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An Inexact Feasible Interior Point Method for Linear Optimization with High Adaptability to Quantum Computers

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Abstract

Exploring the opportunities offered by quantum computing to speed up the solution of challenging optimization problems is a hot research area. This paper applies Quantum Linear System Algorithms (QLSAs) to solve the Newton systems within Interior Point Methods (IPMs) and takes advantage of the provided quantum speed-up in solving Linear Optimization (LO) problems. Since QLSAs inherently produce inexact solutions, we can only use Inexact variants of IPMs. Existing IPMs with inexact Newton direction are infeasible methods because inexactness leads to infeasibility. Here, we propose an Inexact-Feasible IPM (IF-IPM) for LO problems in which a novel system is used to generate inexact but feasible steps. We show that this method has $\mathcal{O}(\sqrt{n}L)$ iteration complexity, analogous to the best exact IPMs. After analyzing the total time complexity of the proposed method with classical solvers such as the Conjugate Gradient method, we also discuss how QLSAs can be used to solve the proposed system efficiently in an Iterative Refinement scheme to find an exact solution without excessive time of QLSAs. The IF-IPM is implemented with both classical and quantum solvers to investigate its efficiency empirically.

1 Introduction

Recently, many scholars in different fields of knowledge are hardly working on building efficient quantum computers and using them to solve crucial real-world problems. Starting with Deutsch's method for determining a binary oracle is constant or balanced (Deutsch and Penrose, 1985), quantum computing shows exponential speed-up compared to conventional computers for some hard mathematical problems such as integer factorization problem (Shor, 1994) and unstructured search problem (Grover, 1996). Since mathematical optimization has widespread applications along with computational challenges, many researchers have attempted to develop quantum optimization methods in different directions from Quantum Approximation Optimization Algorithm (QAOA) for unconstrained binary optimization (Farhi et al., 2014) to Quantum Interior Point Method (QIPM) for linear optimization (LO) problems (Mohammadisiahroudi et al., 2021).

Specifically, QIPMs are the classical Interior Points Methods (IPMs) using Quantum Linear System Algorithms (QLSAs) to solve the Newton system. Since QLSAs are efficient for finding inexact solutions, the appropriate versions of IPMs for this purpose are Inexact IPMs, i.e., IPMs with inexact Newton steps. Since inexactness in the solution of the Newton system can lead to infeasibility, Inexact Infeasible IPM was applied to use the QLSAs efficiently (Mohammadisiahroudi et al., 2021). Motivated by efficient use of QLSA in IPMs, we develop an Inexact Feasible IPM (IF-IPM) using a novel system. The proposed IF-QIPM starts from a feasible point and remains in the feasible region even with an inexact solution of the proposed system. By IF-IPM, we efficiently use QLSA to accelerate solution of LO problems. First, we define the LO problem.

Definition 1 (Linear Optimization Problem: Standard Formulation)**.**

$$
\begin{aligned}
(P)\quad &\min\ c^T x, & \qquad (D)\quad &\max\ b^T y,\\
&\text{s.t. } Ax = b, & &\text{s.t. } A^T y + s = c,\\
&\qquad x \ge 0, & &\qquad s \ge 0,
\end{aligned}
$$

where vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and matrix $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = m$. The problem (P) is called primal problem and (D) called dual problem.

The set of feasible primal-dual solutions is defined as

$$\mathcal{PD} = \left\{(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n |\ Ax = b,\ A^T y + s = c,\ (x, s) \ge 0\right\}.$$

Then, the set of all feasible interior solutions is

$$\mathcal{PD}^0 = \{(x, y, s) \in \mathcal{PD} \mid (x, s) > 0\}.$$

By the Strong Duality theorem, all optimal solutions, if there exist any, belong to the set $\mathcal{PD}^*$ defined as

$$\mathcal{PD}^* = \left\{(x, y, s) \in \mathcal{PD} \mid x^T s = 0\right\}.$$

Let $\zeta \ge 0$, the set of ζ-optimal solutions can be defined as

$$\mathcal{PD}_\zeta = \left\{(x, y, s) \in \mathcal{PD} \;\middle|\; \frac{x^T s}{n} \le \zeta\right\}.$$

Dantzig's Simplex method was the first efficient algorithm to solve LO problems (Dantzig, 1990). In each iteration of Simplex methods, one pivots on matrix A augmented with vectors b and c and move from one vertex to another vertex converging to an optimal solution. Klee and Minty (1972) showed that Simplex algorithms have an exponential time complexity despite their practical efficiency. Khachiyan (1979) proposed the Ellipsoid method that solves LO problems with integer input data with a polynomial time complexity. However, the Ellipsoid method was not as efficient as simplex methods. Karmarkar (1984) developed a practically efficient Interior Point Method (IPM) for solving LO problems with polynomial time complexity. After that, many theoretically and practically efficient IPMs were developed (e.g., see Wright, 1997; Roos et al., 2005; Terlaky, 2013).

A feasible IPM reaches an optimal solution by traversing the interior of the feasible region (Roos et al., 2005). Generally, all IPMs reach an optimal solution by starting from an interior point and following the so-called central path (Roos et al., 2005). Most of them are primal-dual methods meaning they attempt to satisfy the optimality condition with primal and dual feasibility. To have IF-IPM, we use the scheme of primal-dual feasible interior point method. Let $\mu = \frac{x^T s}{n}$, the definition of the central path is

$$\mathcal{CP} = \left\{ (x, y, s) \in \mathcal{PD} \;\middle|\; x_i s_i = \mu \;\text{ for }\; i \in \{1, \dots, n\} \right\}.$$

For any $\theta \in [0, 1)$, a neighborhood of the central path can be defined as

$$\mathcal{N}(\theta) = \left\{ (x, y, s) \in \mathcal{PD}^0 \;\middle|\; \|XSe - \mu e\|_2 \leq \theta\mu \;\text{ for }\; i \in \{1, \dots, n\} \right\},$$

where X and S are diagonal matrices of x and s, respectively. As illustrated in Figure 1, the Newton step is made inside a neighborhood of the central path to decrease the optimality gap in each iteration of IPMs.

