x_2^1)^2 + (x_3^1)^2}, \\ & x_1^2 \geq \sqrt{(x_2^2)^2 + (x_3^2)^2}, \end{array} \tag{10}$$

where $\mathcal{E} = \mathbb{R}$. On the interval $(-\frac{1}{2}, \frac{3}{2})$, a strictly complementary solution is given by

$$x^*(\epsilon) = \begin{pmatrix} 4\epsilon^3 - 6\epsilon^2 + \epsilon + \frac{5}{2} \\ 4\epsilon^2 - 2\epsilon - 2 \\ -4\epsilon^3 + 6\epsilon^2 + \epsilon - \frac{3}{2} \\ -4\epsilon^3 + 6\epsilon^2 - \epsilon + \frac{3}{2} \\ 6\epsilon - 4\epsilon^2 \\ 4\epsilon^3 - 6\epsilon^2 - \epsilon + \frac{3}{2} \end{pmatrix}, \quad s^*(\epsilon) = \begin{pmatrix} \frac{1}{2}\epsilon^2 - \epsilon + \frac{5}{8} \\ \frac{1}{2} - \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon^2 - \epsilon + \frac{3}{8} \\ \frac{1}{2}\epsilon^2 + \frac{1}{8} \\ -\frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon^2 - \frac{1}{8} \end{pmatrix},$$

which is the unique primal optimal solution for every $\epsilon \in (-\frac{1}{2}, \frac{3}{2}) \setminus \{\frac{1}{2}\}$. Then one can easily verify that $(-\frac{1}{2}, \frac{3}{2})$ is a nonlinearity interval in $\operatorname{int}(\mathcal{E})$, while the primal optimal set mapping fails to be continuous at $\epsilon = \frac{1}{2}$.

A numerical procedure If the Jacobian (4) is nonsingular at $\bar{\epsilon}$, then the existence of a nonlinearity interval around $\bar{\epsilon}$ follows from the implicit function theorem [12, Theorem 10.2.1], see [25, Lemma 3.9] and its subsequent discussion. In general, however, the Jacobian of the optimality conditions might be singular on a given subinterval of $\operatorname{int}(\mathcal{E})$, see e.g., the interval $[1, \infty)$ in the parametric SOCO problem (8). Even the continuity condition of Lemma 4 may either fail to exist or may be impossible to check at a given point $\bar{\epsilon}$. On the other hand, since the transition points are isolated in $\operatorname{int}(\mathcal{E})$, see Lemma 3, and the magnitude of optimal solutions could be doubly exponentially small [26, Example 3.2], it may be impractical to obtain the desired nonlinearity interval by simply solving (P_ϵ) and (D_ϵ) at arbitrary values of ϵ.

Under strict complementarity condition, we present a numerical procedure to compute a nonlinearity interval in $\operatorname{int}(\mathcal{E})$. The procedure starts from $\bar\epsilon$ and a given strictly complementary solution with the goal to find the nonlinearity interval surrounding $\bar\epsilon$. The procedure iteratively generates a sequence of subintervals of $\operatorname{int}(\mathcal{E})$ by solving a pair of nonlinear auxiliary problems. At every iteration of the procedure, a method of real algebraic geometry is invoked in order to check the existence of a transition point in the given subinterval.

The nonlinear auxiliary problems in the above procedure are defined locally w.r.t. a given strictly complementary solution $\big(x^*(\bar\epsilon); y^*(\bar\epsilon); s^*(\bar\epsilon)\big)$. Let us define

$$\begin{aligned}
\delta_{\mathcal{B}(\bar\epsilon)} &:= \frac{\sqrt{2}}{2} \min_{i\in\mathcal{B}(\bar\epsilon)} \big\{x_1^{*i}(\bar\epsilon) - \|x_{2:n_i}^{*i}(\bar\epsilon)\|\big\},\\
\delta_{\mathcal{N}(\bar\epsilon)} &:= \frac{\sqrt{2}}{2} \min_{i\in\mathcal{N}(\bar\epsilon)} \big\{s_1^{*i}(\bar\epsilon) - \|s_{2:n_i}^{*i}(\bar\epsilon)\|\big\},\\
\delta_{\mathcal{R}(\bar\epsilon)} &:= \min\Big\{\min_{i\in\mathcal{R}(\bar\epsilon)} \big\{x_1^{*i}(\bar\epsilon)\big\}, \min_{i\in\mathcal{R}(\bar\epsilon)} \big\{s_1^{*i}(\bar\epsilon)\big\}\Big\},\\
\delta(\bar\epsilon) &:= \min\big\{\delta_{\mathcal{B}(\bar\epsilon)}, \delta_{\mathcal{N}(\bar\epsilon)}, \delta_{\mathcal{R}(\bar\epsilon)}\big\},
\end{aligned}$$

and for a given $\delta > 0$ let a closed basic semialgebraic set $\mathcal{S}(\delta, \bar\epsilon)$ be defined by

$$\begin{aligned}
\mathcal{S}(\delta,\bar\epsilon) := \big\{\epsilon \mid \exists\ (x;y;s) \ \text{ s.t. } \ Ax = b,\ A^Ty = c + \epsilon\bar c,\ x\circ s = 0,&\\
\|x - x^*(\bar\epsilon)\|^2 \le \delta^2,&\\
\|s - s^*(\bar\epsilon)\|^2 \le \delta^2\big\},&
\end{aligned} \tag{11}$$

which is nonempty, and it has a finite number of connected components, see Lemma 3. Then it is immediate that $\mathcal{S}(\delta', \bar\epsilon) \subseteq \mathcal{S}(\delta, \bar\epsilon)$ for every $0 < \delta' < \delta$.

Lemma 5 *Let $\bar\epsilon$ be a singleton invariancy set and $\big(x^*(\bar\epsilon); y^*(\bar\epsilon); s^*(\bar\epsilon)\big)$ be a strictly complementary solution. If $0 < \delta < \delta(\bar\epsilon)$, then we have $\pi(\epsilon) = \pi(\bar\epsilon)$ for every $\epsilon \in \mathcal{S}(\delta, \bar\epsilon) \cap \operatorname{int}(\mathcal{E})$.*

Proof The last two constraints in (11) ensure that the strict complementarity condition holds at every $\epsilon \in \mathcal{S}(\delta,\bar\epsilon) \cap \operatorname{int}(\mathcal{E})$. For the given ϵ let $\big(\tilde x(\epsilon); \tilde y(\epsilon); \tilde s(\epsilon)\big)$ be a solution which satisfies the equations and inequalities in (11). Then we have

$$\big|\tilde x_1^i(\epsilon) - x_1^{*i}(\bar\epsilon) - \big\|\tilde x_{2:n_i}^i(\epsilon) - x_{2:n_i}^{*i}(\bar\epsilon)\big\|\big| \le \sqrt{2}\|\tilde x^i(\epsilon) - x^{*i}(\bar\epsilon)\|,$$

which, together with $\|\tilde x^i(\epsilon) - x^{*i}(\bar\epsilon)\| < \delta$ for every $i \in \mathcal{B}(\bar\epsilon)$, gives

$$\begin{aligned}
\|\tilde x_{2:n_i}^i(\epsilon)\| - \|x_{2:n_i}^{*i}(\bar\epsilon)\| - \tilde x_1^i(\epsilon) + x_1^{*i}(\bar\epsilon) &\le \|\tilde x_{2:n_i}^i(\epsilon) - x_{2:n_i}^{*i}(\bar\epsilon)\| - \tilde x_1^i(\epsilon) + x_1^{*i}(\bar\epsilon)\\
&\le \sqrt{2}\|\tilde x^i(\epsilon) - x^{*i}(\bar\epsilon)\| \le \sqrt{2}\delta.
\end{aligned}$$

Consequently,

$$0 < x_1^{*i}(\bar\epsilon) - \|x_{2:n_i}^{*i}(\bar\epsilon)\| - \sqrt{2}\delta \le \tilde x_1^i(\epsilon) - \|\tilde x_{2:n_i}^i(\epsilon)\|, \qquad i \in \mathcal{B}(\epsilon). \tag{12}$$