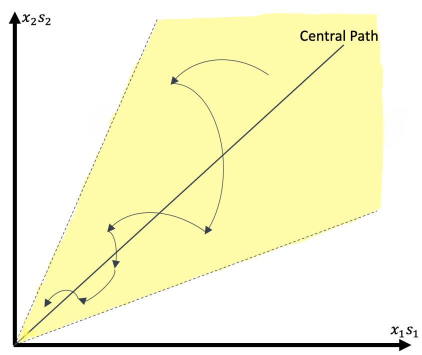

Figure 1: The iterations of IPMs

IPMs can be categorized into two groups: Feasible IPMs and Infeasible IPMs. A Feasible IPM (F-IPM) needs an initial feasible interior point to start the algorithm. They usually employ a self-dual embedding formulation of the LO problem where a feasible interior solution can be easily constructed Roos et al., 2005. Instead, Infeasible Interior Point Methods (I-IPMs) start with an infeasible but strictly positive solution. Theoretical analysis shows that the best time complexity of F-IPMs is $\mathcal{O}(\sqrt{n}L)$ where L is the binary length of the input data. On the other hand, the best time complexity of I-IPMs is $\mathcal{O}(nL)$. Although theoretical analysis shows that F-IPMs are faster than I-IPMs, both feasible and infeasible IPMs can solve LO problems efficiently in practice (Wright, 1997).

Recent studies have noticed the convergence of IPMs with inexact search directions since even classical solvers are not exact due to floating-point arithmetic errors. First, Mizuno and his colleagues did series of research on convergence of Inexact Infeasible IPMs (II-IPMs) (Mizuno and Jarre, 1999; Freund et al., 1999). The convergence of I-IPM proposed by Kojima et al. (1993) with inexact Newton step is proved by Baryamureeba and Steihaug (2006). Korzak (2000) also showed that it has a polynomial time complexity.

Several authors studied the use of Preconditioned Conjugate Gradient (PCG) method in II-IPM (Al-Jeiroudi and Gondzio, 2009; Monteiro and O'Neal, 2003). Al-Jeiroudi and Gondzio (2009) used I-IPM of (Wright, 1997) solving Augmented system by PCG method Monteiro and O'Neal (2003) applied the PCG method on the Normal Equation System (NES). Bellavia (1998) studied the convergence of the II-IPM for general convex optimization. Zhou and Toh (2004) also proved that the II-IPM converges to an optimal solution for the Semi-Definite Optimization (SDO) problems. The best bound for iterations of II-IPMs is $\mathcal{O}(n^2L)$ for LO problems. With the emergence of QIPMs, IPMs with inexact Newton step received more attention since QLSAs inherently find an inexact solution. Mohammadisiahroudi et al. (2021) and Augustino et al. (2021) used II-IPM framework to develop quantum IPMs for LO and SDO problems, respectively. In this paper, we introduce a new formulation of the Newton system for finding an inexact but feasible step and develop an IF-IPM using both QLSA and classical solvers such as the Conjugate Gradient method. We prove the convergence of the proposed IF-IPM. We also show the efficiency of the proposed algorithm through computational experiments.

- Section 2: proposing a novel system to produce inexact but feasible Newton step along and developing a short-step Inexact Feasible IPM.
- Section 5: adapting IF-IPM to self-dual embedding formulation of LO problems.
- Section 3: analyzing the novel system and its performance in classical IPMs.
- Section 4: using QLSA to solve the novel system and developing an Inexact Feasible IPM.

2 Inexact Feasible IPM

To speed up IPMs, we want to use QLSAs to solve the Newton system at each iteration of IPMs since QLSAs have better dependence on the size of the problem(Harrow et al., 2009). One approach to use QLSA efficiently is to develop an Inexact IPM. There are three choices of linear systems to calculate the Newton step: Augmented system, Normal Equation System, and full Newton system. Solving any of three systems inexactly leads to residuals in the primal and or dual feasibility equations. In this paper, we develop an inexact IPM to avoid the infeasibility caused by residuals. In essence, feasible IPMs have better complexity than infeasible IPMs. To avoid the extra cost of infeasible IPMs, we define a new system resulting a primal-dual feasible step using orthogonal vectors from orthogonal sub-spaces.

With this motivation, we utilize a short-step feasible IPM with the inexact Newton steps. For a feasible interior solution $(x^k, y^k, s^k) \in \mathcal{PD}^0$, the Newton system is defined as

$$\begin{aligned} A\Delta x^k &= 0, \\ A^T\Delta y^k + \Delta s^k &= 0, \\ X^k\Delta s^k + S^k\Delta x^k &= \beta\mu^k e - X^k s^k. \end{aligned} \tag{1}$$

Also, $\beta \in [0,1]$ is the centering parameter and $\mu^k = \frac{{x^k}^T s^k}{n}$.

Throughout the paper, we assume that matrix A has full row rank. By applying Gaussian elimination method on matrix A, we can always convert the LO problem to an LO problem that satisfies Assumption 1.

Assumption 1. *Matrix $A \in \mathbb{R}^{m\times n}$ has full row rank $m \leq n$.*

Let a_i be the i'th column of matrix A. We define set $B \subset \{1,\ldots,m\}$ as the index set of the first m linearly independent columns of A and $A_B = [a_i]_{i\in B}$. Then, matrix A_B is non-singular and A_B^{-1} is the inverse of A_B. By pivoting on matrix $A = \begin{bmatrix} A_B & A_N \end{bmatrix}$, we can construct matrix $\begin{bmatrix} I & A_B^{-1}A_N \end{bmatrix} \in \mathbb{R}^{m\times n}$.

We also define matrices $V \in \mathbb{R}^{n\times(n-m)}$ and $W \in \mathbb{R}^{n\times m}$ as follows

$$V = \begin{bmatrix} A_B^{-1}A_N \\ -I \end{bmatrix}, \qquad W = A^T.$$

Vector w_j is the j'th column of matrix W (or the j'th row of matrix A), and vector v_i denotes the i'th column of matrix V.