Analogously, we can show that $\tilde s_1^i(\epsilon) - \|\tilde s_{2:n_i}^i(\epsilon)\| > 0$ for every $i \in \mathcal{N}(\epsilon)$. For any given $i \in \mathcal{R}(\epsilon)$ we can derive

$$|\tilde x_1^i(\epsilon) - x_1^{*i}(\bar\epsilon)| \le \|\tilde x^i(\epsilon) - x^{*i}(\bar\epsilon)\| \le \delta \qquad \Longrightarrow \tilde x_1^i(\epsilon) \ge x_1^{*i}(\bar\epsilon) - \delta > 0, \tag{13}$$

$$|\tilde s_1^i(\epsilon) - s_1^{*i}(\bar\epsilon)| \le \|\tilde s^i(\epsilon) - s^{*i}(\bar\epsilon)\| \le \delta \qquad \Longrightarrow \tilde s_1^i(\epsilon) \ge s_1^{*i}(\bar\epsilon) - \delta > 0. \tag{14}$$

Finally, it follows from $\tilde x^i(\epsilon) \circ \tilde s^i(\epsilon) = 0$ and (3) that

$$\begin{aligned}
0 = \tilde x^i(\epsilon)^T \tilde s^i(\epsilon) = \tilde x_1^i(\epsilon)\tilde s_1^i(\epsilon) + \big(\tilde x_{2:n_i}^i(\epsilon)\big)^T \tilde s_{2:n_i}^i(\epsilon) &= \tilde x_1^i(\epsilon)\tilde s_1^i(\epsilon) - \frac{\tilde s_1^i(\epsilon)\|\tilde x_{2:n_i}^i(\epsilon)\|^2}{\tilde x_1^i(\epsilon)}\\
&= \frac{\tilde s_1^i(\epsilon)\big((\tilde x_1^i(\epsilon))^2 - \|\tilde x_{2:n_i}^i(\epsilon)\|^2\big)}{\tilde x_1^i(\epsilon)},
\end{aligned}$$

which, by (13) and (14), yields $\tilde{x}_1^i(\epsilon) - \|\tilde{x}_{2:n_i}^i(\epsilon)\| = 0$, and analogously, $\tilde{s}_1^i(\epsilon) - \|\tilde{s}_{2:n_i}^i(\epsilon)\| = 0$ for every $i \in \mathcal{R}(\epsilon)$. Thus, by (12) to (14), $\big(\tilde{x}(\epsilon); \tilde{y}(\epsilon); \tilde{s}(\epsilon)\big)$ is a strictly complementary solution and thus the optimal partition $\pi(\epsilon)$ is identical to $\pi(\bar{\epsilon})$ for every $\epsilon \in \mathcal{S}(\delta, \bar{\epsilon}) \cap \mathrm{int}(\mathcal{E})$. This completes the proof. □

If $\mathcal{S}(\delta, \bar{\epsilon})$ is non-singleton, then there always exists a $0 < \delta' < \delta$ such that $\mathcal{S}(\delta', \bar{\epsilon}) \subsetneq \mathcal{S}(\delta, \bar{\epsilon})$, since for every $\bar{\epsilon} \neq \epsilon \in \mathcal{S}(\delta, \bar{\epsilon})$ we have

$$\mathrm{dist}\,\big((x^*(\bar{\epsilon}); y^*(\bar{\epsilon}); s^*(\bar{\epsilon})), \mathcal{P}^*(\epsilon) \times \mathcal{D}^*(\epsilon)\big) > 0.$$

Therefore, by choosing a sufficiently small δ' we can exclude ϵ from $\mathcal{S}(\delta', \bar{\epsilon})$.

Lemma 6 *Assume that the strict complementarity condition holds at $\bar{\epsilon}$. Then for every sequence $\delta_k \to 0$ we have $\lim_{k\to\infty} \mathcal{S}(\delta_k, \bar{\epsilon}) = \{\bar{\epsilon}\}$ in the sense of Painlevé-Kuratowski convergence.*

Proof If $\mathcal{S}(\delta_k, \bar{\epsilon}) = \{\bar{\epsilon}\}$ for some k, then the result is trivial. Otherwise, for any sequence $\delta_k \to 0$, we have a monotone sequence of sets $\mathcal{S}(\delta_k, \bar{\epsilon})$ such that $\mathcal{S}(\delta_k, \bar{\epsilon}) \supseteq \mathcal{S}(\delta_{k+1}, \bar{\epsilon}) \supseteq \cdots \supsetneq \mathcal{S}(\delta_\ell, \bar{\epsilon}) \supseteq \cdots$ for some $\ell > k$. Then, by [33, Exercise 4.3(b)], we get

$$\lim_{k\to\infty} \mathcal{S}(\delta_k, \bar{\epsilon}) = \bigcap_k \mathrm{cl}\,\big(\mathcal{S}(\delta_k, \bar{\epsilon})\big) = \{\bar{\epsilon}\}.$$

This completes the proof. □

Given a strictly complementary solution $\big(x^*(\bar{\epsilon}); y^*(\bar{\epsilon}); s^*(\bar{\epsilon})\big)$, the idea is to explore the semialgebraic set $\mathcal{S}(\delta, \bar{\epsilon})$ by solving the following nonlinear auxiliary problems

$$\begin{aligned} \alpha(\delta)\big(\beta(\delta)\big) := \min(\max)\ & \epsilon \\ \text{s.t.}\ & Ax = b, \\ & A^T y = c + \epsilon\bar{c}, \\ & x \circ s = 0, \\ & \|x - x^*(\bar{\epsilon})\|^2 \leq \delta^2, \\ & \|s - s^*(\bar{\epsilon})\|^2 \leq \delta^2, \\ & \epsilon \in \mathrm{int}(\mathcal{E}), \end{aligned} \tag{15}$$

where $\alpha(\delta)$ and $\beta(\delta)$ denote the optimal values of (15). Since $\mathcal{S}(\delta, \bar{\epsilon})$ has a finite number of connected components, Lemma 6 indicates that when δ is sufficiently small, $\big(\alpha(\delta), \beta(\delta)\big)$ yields either a singleton set or a subinterval of a connected component of $\mathcal{S}(\delta, \bar{\epsilon})$. Lemma 7 guarantees the latter case by requiring the existence of a sequence of optimal solutions converging to $\big(x^*(\bar{\epsilon}); y^*(\bar{\epsilon}); s^*(\bar{\epsilon})\big)$ for every $\epsilon_k \to \bar{\epsilon}$.

Lemma 7 *Let $\big(x^*(\bar{\epsilon}); y^*(\bar{\epsilon}); s^*(\bar{\epsilon})\big)$ be a strictly complementary solution, and assume that*

$$x^*(\bar{\epsilon}) \in \liminf_{k\to\infty} \mathcal{P}^*(\epsilon_k) \quad \text{and} \quad \big(y^*(\bar{\epsilon}); s^*(\bar{\epsilon})\big) \in \liminf_{k\to\infty} \mathcal{D}^*(\epsilon_k) \tag{16}$$

for every sequence $\epsilon_k \to \bar{\epsilon}$. If $0 < \delta < \delta(\bar{\epsilon})$, then we have $\alpha(\delta) < \bar{\epsilon} < \beta(\delta)$ such that $\pi\big(\alpha(\delta)\big) = \pi(\bar{\epsilon}) = \pi\big(\beta(\delta)\big)$.

Proof By [32, Proposition 3A.1], conditions (16) are equivalent to

$$\lim_{k\to\infty} \mathrm{dist}\big(x^*(\bar{\epsilon}), \mathcal{P}^*(\epsilon_k)\big) = 0 \quad \text{and} \quad \lim_{k\to\infty} \mathrm{dist}\big(\big(y^*(\bar{\epsilon}); s^*(\bar{\epsilon})\big), \mathcal{D}^*(\epsilon_k)\big) = 0$$

for any sequence $\epsilon_k \to \bar{\epsilon}$. Hence, for the given δ there exists a $\xi > 0$ such that

$$(\bar{\epsilon} - \xi, \bar{\epsilon} + \xi) \subset \mathcal{S}(\delta, \bar{\epsilon}).$$

The second part of the lemma follows from Lemma 5. □

Note that the conditions of Lemma 7 induce the existence of a nonlinearity interval around $\bar{\epsilon}$, while the conditions presented in (16) are weaker than the continuity condition in Lemma 4. The following corollary is then immediate from Lemma 7.

Corollary 1 *Assume that the strict complementarity condition holds at $\bar{\epsilon}$. If either $\alpha(\delta) = \bar{\epsilon}$, or $\beta(\delta) = \bar{\epsilon}$, or both holds for some $\delta > 0$, then there exists a sequence $\epsilon_k \to \bar{\epsilon}$ at which the conditions (16) fail.*

Consequently, by Lemma 6, Corollary 1, and the finiteness of the number of transition points, a sufficiently small value of δ allows us to decide whether $\bar{\epsilon}$ belongs to a nonlinearity interval, or it is a singleton invariancy set at which conditions (16) fail to hold. Note that the later case would not necessarily lead to the existence of a transition point, see e.g., [22, problem (10)] which can be indeed represented as a parametric SOCO problem.

Remark 5 In Lemma 7, $\big(\alpha(\delta), \beta(\delta)\big)$ contains a subinterval of the nonlinearity interval surrounding $\bar{\epsilon}$. If the given nonlinearity interval is unbounded or the optimal partitions on $\operatorname{int}(\mathcal{E})$ are distinct, then $\mathcal{S}(\delta, \bar{\epsilon})$ is connected in $\mathbb{R}$ and thus $\big(\alpha(\delta), \beta(\delta)\big)$ yields a subinterval of the given nonlinearity interval for any $\delta > 0$.

Outline of the numerical procedure Based on auxiliary problems (15) and the above description, Algorithm 1 presents the outline of our numerical procedure for the computation of a nonlinearity interval in $\operatorname{int}(\mathcal{E})$. Given the initial point $\bar{\epsilon}$ at which the strict complementarity condition holds, Algorithm 1 tracks forwards and backwards by iteratively solving auxiliary problems (15). The procedure stops only when it reaches a point at which conditions (16) fail. The connectivity of the subintervals generated by Algorithm 1 can be investigated by using the so-called roadmap algorithm [5, Chapter 16]. The roadmap algorithm has singly exponential complexity and its goal is to make a decision on whether the two points α_k and $\alpha(\delta)$, or β_k and $\beta(\delta)$ belong to the same connected component of $\mathcal{S}(\delta, \bar{\epsilon})$. We omit the description here and refer the reader to [5, Chapter 16] for details.

Let $\mathcal{E}_{\text{non}}$ be a bounded nonlinearity interval. Then, starting at an arbitrary $\bar{\epsilon} \in \mathcal{E}_{\text{non}}$, the algorithmic map of Algorithm 1 generates a nonincreasing sequence of α_k and a nondecreasing sequence of β_k which converge to $\hat{\alpha}$ and $\hat{\beta}$, respectively, in the closure of $\mathcal{E}_{\text{non}}$, as $k \to \infty$. If the strict complementarity condition fails at either $\hat{\alpha}$ or $\hat{\beta}$, then $\hat{\alpha}$ or $\hat{\beta}$ must be a transition point in $\operatorname{int}(\mathcal{E})$.

3.3 On the identification of a transition point

Algorithm 1 relies on the existence of a strictly complementary solution at a given initial point $\bar{\epsilon}$. However, both a nonlinearity interval and a transition point might coexist with the failure of the strict complementarity condition, see problem (8), and thus Lemma 4 is no longer applicable.

Under both the primal and dual nondegeneracy conditions, which imply a unique primal-dual optimal solution, we present an alternative approach to check the existence of a transition point. To that end, we evaluate the higher-order derivatives of the Lagrange multipliers associated with a nonlinear optimization reformulation

Algorithm 1 Computation of a nonlinearity interval

Input $\bar{\epsilon}$

Set $\alpha_1 = \bar{\epsilon}$, $\alpha_0 = -\infty$, $k = 1$

while $\alpha_k \neq \alpha_{k-1}$ **do** ▷ Move backwards
    Compute a strictly complementary solution $\big(x^*(\alpha_k); y^*(\alpha_k); s^*(\alpha_k)\big)$ and $\delta(\alpha_k)$
    Set $\delta = 2\delta(\alpha_k)$
    repeat ▷ Connectivity subroutine
        Set $\delta = \delta/2$
        Solve the minimization auxiliary problem in (15) to compute $\alpha(\delta)$
    until $\alpha(\delta)$ and α_k are in the same connected component
    Set $k = k+1$, $\alpha_k = \alpha(\delta)$
end while
return $\hat{\alpha} = \alpha_k$ and its associated optimal solution $\big(x^*(\hat{\alpha}); y^*(\hat{\alpha}); s^*(\hat{\alpha})\big)$

Set $\beta_1 = \bar{\epsilon}$, $\beta_0 = \infty$, $k = 1$

while $\beta_k \neq \beta_{k-1}$ **do** ▷ Move forwards
    Compute a strictly complementary solution $\big(x^*(\beta_k); y^*(\beta_k); s^*(\beta_k)\big)$ and $\delta(\beta_k)$
    Set $\delta = 2\delta(\beta_k)$
    repeat ▷ Connectivity subroutine
        Set $\delta = \delta/2$
        Solve the maximization auxiliary problem in (15) to compute $\beta(\delta)$
    until $\beta(\delta)$ and β_k are in the same connected component
    Set $k = k+1$, $\beta_k = \beta(\delta)$
end while
return $\hat{\beta} = \beta_k$ and its associated optimal solution $\big(x^*(\hat{\beta}); y^*(\hat{\beta}); s^*(\hat{\beta})\big)$

if $\hat{\alpha} < \bar{\epsilon} < \hat{\beta}$ **then** ▷ $\bar{\epsilon}$ belongs to a nonlinearity interval.
    $(\hat{\alpha}, \hat{\beta})$ is a subinterval of the nonlinearity interval containing $\bar{\epsilon}$
else ▷ $\bar{\epsilon}$ might be a transition point.
    $\bar{\epsilon}$ is a singleton invariancy set at which the conditions (16) fail.
end if

of (D_ϵ). Obviously, we assume the failure of the strict complementarity condition, since otherwise we would have a nonlinearity interval by Lemma 4.

From this point on, we fix $\bar{\epsilon}$ and assume that both the primal and dual nondegeneracy conditions hold at $\bar{\epsilon}$, i.e., there exists a unique optimal solution $\big(x^*(\bar{\epsilon}); y^*(\bar{\epsilon}); s^*(\bar{\epsilon})\big)$ which is both primal and dual nondegenerate. Further, we define

$$\bar{\pi} := (\bar{\mathcal{B}}, \bar{\mathcal{N}}, \bar{\mathcal{R}}, \bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2, \bar{\mathcal{T}}_3) = \pi(\bar{\epsilon}).$$

Nonlinear reformulation As shown in [27], the unique optimal solution of ($P_{\bar{\epsilon}}$) can be obtained by solving the following nonlinear optimization problem at $\bar{\epsilon}$:

$$\begin{aligned}
(\mathrm{PN}_\epsilon) \quad \min \quad & \sum_{i \in \bar{\mathcal{B}} \cup \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2} (c^i + \epsilon \bar{c}^i)^T \nu^i \\
\text{s.t.} \quad & \sum_{i \in \bar{\mathcal{B}} \cup \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2} A_i \nu^i = b, \\
& (\nu^i)^T R^i \nu^i = 0, \qquad i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2, \\
& \nu \in \bar{\mathcal{V}},
\end{aligned}$$

where $\nu^i := (\nu_1^i; \nu_{2:n_i}^i) \in \mathbb{R}^{n_i}$ for $i \in \bar{\mathcal{B}} \cup \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2$, $\bar{\mathcal{V}}$ is a nonempty open convex cone given by

$$\bar{\mathcal{V}} := \big\{ \nu \mid \nu_1^i > 0, \ i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2, \quad \nu^i \in \mathrm{int}(\mathbb{L}_+^{n_i}), \ i \in \bar{\mathcal{B}} \big\},$$

and R^i is defined in (1). Since ($\mathrm{P}_{\bar\epsilon}$) has a unique optimal solution, ($\mathrm{PN}_{\bar\epsilon}$) has a unique globally optimal solution. In a similar manner, the unique optimal solution of ($\mathrm{D}_{\bar\epsilon}$) can be retrieved from a globally optimal solution of (DN_ϵ) at $\bar\epsilon$:

$$
\begin{array}{llll}
(\mathrm{DN}_\epsilon) & \min & -b^T w & \\
& \text{s.t.} & A_i^T w = c^i + \epsilon \bar c^i, & i \in \bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2, \\
& & A_i^T w + z^i = c^i + \epsilon \bar c^i, & i \in \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3, \\
& & (z^i)^T R^i z^i = 0, & i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3, \\
& & z \in \bar{\mathcal{W}}, &
\end{array}
$$

where $w \in \mathbb{R}^m$, $z^i := (z_1^i; z_{2:n_i}^i) \in \mathbb{R}^{n_i}$ for $i \in \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3$, and $\bar{\mathcal{W}}$ is a nonempty open convex cone defined as

$$
\bar{\mathcal{W}} := \big\{ z \mid z_1^i > 0,\ i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3,\quad z^i \in \mathrm{int}(\mathbb{L}_+^{n_i}),\ i \in \bar{\mathcal{N}} \big\}.
$$

Analogously, ($\mathrm{DN}_{\bar\epsilon}$) has a unique globally optimal solution because ($\mathrm{D}_{\bar\epsilon}$) has a unique optimal solution. For brevity, we only consider (DN_ϵ) from this point on.