Lemma 1. *The vectors w_i are basis for range space of A^T (or row space of A) and the vectors v_j are basis for null space of A. Furthermore, for any $i \in \{1,\ldots,m\}$ and any $j \in \{1,\ldots,n-m\}$, we have $w_i^T v_j = 0$.*

Proof. Based on Assumption 1, A has full row rank and A^T has full column rank. Since the vectors w_i are rows of A and columns of A^T, the vectors w_i are basis for row space of A and range space of A^T. On the other hand, matrix V has full column rank because the vectors v_j are linearly independent. Also, we have

$$AV = \begin{bmatrix} A_B & A_N \end{bmatrix} \begin{bmatrix} A_B^{-1}A_N \\ -I \end{bmatrix} = A_N - A_N = 0.$$

We can conclude that the vectors v_j are basis for null space of A and $w_i^T v_j = 0$ for any $i \in \{1,\ldots,m\}$ and any $j \in \{1,\ldots,n-m\}$. □

Using Lemma 1, we define linear system of equations (2).

$$\Delta x^k = \sum_{i=1}^{n-m} \lambda_i^k v_i, \tag{2a}$$

$$\Delta s^k = -\sum_{j=1}^{m} \Delta y_j^k w_j, \tag{2b}$$

$$X^k\Delta s^k + S^k\Delta x^k = \beta\mu^k e - X^k s^k. \tag{2c}$$

Here, vectors λ^k and Δy^k are unknown. Substituting Δx^k defined in equation (2a) and Δs^k defined in equation (2b) in equation (2c) leading to

$$-X^k A^T \Delta y^k + S^k V \lambda^k = \beta \mu^k e - X^k s^k. \tag{3}$$

One can rewrite equation (3) as $M^k z^k = \sigma^k$ where

$$M^k = \begin{bmatrix} -X^k A^T & S^k V \end{bmatrix}, \qquad z^k = \begin{pmatrix} \Delta y^k \\ \lambda^k \end{pmatrix}, \qquad \sigma^k = \beta \mu^k e - X^k s^k.$$

We call this new system "Orthogonal Subspaces System" (OSS) which has n equalities, $n-m$ variables λ_j^k, and m variables Δy_i^k.

Lemma 2. *Linear systems* (1) *and* (2) *are equivalent.*

The proof of Lemma 2 is trivial due to the construction of the OSS. An immediate result of Lemma 2 is that the OSS has unique solution. We can also show directly that the OSS has unique solution as Lemma 3.

Lemma 3. *If $(x^k, s^k, y^k) \in \mathcal{PD}^0$, then the System (2) has a unique solution.*

Proof. It is enough to show that M is non singular and its columns are linearly independent. We want to show that if $M\alpha = 0$ then $\alpha = 0$. We now that columns of V are linearly independent then the columns of $M_2 = S^k V$ are linearly independent. Similarly the columns of $M_1 = -X^k A^T$ are linearly independent. Now let $[M_1, M_2] \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$ and m_i, then $(M_1\alpha_1)^T M\alpha = 0$. So we have

$$\begin{aligned} (M_1\alpha_1)^T M\alpha &= 0 \\ (M_1\alpha_1)^T M_1\alpha_1 + (M_1\alpha_1)^T M_2\alpha_2 &= 0 \\ (M_1\alpha_1)^T M_1\alpha_1 &= 0 \\ \alpha_1 &= 0 \end{aligned}$$

Since $(M_2\alpha_2)^T M\alpha = 0$, with similar approach we have $\alpha 2 = 0$. We can conclude that $\alpha = 0$ and M is full rank and non singular. □

Let $(\widetilde{\lambda^k}, \widetilde{\Delta y^k})$ be an inexact solution of the system 3. Then, we can easily find $\widetilde{\Delta x^k}$ and $\widetilde{\Delta s^k}$ by using the first and the second equations of System 2. For the solution $(\widetilde{\Delta x^k}, \widetilde{\Delta s^k}, \widetilde{\Delta y^k})$, we have

$$\begin{aligned} \widetilde{\Delta x^k} &= \sum_{j=1}^{n-m} \widetilde{\lambda_j^k} v_j \\ \widetilde{\Delta s^k} &= -\sum_{i=1}^{m} \widetilde{\Delta y_i^k} w_i, \\ X^k \widetilde{\Delta s^k} + S^k \widetilde{\Delta x^k} &= \beta \mu^k e - X^k s^k + r^k, \end{aligned} \tag{4}$$

where $r^k = S^k V(\widetilde{\lambda^k} - \lambda^k) - X^k A^T(\widetilde{\Delta y^k} - \Delta y^k)$. Regardless of the error of solution, we have $\widetilde{\Delta x^k} \in \text{Null}(A)$ and $\widetilde{\Delta s^k} \in \text{Range}(A^T)$. Thus, for any step length $\alpha \in (0, 1]$, we have

$$\begin{aligned} A(x^k + \alpha\widetilde{\Delta x^k}) &= 0 \\ A^T(y^k + \alpha\widetilde{\Delta y^k}) + (s^k + \alpha\widetilde{\Delta s^k}) &= 0. \end{aligned} \tag{5}$$

With Assumption 2 for residual of the solution, we can develop a convergent IF-IPM.

Assumption 2. *Let assume in all iterations of IF-IPM, we have*

$$\|r^k\|_2 \leq \eta\mu^k,$$

where η is an enforcing parameter with $0 \leq \eta \leq 1$.

Let ϵ^k be the target error of the solution such that

$$\|(\widetilde{\lambda^k} - \lambda^k, \widetilde{\Delta y^k} - \Delta y^k)\|_2 \leq \epsilon^k.$$

To satisfy the Assumption 2, we need

$$\epsilon^k \leq \eta\frac{\mu^k}{\|M^k\|_2}.$$

Algorithm 1 is a Short-step IF-IPM for solving LO problems using system 3.