Let us define

$$
\bar{\mathcal{I}} := \bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2 \cup \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3.
$$

The Lagrange multipliers associated with the constraints in (DN_ϵ) are denoted by u^i for $i \in \bar{\mathcal{I}}$ and $v \in \mathbb{R}^{|\bar{\mathcal{R}}|+|\bar{\mathcal{T}}_3|}$, respectively. Further, the concatenation of the column vectors z^i for $i \in \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3$ and the concatenation of the column vectors u^i for $i \in \bar{\mathcal{I}}$ are denoted by z and u, respectively. The first-order optimality conditions for (DN_ϵ) are given by

$$
\begin{array}{rll}
-(A^i)_{\bar{\mathcal{I}}} u = b, & & \\
-u^i - 2v_i R^i z^i = 0, & i \in \bar{\mathcal{R}}, & \\
-u^i = 0, & i \in \bar{\mathcal{N}}, & \\
-u^i - 2v_i R^i z^i = 0, & i \in \bar{\mathcal{T}}_3, & \\
(A^i)^T w = c^i + \epsilon \bar c^i, & i \in \bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2, & \\
(A^i)^T w + z^i = c^i + \epsilon \bar c^i, & i \in \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3, & \\
(z^i)^T R^i z^i = 0, & i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3, & \\
z \in \bar{\mathcal{W}}. & &
\end{array}
\tag{17}
$$

For the unique globally optimal solution $\big(w^*(\bar\epsilon); z^*(\bar\epsilon)\big)$ with $z^*(\bar\epsilon) \in \bar{\mathcal{W}}$, there exist unique [27, Lemma 3.2] Lagrange multipliers $u^*(\bar\epsilon)$ and $v^*(\bar\epsilon)$, such that $\big(w^*(\bar\epsilon); z^*(\bar\epsilon); u^*(\bar\epsilon); v^*(\bar\epsilon)\big)$ satisfies the first-order optimality conditions (17).

The set of solutions to (17) can be represented by $G\big((w; z; u; v), \epsilon\big) = 0, z \in \bar{\mathcal{W}}$, where the mapping $G : \mathbb{R}^{\bar n_c} \times \mathbb{R} \to \mathbb{R}^{\bar n_c}$ is defined as

$$
G\big((w; z; u; v), \epsilon\big) := \begin{pmatrix}
-(A^i)_{\bar{\mathcal{I}}} u - b & \\
-u^i - 2v_i R^i z^i & i \in \bar{\mathcal{R}} \\
-u^i & i \in \bar{\mathcal{N}} \\
-u^i - 2v_i R^i z^i & i \in \bar{\mathcal{T}}_3 \\
(A^i)^T w - c^i - \epsilon \bar c^i & i \in \bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2 \\
(A^i)^T w + z^i - c^i - \epsilon \bar c^i & i \in \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3 \\
(z^i)^T R^i z^i & i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3
\end{pmatrix},
$$

and

$$
\bar n_c := \sum_{i=1}^{p} n_i + \sum_{i \in \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3} n_i + |\bar{\mathcal{R}}| + |\bar{\mathcal{T}}_3| + m.
$$

The following theorem is in order.

Theorem 3 (Lemmas 3.2, 3.3, and 3.5 in [27]) . *The Jacobian ∇G is nonsingular at $\big((w^*(\bar\epsilon); z^*(\bar\epsilon); u^*(\bar\epsilon); v^*(\bar\epsilon)), \bar\epsilon\big)$ if both the primal and dual nondegeneracy conditions hold at $\bar\epsilon$.*

Stability of primal-dual nondegeneracy The nonsingularity of the Jacobian $\nabla G\big((w^*(\bar\epsilon); z^*(\bar\epsilon); u^*(\bar\epsilon); v^*(\bar\epsilon)), \bar\epsilon\big)$ guarantees quadratic convergence of Newton's method to the unique optimal solution of ($\mathrm{P}_{\bar\epsilon}$) and ($\mathrm{D}_{\bar\epsilon}$) [27, Theorem 3.9]. Under both the primal and dual nondegeneracy conditions at $\bar\epsilon$, the uniqueness of an optimal solution $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ is not only guaranteed at $\bar\epsilon$ but also on a neighborhood of $\bar\epsilon$.

Lemma 8 *Both the primal and dual nondegeneracy conditions hold at $\bar\epsilon$, if and only if, they hold on an open neighborhood of $\bar\epsilon$.*

Proof We provide an illustrative proof on the basis of the primal-dual nondegeneracy conditions. Assume that both the primal and the dual nondegeneracy conditions hold at $\bar\epsilon$. Then there exists a unique optimal solution $\big(x^*(\bar\epsilon); y^*(\bar\epsilon); s^*(\bar\epsilon)\big)$ and thus, by Lemma 2, both $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ are continuous at $\bar\epsilon$. Therefore, for every $\epsilon_k \to \bar\epsilon$ there exists a sequence of optimal solutions $\big(x(\epsilon_k); y(\epsilon_k); s(\epsilon_k)\big)$ converging to $\big(x^*(\bar\epsilon); y^*(\bar\epsilon); s^*(\bar\epsilon)\big)$.

Given a fixed ϵ and a corresponding optimal solution $\big(x(\epsilon); y(\epsilon); s(\epsilon)\big)$, let us define

$$\begin{aligned}
\mathcal{I}(\bar\epsilon,\epsilon) &:= \big\{i \in \bar{\mathcal{T}}_1 \mid x_1^i(\epsilon) > \|x_{2:n_i}^i(\epsilon)\|\big\},\\
\mathcal{I}'(\bar\epsilon,\epsilon) &:= \big\{i \in \bar{\mathcal{T}}_1 \mid x_1^i(\epsilon) = \|x_{2:n_i}^i(\epsilon)\|, x_1^i(\epsilon) > 0\big\},\\
\mathcal{I}''(\bar\epsilon,\epsilon) &:= \big\{i \in \bar{\mathcal{T}}_1 \mid s_1^i(\epsilon) > \|s_{2:n_i}^i(\epsilon)\|\big\},\\
\mathcal{I}'''(\bar\epsilon,\epsilon) &:= \big\{i \in \bar{\mathcal{T}}_1 \mid s_1^i(\epsilon) = \|s_{2:n_i}^i(\epsilon)\|, s_1^i(\epsilon) > 0\big\},\\
\mathcal{J}(\bar\epsilon,\epsilon) &:= \big\{i \in \bar{\mathcal{T}}_2 \mid x_1^i(\epsilon) > \|x_{2:n_i}^i(\epsilon)\|\big\},\\
\mathcal{J}'(\bar\epsilon,\epsilon) &:= \big\{i \in \bar{\mathcal{T}}_2 \mid s_1^i(\epsilon) = \|s_{2:n_i}^i(\epsilon)\|, s_1^i(\epsilon) > 0\big\},\\
\mathcal{K}(\bar\epsilon,\epsilon) &:= \big\{i \in \bar{\mathcal{T}}_3 \mid x_1^i(\epsilon) = \|x_{2:n_i}^i(\epsilon)\|, x_1^i(\epsilon) > 0\big\},\\
\mathcal{K}'(\bar\epsilon,\epsilon) &:= \big\{i \in \bar{\mathcal{T}}_3 \mid s_1^i(\epsilon) > \|s_{2:n_i}^i(\epsilon)\|\big\}.
\end{aligned}$$

When ϵ is sufficiently close to $\bar\epsilon$, it follows from the continuity of the primal optimal set mapping at $\bar\epsilon$ that

$$\begin{aligned}
x^i(\epsilon) \in \mathrm{int}(\mathbb{L}_+^{n_i}) &\iff i \in \bar{\mathcal{B}} \cup \mathcal{I}(\bar\epsilon,\epsilon) \cup \mathcal{J}(\bar\epsilon,\epsilon),\\
x^i(\epsilon) \in \mathrm{bd}(\mathbb{L}_+^{n_i}) \setminus \{0\} &\iff i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2 \cup \mathcal{I}'(\bar\epsilon,\epsilon) \cup \mathcal{K}(\bar\epsilon,\epsilon) \setminus \mathcal{J}(\bar\epsilon,\epsilon).
\end{aligned} \tag{18}$$

Analogously, with a close enough ϵ, the continuity of the dual optimal set mapping at $\bar\epsilon$ implies

$$\begin{aligned}
s^i(\epsilon) \in \mathrm{bd}(\mathbb{L}_+^{n_i}) \setminus \{0\} &\iff i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3 \cup \mathcal{I}'''(\bar\epsilon,\epsilon) \cup \mathcal{J}'(\bar\epsilon,\epsilon) \setminus \mathcal{K}'(\bar\epsilon,\epsilon),\\
s^i(\epsilon) = 0 &\iff i \in \bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2 \setminus \big\{\mathcal{I}''(\bar\epsilon,\epsilon) \cup \mathcal{I}'''(\bar\epsilon,\epsilon) \cup \mathcal{J}'(\bar\epsilon,\epsilon)\big\}.
\end{aligned} \tag{19}$$

In what follows, we show that $\big(x(\epsilon); y(\epsilon); s(\epsilon)\big)$ is primal-dual nondegenerate when ϵ belongs to a sufficiently small neighborhood of $\bar\epsilon$.

Primal nondegeneracy Recall from the primal nondegeneracy condition at $\bar\epsilon$ that

$$\Big(\big(A^i \bar P^{*i}(\bar\epsilon)\big)_{\bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2},\ (A^i)_{\bar{\mathcal{B}}}\Big) \tag{20}$$

has full row rank. By the continuity of the optimal set mapping at $\bar\epsilon$ and the perturbation theory of invariant subspaces [36, Theorem 4.11][5], $\bar P^i(\epsilon)$ stays near

[5] The perturbation theory of invariant subspaces states that the eigenspace associated with the cluster of positive eigenvalues of $L\big(x^{*i}(\bar\epsilon)\big)$ stays near that of $L\big(x^i(\epsilon)\big)$.

$\bar{P}^{*i}(\bar{\epsilon})$ for $i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2$, where the columns of $\bar{P}^i(\epsilon) \in \mathbb{R}^{n_i \times n_i - 1}$ are normalized eigenvectors of the positive eigenvalues of $L\big((x^i(\epsilon)\big)$. Thus the matrix

$$\Big(\big(A^i \bar{P}^i(\epsilon)\big)_{\bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2},\ (A^i)_{\bar{\mathcal{B}}}\Big) \tag{21}$$

can be made arbitrary close to (20). Let us fix ϵ such that (21) is of full row rank, and (18) holds. Then, it is easy to verify that

$$\Big(\big(A^i \bar{P}^i(\epsilon)\big)_{\check{\mathcal{I}}},\ (A^i)_{\hat{\mathcal{I}}}\Big)$$

where

$$\check{\mathcal{I}} := \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_2 \cup \mathcal{I}'(\bar{\epsilon}, \epsilon) \cup \mathcal{K}(\bar{\epsilon}, \epsilon) \setminus \mathcal{J}(\bar{\epsilon}, \epsilon),$$
$$\hat{\mathcal{I}} := \bar{\mathcal{B}} \cup \mathcal{I}(\bar{\epsilon}, \epsilon) \cup \mathcal{J}(\bar{\epsilon}, \epsilon)$$

must have full row rank, since otherwise (21) would be rank deficient. This completes the proof for the primal nondegeneracy of $x(\epsilon)$.

Dual nondegeneracy The proof for the dual nondegeneracy condition is analogous. The dual nondegeneracy condition at $\bar{\epsilon}$ holds if

$$\Big(\big(A^i R^i s^{*i}(\bar{\epsilon})\big)_{\bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3},\ (A^i)_{\bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2}\Big)$$

has linearly independent columns. By continuity of the dual optimal set mapping, when ϵ is sufficiently close to $\bar{\epsilon}$, we have $s_1^i(\epsilon) \neq 0$ for all $i \in \bar{\mathcal{T}}_3$ and $s^i(\epsilon)$ stays close to $s^{*i}(\bar{\epsilon})$. Let us fix ϵ such that

$$\Big(\big(A^i R^i s^i(\epsilon)\big)_{\bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3},\ (A^i)_{\bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2}\Big) \tag{22}$$

remains full column rank, and (19) holds. Then the matrix

$$\Big(\big(A^i R^i s^i(\epsilon)\big)_{\check{\mathcal{J}}},\ (A^i)_{\hat{\mathcal{J}}}\Big)$$

where

$$\check{\mathcal{J}} := \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3 \cup \mathcal{I}'''(\bar{\epsilon}, \epsilon) \cup \mathcal{J}'(\bar{\epsilon}, \epsilon) \setminus \mathcal{K}'(\bar{\epsilon}, \epsilon),$$
$$\hat{\mathcal{J}} := \bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2 \setminus \big\{\mathcal{I}''(\bar{\epsilon}, \epsilon) \cup \mathcal{I}'''(\bar{\epsilon}, \epsilon) \cup \mathcal{J}'(\bar{\epsilon}, \epsilon)\big\}$$

must have linearly independent columns, since otherwise the columns of (22) would be linearly dependent. This completes the proof for the dual nondegeneracy of $\big(y(\epsilon); s(\epsilon)\big)$. □

As a result of Lemma 8, there exists $\iota > 0$ such that both the primal and dual optimal set mappings are single-valued, and thus continuous on $(\bar{\epsilon} - \iota, \bar{\epsilon} + \iota)$ by Lemma 2. Given the unique optimal solution $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ for every $\epsilon \in (\bar{\epsilon} - \iota, \bar{\epsilon} + \iota)$, we define a continuous[6] mapping $\vartheta(\epsilon) : \mathbb{R} \to \mathbb{R}^{\bar{n}_c}$ on $(\bar{\epsilon} - \iota, \bar{\epsilon} + \iota)$ by

$$\vartheta(\epsilon) := \big(\hat{w}(\epsilon); \hat{z}(\epsilon); \hat{u}(\epsilon); \hat{v}(\epsilon)\big),$$

where

$$\begin{aligned}
&\hat{w}(\epsilon) := y^*(\epsilon), && \\
&\hat{z}^i(\epsilon) := s^{*i}(\epsilon), && i \in \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3, \\
&\hat{u}^i(\epsilon) := -x^{*i}(\epsilon), && i \in \bar{\mathcal{I}}, \\
&\hat{v}_i(\epsilon) := \frac{1}{2}\frac{x_1^{*i}(\epsilon)}{s_1^{*i}(\epsilon)}, && i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3.
\end{aligned}$$

It is easy to verify from (17) that $\vartheta(\bar{\epsilon})$ yields the unique globally optimal solution of $(\mathrm{DN}_{\bar{\epsilon}})$ along with its unique Lagrange multipliers [27, Section 3], i.e., we have

$$\vartheta(\bar{\epsilon}) = \big(w^*(\bar{\epsilon}); z^*(\bar{\epsilon}); u^*(\bar{\epsilon}); v^*(\bar{\epsilon})\big).$$

[6] Notice that the continuity of $\vartheta(\epsilon)$ follows from the continuity of $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ on $(\bar{\epsilon} - \iota, \bar{\epsilon} + \iota)$.