Algorithm 1 Short-step IF-IPM

Choose $\zeta > 0$, $\eta = 0.1$, $\theta = 0.3$ and $\beta = (1 - \frac{0.3}{\sqrt{n}})$.
$k \leftarrow 0$
Choose initial feasible interior solution $(x^0, y^0, s^0) \in \mathcal{N}(\theta)$
while $(x^k, y^k, s^k) \notin \mathcal{PD}_\zeta$ **do**
    $\mu^k \leftarrow \frac{(x^k)^T s^k}{n}$
    $\epsilon^k \leftarrow \eta\frac{\mu^k}{\|M^k\|_2}$
    $(\lambda^k, \Delta y^k) \leftarrow$ **solve** System (3) with error bound ϵ^k
    $\Delta x^k = V\lambda^k$ and $\Delta s^k = -A^T\Delta y^k$
    $(x^{k+1}, y^{k+1}, s^{k+1}) \leftarrow (x^k, y^k, s^k) + (\Delta x^k, \Delta y^k, \Delta s^k)$
    $k \leftarrow k + 1$
return (x^k, y^k, s^k)

2.1 Polynomial Complexity of Short-step IF-IPM

To prove the convergence of IF-IPM, we show that μ_k which is a measure for the optimality gap decreases linearly in Theorem 1. Lemma 5 indicates that IF-IPM does not go out of the neighborhood of the central path with full step in each iteration. The main step is Theorem 1 showing that IF-IPM converges to a ζ-optimal solution after polynomial number of iterations. Finally, the complexity of IF-IPM to find an exact solution is calculated using Lemma 6. To do the proofs, we need some basic results in Lemma 4

Lemma 4. *Let the step $(\Delta x, \Delta y, \Delta s)$ calculated by the OSS system in each iteration of the IF-IPM. Then*

$$\Delta x^T \Delta s = 0 \tag{6a}$$

$$(x + \Delta x)^T (s + \Delta s) \leq (\beta + \frac{\eta}{\sqrt{n}}) x^T s \tag{6b}$$

$$(x + \Delta x)^T (s + \Delta s) \geq (\beta - \frac{\eta}{\sqrt{n}}) x^T s \tag{6c}$$

Proof. The first step is proving 6a. Based on Lemma 1, we have

$$\Delta x^T \Delta s = -\sum_{j=1}^{n-m} \sum_{i=1}^{m} \lambda_j \Delta y_i v_j^T w_i = 0.$$

To prove 6b, we have

$$\begin{aligned}
(x + \Delta x)^T (s + \Delta s) &= (x)^T s + (x)^T \Delta s + (s)^T \Delta x + (\Delta x)^T \Delta s && \text{(7a)}\\
&\leq (x)^T s + n\beta\mu - (x)^T s + \|r\|_1 && \text{(7b)}\\
&\leq n\beta\mu + \sqrt{n}\eta\mu && \text{(7c)}\\
&= (\beta + \frac{\eta}{\sqrt{n}})(x)^T s. && \text{(7d)}
\end{aligned}$$

Based on Lemma 3, we can use the last equation of System 1 in line (7b). Inequality (7c) comes from Assumption 2, and (7d) from the definition of μ. Similarly, we can show that

$$\begin{aligned}
(x + \Delta x)^T (s + \Delta s) &\geq (x)^T s + n\beta\mu - (x)^T s - \|r\|_1\\
&\geq (\beta - \frac{\eta}{\sqrt{n}}) x^T s
\end{aligned}$$

□

Now, we can show that the IF-IPM remains in the neighborhood of the central path in Lemma 5, by using results of Lemma 4.

Lemma 5. *Let $(x^0, s^0, y^0) \in \mathcal{N}(\theta)$ for a given $\theta \in [0, 1)$, then $(x^k, s^k, y^k) \in \mathcal{N}(\theta)$ for any $k \in \mathbb{N}$.*

Proof. It is enough to show that if $(x^k, s^k, y^k) \in \mathcal{N}(\theta)$, then we have

$$\begin{aligned}
Ax^{k+1} &= b, && \text{(8a)}\\
A^T y^{k+1} + s^{k+1} &= c, && \text{(8b)}\\
(x^{k+1}, s^{k+1}) &> 0, && \text{(8c)}\\
\|X^{k+1} S^{k+1} e - \mu^{k+1} e\|_2 &\leq \theta\mu^{k+1} \quad \text{for} \quad i \in \{1, \ldots, n\}. && \text{(8d)}
\end{aligned}$$

We can easily conclude the feasibility conditions (8a) and (8b) from equation (5). To prove (8d), first we show that $\|\Delta X^k \Delta S^k e\| \leq \frac{\theta^2+n(1-\beta)^2+\eta^2}{2^{3/2}(1-\theta)}\mu^k$. Let $D = (X^k)^{1/2}(S^k)^{-1/2}$, then we have

$$
\begin{aligned}
\|\Delta X^k \Delta S^k e\| &= \|(D^{-1}\Delta X^k)(D\Delta S^k)e\| && \text{(9a)}\\
&\leq 2^{-3/2}\|D^{-1}\Delta x^k + D\Delta s^k\|^2 && \text{(9b)}\\
&= 2^{-3/2}\|(X^k S^k)^{-1/2}(S^k \Delta x^k + X^k \Delta s^k)\|^2 && \text{(9c)}\\
&= 2^{-3/2}\|(X^k S^k)^{-1/2}(\beta\mu^k e - X^k S^k e + r^k)\|^2 && \text{(9d)}\\
&= 2^{-3/2}\sum_{i=1}^{n}\frac{(\beta\mu^k - x_i^k s_i^k + r_i^k)^2}{x_i^k s_i^k} && \text{(9e)}\\
&\leq 2^{-3/2}\frac{\|\beta\mu^k e - X^k S^k e + r^k\|^2}{\min_i x_i^k s_i^k} && \text{(9f)}\\
&\leq 2^{-3/2}\frac{\|\beta\mu^k e - X^k S^k e\|^2 + \|r^k\|^2}{(1-\theta)\mu^k} && \text{(9g)}\\
&\leq 2^{-3/2}\frac{\|(X^k S^k e - \mu^k e) + (1-\beta)\mu^k e\|^2 + (\eta\mu^k)^2}{(1-\theta)\mu^k} && \text{(9h)}\\
&\leq 2^{-3/2}\frac{\|(X^k S^k e - \mu^k e)\|^2 + 2(1-\beta)\mu^k e^T(X^k S^k e - \mu^k e) + n((1-\beta)\mu^k)^2 + (\eta\mu^k)^2}{(1-\theta)\mu^k} && \text{(9i)}\\
&\leq 2^{-3/2}\frac{(\theta\mu^k)^2 + n((1-\beta)\mu^k)^2 + (\eta\mu^k)^2}{(1-\theta)\mu^k} && \text{(9j)}\\
&\leq \frac{\theta^2 + n(1-\beta)^2 + \eta^2}{2^{3/2}(1-\theta)}\mu^k. && \text{(9k)}
\end{aligned}
$$