Furthermore, if $\pi(\epsilon)$ is constant on a neighborhood of $\bar{\epsilon}$, then we can prove that $\vartheta(\epsilon)$ is a unique real analytic mapping[7] such that $G(\vartheta(\epsilon), \epsilon) = 0$.

Lemma 9 *Suppose that both the primal and dual nondegeneracy conditions hold at $\bar{\epsilon}$, and the optimal partition is constant on a neighborhood of $\bar{\epsilon}$. Then there exists $0 < \varsigma \leq \iota$ such that $\vartheta(\epsilon)$ is a unique real analytic mapping on $(\bar{\epsilon} - \varsigma, \bar{\epsilon} + \varsigma)$ with $G(\vartheta(\epsilon), \epsilon) = 0$ for every $\epsilon \in (\bar{\epsilon} - \varsigma, \bar{\epsilon} + \varsigma)$.*

Proof Recall from the discussion after Lemma 8 that both $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ are single-valued and continuous on $(\bar{\epsilon} - \iota, \bar{\epsilon} + \iota)$. Since $\nabla G\big(\vartheta(\bar{\epsilon}), \bar{\epsilon}\big)$ is nonsingular, see Theorem 3, the analyticity of $\vartheta(\epsilon)$ follows from the analytic implicit function theorem [12, Theorem 10.2.4] and the invariancy of the optimal partition. More specifically, there exists $\varrho > 0$ and a unique real analytic mapping $\chi(\epsilon) = \big(w(\epsilon); z(\epsilon); u(\epsilon); v(\epsilon)\big)$ on $(\bar{\epsilon} - \varrho, \bar{\epsilon} + \varrho)$ such that $G(\chi(\epsilon), \epsilon) = 0$ for all $\epsilon \in (\bar{\epsilon} - \varrho, \bar{\epsilon} + \varrho)$ and $\chi(\bar{\epsilon}) = \big(w^*(\bar{\epsilon}); z^*(\bar{\epsilon}); u^*(\bar{\epsilon}); v^*(\bar{\epsilon})\big)$. Further, the invariancy of the optimal partition implies that $G(\vartheta(\epsilon), \epsilon) = 0$ on a small neighborhood of $\bar{\epsilon}$ [27, Section 3]. Therefore, by the continuity and uniqueness of $\chi(\epsilon)$, there exists $0 < \varsigma \leq \min\{\varrho, \iota\}$ such that the analytic mapping $\chi(\epsilon)$ and the continuous mapping $\vartheta(\epsilon)$ coincide on $(\bar{\epsilon} - \varsigma, \bar{\epsilon} + \varsigma)$. This completes the proof. □

Consequently, the derivatives of $\chi(\epsilon)$ are analytic and well-defined on $(\bar{\epsilon} - \varrho, \bar{\epsilon} + \varrho)$, where ϱ is defined in Lemma 9. Further, when $\pi(\epsilon)$ is constant on $(\bar{\epsilon} - \varsigma, \bar{\epsilon} + \varsigma)$, $\vartheta(\epsilon)$ yields a real analytic mapping for the unique globally optimal solution of (DN_ϵ) and its Lagrange multipliers on $(\bar{\epsilon} - \varsigma, \bar{\epsilon} + \varsigma)$.

Computation of the higher-order derivatives The continuity of $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ at $\bar{\epsilon}$ yields the existence of a neighborhood around $\bar{\epsilon}$ on which

$$\bar{\mathcal{B}} \subseteq \mathcal{B}(\epsilon), \quad \bar{\mathcal{N}} \subseteq \mathcal{N}(\epsilon), \quad \bar{\mathcal{R}} \subseteq \mathcal{R}(\epsilon) \tag{23}$$

for every ϵ in the neighborhood, see also Lemma 4. Hence, in order to identify a transition point, we only need to know how the index sets $\mathcal{T}_1(\epsilon)$, $\mathcal{T}_2(\epsilon)$, and $\mathcal{T}_3(\epsilon)$ change near $\bar{\epsilon}$. This can be done by evaluating the higher-order derivatives of the Lagrange multipliers given by $\chi(\epsilon)$ at $\bar{\epsilon}$, as stated in Theorem 4.

Theorem 4 *Suppose that both the primal and dual nondegeneracy conditions hold at $\bar{\epsilon}$. Then $\bar{\epsilon}$ belongs to a nonlinearity interval, if and only if*

$$\begin{aligned} \big(u_j^i(\epsilon)\big)^{(k)}\big|_{\epsilon=\bar{\epsilon}} &= 0, \quad \forall i \in \bar{\mathcal{T}}_1, \forall j = 1, \ldots, n_i, \forall k, \\ \big((u_1^i(\epsilon))^2 - \|u_{2:n_i}^i(\epsilon)\|^2\big)^{(k)}\big|_{\epsilon=\bar{\epsilon}} &= 0, \quad \forall i \in \bar{\mathcal{T}}_2, \forall k, \\ \big(v_i(\epsilon)\big)^{(k)}\big|_{\epsilon=\bar{\epsilon}} &= 0, \quad \forall i \in \bar{\mathcal{T}}_3, \forall k, \end{aligned} \tag{24}$$

where $\big(u(\epsilon); v(\epsilon)\big)$ is given by the analytic mapping $\chi(\epsilon)$ and $(.)^{(k)}$ denotes the k^{th}-order derivative w.r.t. ϵ.

[7] On an open set $U \subseteq \mathbb{R}$, a mapping $f(x)$ is real analytic if for any given $x_0 \in U$

$$f(x) = \sum_{k=0}^{\infty} \frac{\big(f(x_0)\big)^{(k)}}{k!}(x - x_0)^k$$

for all x in a neighborhood of x_0. See [12, Chapter IX] for further properties of an analytic mapping.

Proof $\Rightarrow$ Recall from Lemma 9 that for the analytic mapping $\chi(\epsilon)$ we have

$$\begin{aligned} u^i(\bar\epsilon) &= 0, \quad \forall i \in \bar{\mathcal{T}}_1, \\ (u_1^i(\bar\epsilon))^2 - \|u_{2:n_i}^i(\bar\epsilon)\|^2 &= 0, \quad \forall i \in \bar{\mathcal{T}}_2, \\ v_i(\bar\epsilon) &= 0, \quad \forall i \in \bar{\mathcal{T}}_3. \end{aligned} \tag{25}$$

Assume that $\pi(\epsilon) = \pi(\bar\epsilon)$ on $(\bar\epsilon - \varsigma, \bar\epsilon + \varsigma)$, i.e., $\bar\epsilon$ is not a transition point, where ς is defined in Lemma 9. Then, for every $\epsilon \in (\bar\epsilon - \varsigma, \bar\epsilon + \varsigma)$ there exists a unique optimal solution $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ such that

$$\begin{aligned} x^{*i}(\epsilon) &= 0, \quad \forall i \in \bar{\mathcal{T}}_1, \\ x_1^{*i}(\epsilon) - \|x_{2:n_i}^{*i}(\epsilon)\| &= 0, \quad \forall i \in \bar{\mathcal{T}}_2, \\ x^{*i}(\epsilon) &= 0, \quad \forall i \in \bar{\mathcal{T}}_3. \end{aligned} \tag{26}$$

In the sequel, from the coincidence of $\chi(\epsilon)$ and $\vartheta(\epsilon)$ on $(\bar\epsilon - \varsigma, \bar\epsilon + \varsigma)$ and (26) we obtain

$$\begin{aligned} u^i(\epsilon) &= 0, \quad \forall i \in \bar{\mathcal{T}}_1, \\ (u_1^i(\epsilon))^2 - \|u_{2:n_i}^i(\epsilon)\|^2 &= 0, \quad \forall i \in \bar{\mathcal{T}}_2, \\ v_i(\epsilon) &= 0, \quad \forall i \in \bar{\mathcal{T}}_3 \end{aligned} \tag{27}$$

for every $\epsilon \in (\bar\epsilon - \varsigma, \bar\epsilon + \varsigma)$, which confirm (24).

$\Leftarrow$ Let all the higher-order derivatives in (24) be equal to zero. Then the analyticity of $\chi(\epsilon)$ on $(\bar\epsilon - \varrho, \bar\epsilon + \varrho)$, where ϱ is defined in Lemma 9, and (25) imply (27) for every $\epsilon \in (\bar\epsilon - \varrho, \bar\epsilon + \varrho)$. Therefore, if $\epsilon \in (\bar\epsilon - \varrho, \bar\epsilon + \varrho)$ is so close to $\bar\epsilon$ that (23) holds, then by (27) and the continuity of $\chi(\epsilon)$ at $\bar\epsilon$ there exists[8] a unique optimal solution $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ such that (26) holds, and thus $\pi(\epsilon) = \pi(\bar\epsilon)$. The proof is complete. $\square$

Under the primal and dual nondegeneracy conditions, Theorem 4 provides a complete characterization, in terms of higher-order derivatives, for the identification of a transition point. The higher-order derivatives of $\chi(\epsilon)$ can be computed by

$$\begin{aligned} \big(\chi(\epsilon)\big)'|_{\epsilon=\bar\epsilon} &:= \nabla G^{-1}\big(\chi(\bar\epsilon), \bar\epsilon\big) \begin{pmatrix} 0 & \\ \bar c^i & i \in \bar{\mathcal{B}} \cup \bar{\mathcal{T}}_1 \cup \bar{\mathcal{T}}_2 \\ \bar c^i & i \in \bar{\mathcal{R}} \cup \bar{\mathcal{N}} \cup \bar{\mathcal{T}}_3 \\ 0 & \end{pmatrix}, \\ \big(\chi(\epsilon)\big)^{(k)}|_{\epsilon=\bar\epsilon} &:= \nabla G^{-1}\big(\chi(\bar\epsilon), \bar\epsilon\big)\eta_k(\bar\epsilon), \qquad k > 1, \end{aligned} \tag{28}$$

where

$$\eta_k(\bar\epsilon) := \begin{pmatrix} 0 & \\ 2\sum_{j=1}^{k-1} \big(v_i(\epsilon)\big)^{(j)}|_{\epsilon=\bar\epsilon} R^i \big(z^i(\epsilon)\big)^{(k-j)}|_{\epsilon=\bar\epsilon} & i \in \bar{\mathcal{R}} \\ 0 & \\ 2\sum_{j=1}^{k-1} \big(v_i(\epsilon)\big)^{(j)}|_{\epsilon=\bar\epsilon} R^i \big(z^i(\epsilon)\big)^{(k-j)}|_{\epsilon=\bar\epsilon} & i \in \bar{\mathcal{T}}_3 \\ 0 & \\ -\sum_{j=1}^{k-1} \Big(\big(z^i(\epsilon)\big)^{(j)}|_{\epsilon=\bar\epsilon}\Big)^T R^i \big(z^i(\epsilon)\big)^{(k-j)}|_{\epsilon=\bar\epsilon} & i \in \bar{\mathcal{R}} \cup \bar{\mathcal{T}}_3 \end{pmatrix}.$$

[8] In fact, using (23) and (27) we can generate an optimal solution $\big(x^*(\epsilon); y^*(\epsilon); s^*(\epsilon)\big)$ for (P_ϵ) and (D_ϵ), see [27, Section 3] for details, which then proves to be unique for every $\epsilon \in (\bar\epsilon - \varrho, \bar\epsilon + \varrho)$.

4 Numerical results

In this section, we numerically evaluate the convergence of the boundaries generated by Algorithm 1, and the magnitude of the derivatives introduced in Section 3.3. For the simplicity of computation, we invoke Algorithm 1 without the connectivity subroutine. We will show that on the given parametric SOCO problems, a subinterval of a nonlinearity interval is properly generated without a need for the connectivity subroutine.

We call the SQP algorithm included in the "fmincon" solver of MATLAB to solve the auxiliary problems in (15), and we employ the CVX optimization package [19, 20] to solve the SOCO problems (P_ϵ) and (D_ϵ). The outer loops of Algorithm 1 continue as long as $|\alpha_k - \alpha_{k-1}| > 10^{-7}$ and $|\beta_k - \beta_{k-1}| > 10^{-7}$ hold. Furthermore, in order to accurately compute the higher-order derivatives of the Lagrange multipliers at $\bar{\epsilon}$, we first round the near zero solutions obtained from CVX according to the optimal partition at $\bar{\epsilon}$. We then take a Newton step to solve $G\big((w; z; u; v), \bar{\epsilon}\big) = 0$ and thus correct the resulting errors, see [27, Section 3.1]. All the codes are run in MATLAB 9.6 environment on a MacBook Pro with Intel Core i5 CPU @ 2.3 GHz and 8GB of RAM.

4.1 Computation of a nonlinearity interval

We apply Algorithm 1 to the parametric SOCO problems (8) and (9) for the computation of a nonlinearity interval. Additionally, we consider solving the following parametric SOCO problem which fails the primal nondegeneracy condition on nonlinearity intervals:

$$
\begin{aligned}
\min \quad & -\epsilon x_2^1 - (1-\epsilon) x_3^1 \\
\text{s.t.} \quad & x_1^1 = 1, \\
& x_1^2 + x_1^3 = 2, \\
& x_2^1 - x_2^3 = 0, \\
& x_3^1 - x_3^3 = -1, \\
& x_1^1 \geq \sqrt{(x_2^1)^2 + (x_3^1)^2}, \\
& x_1^2 \geq 0, \\
& x_1^3 \geq \sqrt{(x_2^3)^2 + (x_3^3)^2},
\end{aligned}
\tag{29}
$$

for which $(-\infty, 0)$ and $(0, \infty)$ are the nonlinearity intervals and $\epsilon = 0$ is a transition point.

For the parametric SOCO problem (8), a one-time application of the auxiliary problems (15) at $\epsilon = \frac{1}{2}$ yields $[0.3947, 0.6053]$ as a subinterval of the nonlinearity interval containing $\epsilon = \frac{1}{2}$. By invoking Algorithm 1, we get the boundary points of the nonlinearity interval, up to our desired precision, in 26 and 25 iterations. The numerical results are demonstrated in Tables 1 and 2, where "Optim." and "Viol." denote the optimality and feasibility of α_k or β_k w.r.t. the auxiliary problems in (15), and $\sigma_{\min}(.)$ is the minimum singular value. One can observe from the entries of Tables 1 and 2 that α_k and β_k always remain within $(0, 1)$, even without the connectivity subroutine, and converge to 0 and 1 at almost linear rate. Notice

Table 1: The convergence of α_k for the parametric SOCO problem (8).

k	α_k	Optim.	Viol.	$\delta(\alpha_k)$	$\sigma_{\min}(\nabla F)$	$\lvert\alpha_k - \hat{\alpha}\rvert$
0	0.5			2.93E-01	1.69E-01	
1	0.394746	3.33E-16	2.22E-16	2.71E-01	1.57E-01	3.95E-01
2	0.288771	2.63E-09	2.22E-16	2.14E-01	1.25E-01	2.89E-01
3	0.192602	4.02E-16	1.11E-16	1.45E-01	8.71E-02	1.93E-01
4	0.117015	3.86E-16	1.45E-16	8.68E-02	5.34E-02	1.17E-01
5	0.065864	1.61E-15	1.11E-16	4.80E-02	2.99E-02	6.59E-02
22	8.10E-06	2.90E-12	2.69E-13	5.73E-06	3.62E-06	8.10E-06
23	4.78E-06	7.32E-12	2.90E-13	3.38E-06	2.14E-06	4.78E-06
24	2.80E-06	2.46E-11	2.61E-13	1.98E-06	1.25E-06	2.80E-06
25	1.38E-06	3.39E-11	2.17E-12	9.71E-07	6.14E-07	1.38E-06
26	2.78E-07	1.94E-10	2.68E-12	1.91E-07	1.21E-07	2.78E-07

Table 2: The convergence of β_k for the parametric SOCO problem (8).

k	β_k	Optim.	Viol.	$\delta(\beta_k)$	$\sigma_{\min}(\nabla F)$	$\lvert\beta_k - \hat{\beta}\rvert$
0	0.5			2.93E-01	1.69E-01	
1	0.605254	2.22E-16	3.52E-17	2.71E-01	1.56E-01	3.95E-01
2	0.711229	2.22E-16	1.11E-16	2.14E-01	1.25E-01	2.89E-01
3	0.807398	3.33E-16	1.11E-16	1.45E-01	8.65E-02	1.93E-01
4	0.882985	3.33E-16	1.11E-16	8.68E-02	5.31E-02	1.17E-01
5	0.934136	2.22E-16	1.35E-19	4.80E-02	2.98E-02	6.59E-02
22	0.999992	2.69E-13	2.69E-13	5.72E-06	3.62E-06	8.00E-06
23	0.999995	2.88E-13	2.88E-13	3.38E-06	2.14E-06	5.00E-06
24	0.999997	2.62E-13	2.62E-13	1.99E-06	1.26E-06	3.00E-06
25	0.999999	2.21E-12	2.21E-12	9.72E-07	6.15E-07	1.00E-06

that the continuity of $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ at $\epsilon = 0$ and $\epsilon = 1$ leads to accurate approximations of the transition points.