Equation (9b) comes from Lemma 5.3 of (Wright, 1997), (9h) from equation (4), (9g) from $\min_i x_i^k s_i^k \geq (1-\theta)\mu^k$, and (9e) from Assumption 2 and the definition of the neighborhood. The next part of proving equation (8d) is

$$\|X^{k+1}S^{k+1}e - \mu^{k+1}e\|_2 = \sqrt{\sum_{i=1}^{n}((x_i^k + \Delta x_i^k)(s_i^k + \Delta s_i^k) - \mu^{k+1})^2} \tag{10a}$$

$$= \sqrt{\sum_{i=1}^{n}(\beta\mu^k + \Delta x_i^k \Delta s_i^k - \mu^{k+1})^2} \tag{10b}$$

$$\leq \|\Delta X^k \Delta S^k e\| + \sqrt{n}|\beta\mu^k - \mu^{k+1}| \tag{10c}$$

$$\leq \frac{\theta^2 + n(1-\beta)^2 + \eta^2}{2^{3/2}(1-\theta)}\mu^k + \sqrt{n}|\beta\mu^k - \mu^{k+1}| \tag{10d}$$

$$\leq \frac{\theta^2 + n(1-\beta)^2 + \eta^2}{2^{3/2}(1-\theta)}\mu^k + \eta\mu^k \tag{10e}$$

$$\leq (\frac{\theta^2 + n(1-\beta)^2 + \eta^2}{2^{3/2}(1-\theta)} + \eta)\frac{\mu^{k+1}}{\beta - \frac{\eta}{\sqrt{n}}} \tag{10f}$$

$$\leq \theta\mu^{k+1}. \tag{10g}$$

Equation (10b) comes from system 4, (10c) form triangular inequality, and (10d) form Lemma 4. One can easily verify that inequality (10g) holds for $(\eta, \theta, \beta) = (0.1, 0.3, 1 - \frac{0.3}{\sqrt{n}})$. Let $x_i^k(\alpha) = x_i^k + \alpha(\Delta x_i^k)$ and $s_i^k(\alpha) = s_i^k + \alpha(\Delta s_i^k)$, we can also conclude that $x_i^k(\alpha)s_i^k(\alpha) > 0$ for any $i \in \{1, \dots, n\}$ since $\mu(\alpha) > 0$. We have $(x_i^{k+1}, s_i^{k+1}) > 0$ because we can not have $x_i^k(\alpha) = 0$ or $s_i^k(\alpha) = 0$ for any $i \in \{1, \dots, n\}$ and $(x_i^k(0), s_i^k(0)) > 0$. □

Based on Lemma 5, IF-IPM remains in central path, and it converges to optimal solution if μ_k converges to zero. In Theorem 4, we prove that the algorithm reaches to ζ-optimal solution after a polynomial time.

Theorem 1. *The sequence μ^k converges to zero linearly, and after $\mathcal{O}(\sqrt{n}\log(\frac{\mu_0}{\zeta}))$ iterations, we have $\mu^k \leq \zeta$.*

Proof. By Lemma 4, we have

$$\mu^{k+1} \leq (\beta + \frac{\eta}{\sqrt{n}})\mu^k \leq (1 - \frac{0.2}{\sqrt{n}})\mu^k.$$

Since μ^k is bounded below by zero, and it is monotonically decreasing, it converges linearly to zero. We also have

$$\mu^k \leq (1 - \frac{0.2}{\sqrt{n}})\mu^{k-1} \leq (1 - \frac{0.2}{\sqrt{n}})^k\mu^0.$$

Since we need $\mu^k \leq \zeta$, then we have

$$(1 - \frac{0.2}{\sqrt{n}})^k \leq \frac{\zeta}{\mu^0},$$

$$k\log(1 - \frac{0.2}{\sqrt{n}}) \leq \log(\frac{\zeta}{\mu^0}).$$

We know that $\log(1-\frac{0.2}{\sqrt{n}}) \leq (-\frac{0.2}{\sqrt{n}})$, then we need

$$k(-\frac{0.2}{\sqrt{n}}) \leq \log(\frac{\zeta}{\mu^0}),$$
$$k \geq \frac{\sqrt{n}}{0.2}\log(\frac{\mu^0}{\zeta}).$$

□

As the proof shows the IF-IPM is convergent. We can redo the proof for any values of parameter satisfying conditions (11) and (12).

$$\beta \leq (1-\frac{\eta+0.01}{\sqrt{n}}), \tag{11}$$
$$(\frac{\theta^2+n(1-\beta)^2+\eta^2}{2^{3/2}(1-\theta)}+\eta) \leq \theta(\beta-\frac{\eta}{\sqrt{n}}). \tag{12}$$

It is not hard to check that $\gamma = 0.3$ and $\eta = 0.1$ satisfy these conditions. To find an exact optimal solution $(x^*, y^*, s^*) \in \mathcal{PD}^*$, Lemma 6 is needed to find an appropriate value for ζ. Let L be the binary length of input data defined as

$$L = mn + m + n + \sum_{i,j}\lceil\log(|a_{ij}|+1)\rceil + \sum_{i}\lceil\log(|c_i|+1)\rceil + \sum_{j}\lceil\log(|b_j|+1)\rceil.$$

Lemma 6 (Khachiyan, 1979)**.** *For any* $(x^*, y^*, s^*) \in \mathcal{PD}^*$*, we have*

$$\min_i\{x_i^*|x_i^*>0\} \geq 2^{-L},$$
$$\min_i\{s_i^*|s_i^*>0\} \geq 2^{-L},$$
$$\max_i\{x_i^*\} \leq 2^{L},$$
$$\max_i\{s_i^*\} \leq 2^{L}.$$

Based on Lemma 6, an exact solution can be calculated by rounding (Citation and remarking the quantum solution of system) if

$$\frac{(x^k)^T s^k}{n} \leq 2^{-2L}.$$

Thus, the upper bound for iterations of IF-IPM to find an exact optimal solution is $\mathcal{O}(\sqrt{n}L)$. In next section, we analyze more the OSS and compare it to other Newton systems.

3 Analyzing The New System

In this section, fist we find some bounds for condition number and norm of the matrix of System 3. Then, the new system will be compared to other systems.

3.1 Bounds on the Condition Number and Norm of M

Fist, we analyze the matrix

$$M^k = [-X^k A^T, S^k V]$$

to find a bound for its condition number. If we have degeneracy in optimal solution, there will be a zero row in the matrix for optimal solution and M^k will converge to a singular. We assume that we do not have degeneracy, but the matrix M^k may converge to a singular matrix. Consider the case where $x_i^* > 0$ and $s_i^* = 0$ for any $i \in \{1, \dots, n\}$. Then the matrix M^k will converge to a singular matrix. In worst cases similar to the mentioned case, we have

$$\kappa_{M^k} = \mathcal{O}(\frac{\kappa_T}{\mu^k}) = \mathcal{O}(\frac{\kappa_T}{\zeta}),$$

where $T = \begin{bmatrix} -A^T, V \end{bmatrix}$. We can easily found $\mu^k = \mathcal{O}(\mu_0)$ and $\frac{1}{\mu^k} = \mathcal{O}(\frac{1}{\zeta})$. In addition, we can derive bounds $\|\sigma^k\| = \|\beta\mu^k e - X^k s^k\| = \mathcal{O}(\sqrt{n}\mu^0)$ and $\frac{\|\sigma^k\|}{\mu_k} = \mathcal{O}(\sqrt{n})$.

3.2 Comparing Different Systems

To find the Newton step, one can solve the full Newton system (FNS). For an inexact solution, we have

$$\begin{aligned} A\Delta x^k &= b - Ax^k + r_P, \\ A^T \Delta y^k + \Delta s^k &= c - A^T y^k - s^k + r_D, \\ X^k \Delta s^k + S^k \Delta x^k &= \beta_1 \mu^k e - X^k s^k + r_C, \end{aligned} \tag{13}$$

where (r_P, r_D, r_C) are residuals generated by the inexact solution. As we can see, residuals may lead to infeasiblity. Thus, for the full Newton system we must consider the primal and dual infeasibilities in system (13). The full Newton system can be simplified to the Augmented system (AS). The augmented system for an inexact infeasible solution is

$$\begin{aligned} A\Delta x^k &= b - Ax^k + r'_P, \\ A^T \Delta y^k + {X^k}^{-1} S^k \Delta x^k &= c - A^T y^k - \beta_1 \mu^k {X^k}^{-1} e + r'_D. \end{aligned} \tag{14}$$

After solving the Augmented system, we can calculate Δs^k by the last line of the System (13) where $r_C = 0$. We can simplify the system more to get Normal Equation System (NES) as

$$AX^k {S^k}^{-1} A^T \Delta y^k = AX^k {S^k}^{-1} c - AX^k {S^k}^{-1} A^T y^k - \beta_1 \mu^k A {S^k}^{-1} e + b - Ax^k + r''_P. \tag{15}$$

When we solve the NES instead of the full Newton system, then $r_C = r_D = 0$ since we use the second and third line of the system (13) directly to find Δs^k and Δx^k, respectively. In the NES, we have only primal infeasible because of r''_P. Most of the implementations of IPMs use the NES since it has a small positive definite matrix and can be solved efficiently by Cholesky factorization (Wright, 1997). Although the NES is smaller than the FNS and the AS, the NES is much denser than the two others. We can use the Sherman-Morrison-Woodberry formula to solve NES with sparse matrix efficiently (Andersen et al., 2004). If we solve FNS, AS, or NES inexactly, the potential infeasibility will increase the complexity of IPM. On the other hand, it is more adaptable with inexact solvers such as QLSAs and the CG method. In section 2, we discuss how we can use QLSA efficiently in IF-QIPM and how much the OSS is more adaptable to QLSAs than other systems. Compared to the NES, the proposed OSS is bigger, and it is not positive definite. Thus, we cannot use Cholesky factorization but we can use LU factorization or Gaussian Elimination with $\mathcal{O}(n^3)$ complexity. Since the Conjugate Gradient method has better dependence on the size of the system with complexity $\mathcal{O}(nd\kappa\log(\frac{1}{\zeta}))$, it is a better choice among classical solver for the inexact solution of the OSS. Table 1 compares the charismatics of different systems.

System	Size of system	Symmetric	Positive Definite	d	Condition Number
FNS 13	$2n+m$	✗	✗	$\mathcal{O}(n)$	$\rightarrow\infty$
AS 14	$n+m$	✓	✗	$\mathcal{O}(n)$	$\rightarrow\infty$
NES 15	m	✓	✓	$\mathcal{O}(m)$	$\rightarrow\infty$
OSS 3	n	✗	✗	$\mathcal{O}(n)$	$\rightarrow\infty$

Table 1: Characteristics of Different Newton Systems

IPM	System	System solver	Complexity
Best Theoretical bound	NES	Partial Cholesky Factorization	$\mathcal{O}(Ln^3)$
Feasible IPM	NES	Cholesky Factorization	$\mathcal{O}(Ln^{3.5})$
Infeasible IPM	NES	Cholesky Factorization	$\mathcal{O}(Ln^4)$
II-IPM	NES	Conjugate Gradient	$\mathcal{O}(Ln^5 2^{4L}\kappa_A^2)$
IF-IPM	OSS	Conjugate Gradient	$\mathcal{O}(Ln^{2.5}2^{2L}\kappa_T)$
IF-IPM	OSS	Preconditioned CG	$\mathcal{O}(Ln^{2.5}\bar{\kappa})$
IF-IPM+Iterative Refinement	OSS	Conjugate Gradient	$\mathcal{O}(Ln^{2.5}\kappa_T)$

Table 2: Total time complexity of different IPMs

In Table 2, the best time complexities which can be attained by different IPMs are mentioned to be compared with the proposed IF-IPM. If we use the CG method the dependence on n decreases but the cost of inexactness increases the number of iterations except in IF-IPM. The exponential term 2^L comes from the bound of the condition number. To deal with this issue we can use either a preconditioning technique or an iterative refinement method. With preconditioning, we can get an appropriate bound for condition number $\bar{\kappa}$. On the other hand, we can fix ζ to a large amount such as 10^{-2} and improve the precision with iterative refinement in $\mathcal{O}(L)$ iterations. As we can see, IF-IPM with either iterative refinement or precondition CG has much better dependence on n than other prevailing IPMs and it is near to the theoretical bound. In the next section, we also discuss the effect of using QLSAs to solve the system.