We can observe from Tables 3 and 4 that the bounds given by Algorithm 1 always stay within the corresponding nonlinearity interval, without a need for the connectivity subroutine. However, the convergence of β_k and α_k to their limit points becomes slow, resulting in a large number of iterations even in the presence of primal and dual nondegeneracy conditions. For instance, the sequence of β_k in Table 3 progresses rapidly at the beginning, but the convergence becomes slower than linear as β_k approaches $\frac{1}{2}$. The slow convergence in the vicinity of $\frac{1}{2}$ can be partly explained by the discontinuity of the dual optimal set mapping at $\epsilon = \frac{1}{2}$. In this case, both $\mathcal{P}^*(\epsilon)$ and $\mathcal{D}^*(\epsilon)$ are single-valued and thus continuous on the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, while for any sequence $\epsilon_k \to \frac{1}{2}$ we have $\liminf_{k\to\infty} \mathcal{D}^*(\epsilon_k) \cap \operatorname{ri}\left(\mathcal{D}^*(\frac{1}{2})\right) = \emptyset$. Analogously, the sequence of α_k in Table 4 converges slowly to 0, where the dual optimal set mapping fails to be continuous.

Table 3: The convergence of β_k for the parametric SOCO problem (9).

k	β_k	Optim.	Viol.	$\delta(\beta_k)$	$\sigma_{\min}(\nabla F)$	$\lvert\beta_k - \hat{\beta}\rvert$
0	0.25			1.10E-01	5.49E-02	
1	0.292079	7.77E-16	5.55E-17	8.05E-02	4.02E-02	2.08E-01
2	0.322001	1.55E-15	1.11E-16	6.10E-02	3.05E-02	1.78E-01
3	0.34434	3.34E-15	3.34E-15	4.77E-02	2.39E-02	1.56E-01
4	0.361647	1.36E-15	1.11E-16	3.82E-02	1.92E-02	1.38E-01
5	0.375447	1.25E-15	2.22E-16	3.13E-02	1.57E-02	1.25E-01
196	0.493475	1.79E-12	1.79E-12	9.03E-05	4.51E-05	6.53E-03
197	0.493507	1.94E-12	1.94E-12	8.94E-05	4.47E-05	6.49E-03
198	0.493539	1.38E-14	2.22E-16	8.85E-05	4.43E-05	6.46E-03
199	0.49357	5.73E-15	1.11E-16	8.77E-05	4.38E-05	6.43E-03
200	0.493601	9.49E-15	2.22E-16	8.68E-05	4.34E-05	6.40E-03

Table 4: The convergence of α_k for the parametric SOCO problem (29).

k	α_k	Optim.	Viol.	$\delta(\alpha_k)$	$\sigma_{\min}(\nabla F)$	$\lvert\alpha_k - \hat{\alpha}\rvert$
0	0.5			5.20E-02	5.12E-15	
1	0.481601	3.31E-09	2.22E-16	4.52E-02	5.57E-15	4.82E-01
2	0.465589	3.05E-09	7.78E-17	4.37E-02	5.70E-15	4.66E-01
3	0.450013	3.00E-09	1.68E-16	4.01E-02	9.82E-15	4.50E-01
4	0.435648	2.80E-09	1.77E-15	3.72E-02	6.49E-15	4.36E-01
5	0.422244	2.41E-09	1.31E-14	3.01E-02	5.16E-15	4.22E-01
196	0.070559	3.18E-13	1.95E-14	4.86E-04	5.84E-13	7.06E-02
197	0.07027	1.16E-13	1.16E-14	4.73E-04	1.83E-13	7.03E-02
198	0.069838	3.95E-01	8.69E-07	4.67E-04	2.23E-13	6.98E-02
199	0.069562	3.14E-13	1.11E-16	4.58E-04	9.50E-13	6.96E-02
200	0.069291	2.48E-13	2.57E-15	4.66E-04	2.18E-13	6.93E-02

4.2 Identification of a transition point

In order to illustrate the identification of a transition point, we apply the theoretical results in Section 3.3 to the following parametric SOCO problem:

$$\begin{array}{ll} \min & x_1^1 + (1-\epsilon)x_2^1 \\ \text{s.t.} & x_2^1 + x_2^2 = 0, \\ & x_3^1 + x_1^2 = -1, \\ & x_1^1 \geq \sqrt{(x_2^1)^2 + (x_3^1)^2}, \\ & x_1^2 \geq |x_2^2|, \end{array} \tag{30}$$

which has a singleton invariancy set at $\epsilon = 0$. The optimal partition at $\epsilon = 0$ is given by $\pi(0) = \big(\emptyset, \emptyset, \{1\}, (\emptyset, \emptyset, \{2\})\big)$, which indeed implies the failure of the strict complementarity condition at $\epsilon = 0$. However, one can observe that both the primal and dual nondegeneracy conditions hold at $\epsilon = 0$, and thus Theorem 4 is applicable. By computing the first-order derivative $\big(v(\epsilon)\big)'|_{\epsilon=0} = -\frac{1}{2}$ using the formulas in (28), we can conclude from Theorem 4 that $\epsilon = 0$ is a transition point. This transition point is adjacent to a nonlinearity interval $(1-\sqrt{2}, 0)$ and an invariancy interval $(0, 2)$.

If we change the objective function of (30) to $x_1^1 + (1-\epsilon)x_2^1 + \epsilon x_3^1$, then we obtain an invariancy interval $(-\infty, 1)$ around $\epsilon = 0$, where

$$\pi(\epsilon) = \big(\emptyset, \emptyset, \{1\}, (\emptyset, \emptyset, \{2\})\big), \qquad \forall \epsilon \in (-\infty, 1),$$

and both the primal and dual nondegeneracy conditions hold. The higher-order derivatives of $v(\epsilon)$ at $\epsilon = 0$ are given in Table 5, which stay reliably close to 0 up to the 10^{th}-order derivative. The entries of Table 5 also demonstrate the computation error when solving (28), which iteratively propagates as the order of derivative increases.

Table 5: The higher-order derivatives of $v(\epsilon)$ at $\epsilon = 0$.

k	1	2	3	4	5	6	7	8	9	10
$(v(\epsilon))^{(k)}\|_{\epsilon=0}$	-5.6E-17	-1.1E-16	-3.3E-16	-1.3E-15	-6.7E-15	-4.0E-14	-2.8E-13	-2.2E-12	-2.0E-11	-2.0E-10

5 Conclusions and future research

We studied the parametric analysis of a SOCO problem, where the objective function is perturbed along a fixed direction. We provided sufficient conditions for the existence of a nonlinearity interval and proved that the set of transition points is finite. Furthermore, using a counterexample, we showed that even continuity might fail on a nonlinearity interval. Under strict complementarity condition, we then presented a numerical procedure for the computation of a nonlinearity interval. When the strict complementarity condition fails, but the primal and dual nondegeneracy conditions hold, we showed how to identify a transition point from the higher-order derivatives of the Lagrange multipliers associated with (DN_ϵ). The numerical experiments showed that Algorithm 1 and Theorem 4 can be reliably used to check the existence of a nonlinearity interval and a transition point, respectively.

We present a few topics for future research:

- Algorithm 1 generates sequences of α_k and β_k, each may converge to a transition point. It is worth investigating the properties of this algorithmic map and the rate at which α_k or β_k converges to a transition point.
- Under uniqueness assumption the optimal set mapping is continuous, but it is not known yet if the mapping is smooth.
- It is worth investigating an upper bound on the number of points at which the optimal set mapping fails to be continuous.

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