4 IF-QIPM with QLSAs

To use QLSA inside the IF-IPM efficiently, we apply an approach similar to (Mohammadisiahroudi et al., 2021). The first QLSA was the HHL algorithm by Harrow et al. (2009) solving a quantum linear system with p-by-p Hermitian matrix in $\mathcal{O}(\log(p)\frac{d^2\kappa^2}{\epsilon})$ time complexity, where ϵ is the target error, κ condition number of the coefficient matrix, and d is the maximum number of nonzero entries in any row or column. After the HHL method, many QLSAs proposed with better time complexity than the HHL method. Amplitude Amplification can decrease the dependence on κ, and (Wossnig et al., 2018) proposed a QLSA algorithm independent on sparsity with $\mathcal{O}\Big(\text{polylog}(p)\|M\|\frac{\kappa}{\epsilon}\Big)$ complexity. Childs et al. (Childs et al., 2017) developed a QLSA with exponentially better dependence on error with $\mathcal{O}(\text{polylog}(\frac{p\kappa}{\epsilon})d\kappa)$ complexity. The best QLSA w.r.t time complexity uses Block Encoding with $\mathcal{O}(\text{polylog}(\frac{p}{\epsilon})\kappa)$ complexity (Chakraborty et al., 2018). To encode the linear system in a quantum setting and solve it by QLSA, we need a procedure proposed by Mohammadisiahroudi et al. (2021). To solve the OSS 3, We must build system $M'^k z'^k = \sigma'^k$ where

$$M'^k = \frac{1}{\|M^k\|}\begin{bmatrix} 0 & M^k \\ M^{kT} & 0 \end{bmatrix}, z'^k = \begin{pmatrix} 0 \\ z^k \end{pmatrix}, \text{ and } \sigma'^k = \frac{1}{\|M^k\|}\begin{pmatrix} \sigma^k \\ 0 \end{pmatrix}. \tag{16}$$

The new system can be implemented in a quantum setting and solved by QLSA since M'^k is a Hermitian matrix and $\|M'^k\| = 1$. To the extract classical solution, we use Quantum Tomography Algorithm (QTA) by Kerenidis and Prakash (2020). The Theorem 2 shows how we can adapt QLSA by Chakraborty et al. (2018) to solve the OSS system.

Theorem 2. *Given System (3), QLSA and QTA provide the solution $(\widetilde{\lambda^k}, \widetilde{\Delta y^k})$ with residual r^k, where $\|r^k\| \leq \eta\mu^k$. The total time complexity of finding such solution is $\mathcal{O}(n^2 \text{polylog}(n)\frac{\delta_M}{\delta_L}\kappa_T)$.*

Proof. We can derive the transformed system (16) from the OSS (3). To have $\|r^k\| \leq \eta\mu^k$, the error of linear system ϵ_{LS} must be less than $\frac{\eta\mu^k}{\|M^k\|}$. Since scaling affects the error of QLSA, we need to find appropriate bound for QLSA and QTA (Mohammadisiahroudi et al., 2021). Since the error is linear among QTA and QLSA, we need

$$\epsilon_{QLSA} = \frac{\epsilon_{LS}}{2} = \frac{\eta\mu^k}{2\|\sigma^k\|},$$
$$\epsilon_{QTA} = \frac{\epsilon_{LS}}{2} = \frac{\eta\mu^k}{2\|\sigma^k\|}.$$

The QLSA by Chakraborty et al. (2018) can find such solution with $\mathcal{O}(\text{polylog}(\frac{n}{\epsilon_{QLSA}})\kappa_M) = \mathcal{O}(\text{polylog}(n)\frac{\delta_M}{\delta_L}\kappa_T)$ time complexity. The time complexity of QTA by Kerenidis and Prakash (2020) is $\mathcal{O}(\frac{n}{\epsilon_{QTA}^2}) = \mathcal{O}(n^2)$. □

There are few studies investigating Quantum Interior Point Methods (QIPMs) for LO problems. First, Kerenidis and Prakash (2020) used Block Encoding and QRAM for finding a ζ-optimal solution with $\mathcal{O}(\frac{n^2}{\epsilon^2}\bar{\kappa}^3\log(\frac{1}{\zeta}))$ complexity. Casares and Martin-Delgado (Casares and Martin-Delgado, 2020) used QLSA and developed a Predictor-correcter QIPM with $\mathcal{O}(L\sqrt{n}(n+m)\|\bar{M}\|\frac{\bar{\kappa}^2}{\epsilon^{-2}})$ complexity. Both papers used exact IPMs which is not valid to QLSAs and ϵ and $\bar{\kappa}$ increases exponentially leading to exponential time complexities. To address these problem, Mohammadisiahroudi et al. (2021) developed an II-QIPM using QLSA efficiently with $\mathcal{O}(n^5L\|A\|^4\kappa_A^2)$ time complexity. To improve this time complexity, we can use Algorithm 2 which is a Short-step IF-QIPM for solving LO problems using QLSA and QTA to solve system (3).

Algorithm 2 Short-step IF-QIPM using QLSA

Choose $\zeta > 0$, $\eta = 0.1$, $\gamma = 0.7$and $\beta = (1 - \frac{0.2}{\sqrt{n}})$.
$k \leftarrow 0$
Choose initial feasible interior solution $(x^0, y^0, s^0) \in \mathcal{N}(\gamma)$
while $(x^k, y^k, s^k) \notin \mathcal{PD}_\zeta$ **do**
    $\mu^k \leftarrow \frac{(x^k)^T s^k}{n}$
    $(M^k, \sigma^k) \leftarrow$**build** the OSS (3)
    $\epsilon_{QLSA} \leftarrow \frac{\eta\mu^k}{2\|\sigma^k\|}$
    $\epsilon_{QTA} \leftarrow \frac{\eta\mu^k}{2\|\sigma^k\|}$
    $(\lambda^k, \Delta y^k) \leftarrow$ **solve** the OSS (3) using QLSA and QTA
    $\Delta x^k = V\lambda^k$ and $\Delta s^k = -A^T\Delta y^k$
    $(x^{k+1}, y^{k+1}, s^{k+1}) \leftarrow (x^k, y^k, s^k) + (\Delta x^k, \Delta y^k, \Delta s^k)$
    $k \leftarrow k+1$
return (x^k, y^k, s^k)

We can analyze the IF-QIPM (2) as follow:

- With similar convergence theorem in the Section 2.1, after $\mathcal{O}(\sqrt{n}\log(\frac{\mu^0}{\zeta}))$ IF-QIPM get a ζ-optimal solution.
- The detailed time complexity of IF-QIPM is

$$\mathcal{O}(\sqrt{n}\log(\frac{\mu^0}{\zeta})(n^2 + (\text{polylog}(n)\frac{\delta_M}{\delta_L}\kappa_T)(n^2))).$$

- To get an exact optimal solution, the time complexity contains exponential term 2^L. To address this problem, we can fix $\zeta = 10^{-2}$ and improve the precision by iterative refinement in $\mathcal{O}(L)$ iterations (Mohammadisiahroudi et al., 2021).
- Using Iterative Refinement, the total time complexity of finding exact optimal solution with IR-IF-QIPM is

$$\mathcal{O}(n^{2.5}\text{polylog}(n)L\kappa_T).$$

Algorithm	Time Complexity
QIPM of (Kerenidis and Prakash, 2020)	$\mathcal{O}\Big(n^2 L\bar{\kappa}^3 2^{4L}\Big)$
QIPM of (Casares and Martin-Delgado, 2020)	$\mathcal{O}\big(n^2 L\bar{\kappa}^2 2^{2L} n\frac{\|A\|^2}{2^{-4L}}\big)$
IR-II-QIPM of (Mohammadisiahroudi et al., 2021)	$\mathcal{O}\Big(n^5 L\|A\|^4\kappa_A^2\Big)$
The proposed IR-IF-QIPM	$\mathcal{O}\Big(n^{2.5}\text{polylog}(n)L\kappa_T\Big)$

Table 3: Time complexity of finding exact solution using different QIPMs

As Table 3 shows, the proposed IR-IF-QIPM has much better time complexity than IR-II-QIPM for solving LO problems. It seems that two other QIPMs have better dependence on n but these time complexities can not be attained since they contain iteration complexity of exact IPMs while these QIPMs solve the Newton system inexactly. Both QIPMs of (Kerenidis and Prakash, 2020) and Casares and Martin-Delgado (2020) are not polynomial for finding exact solution and the bound for condition number $\bar{\kappa}$ goes to infinity. The red part of the time complexity of QIPM by Casares and Martin-Delgado (2020) is the time complexity of QTA which was ignored by the authors of that paper. Compared to classical solver IR-IF-QIPM has roughly similar complexity IR-IF-IPM with CG method and both of them are better than other classical and quantum IPMs. However, they are close to the best Theoretical bound.

5 IF-IPM for Self-dual Embedding Formulation

To start the IF-IPM, an initial feasible interior solution is needed. Since in many practical problems, such feasible interior solution is not available, self-dual embedding formulation can be used where all-one vector e is a feasible interior point. To drive a well-organized approach, we use the conical formulation of LO problem as Definition

Definition 2 (Linear Optimization Problem: Cononical Formulation)**.**

$$(P)\quad \begin{aligned} \min\ & c^T x, \\ \text{s.t.}\ & Ax \geq b, \\ & x \geq 0, \end{aligned} \qquad\qquad (D)\quad \begin{aligned} \max\ & b^T y, \\ \text{s.t.}\ & A^T y \leq c, \\ & y \geq 0. \end{aligned}$$

The standard formulation and canonical formulation are equivalent and one can derive both formulation for any LO problem. By finding basic variables, we can drive canonical form from the standard form. In this case, canonical form has $n' = n - m$ variables and $m' = m$ constraints. To keep it simple, we assume the canonical form has n variables and m constraints. Then For any LO problem defined as Definition 2, we can derive the self-dual formulation as

$$
\begin{array}{llllll}
\min & (n+m+2)\theta & & & & \\
 & & Ax & -b\tau & +\bar{b}\theta & \geq 0, \\
-A^T y & & & +c\tau & +\bar{c}\theta & \geq 0, \\
b^T y & -c^T x & & & +\bar{o}\theta & \geq 0, \\
-\bar{b}^T y & -\bar{c}^T x & -\bar{o}\tau & & & \geq -(n+m+2), \\
\multicolumn{6}{l}{x \geq 0, \tau \geq 0, y \geq 0, \text{ and } \theta \geq 0,}
\end{array} \tag{17}
$$

where $\bar{b} = b - Ae + e$, $\bar{c} = A^T e + e - c$, and $\bar{o} = 1 + c^T e - b^T e$. One can verify that dual problem of (17) is itself (Roos et al., 2005). We can also drive the standard form of problem (17) by introducing slack variables (u, s, ϕ, ρ) as

$$
\begin{array}{lllllll}
\min & (n+m+2)\theta & & & & & \\
 & & Ax & -b\tau & +\bar{b}\theta & -u & = 0, \\
-A^T y & & & +c\tau & +\bar{c}\theta & -s & = 0, \\
b^T y & -c^T x & & & +\bar{o}\theta & -\phi & = 0, \\
-\bar{b}^T y & -\bar{c}^T x & -\bar{o}\tau & & & -\rho & = -(n+m+2), \\
\multicolumn{7}{l}{x \geq 0, \tau \geq 0, y \geq 0, \theta \geq 0, s \geq 0, u \geq 0, \phi \geq 0, \rho \geq 0.}
\end{array} \tag{18}
$$

Based on Strong Duality Theorem, an optimal solution satisfies $x^T s + y^T u + \tau\phi + \rho\theta = 0$. One can verify that $(y^0, x^0, \tau^0, \theta^0, u^0, s^0, \phi^0, \rho^0) = (e, e, 1, 1, e, e, 1, 1)$ is a feasible interior solution of problem (18).

Theorem 3 (Roos et al. (2005))**.** *For self-dual formulation (17), the following statements hold.*

1. *Problem (17) has a finite optimal solution $(y^*, x^*, \tau^*, \theta^*, u^*, s^*, \phi^*, \rho^*)$ such that $\theta^* = 0$, $x^* + s^* > 0$, $y^* + u^* > 0$, $\tau^* + \phi^* > 0$, and $\theta^* + \rho^* > 0$.*

2. *If $\tau^* > 0$, $(\frac{x^*}{\tau^*}, \frac{y^*}{\tau^*}, \frac{s^*}{\tau^*}, \frac{u^*}{\tau^*})$ is the optimal solution of the original LO problem.*

3. *If $\tau^* = 0$, $c^T x^* < 0$, and $b^T y^* \leq 0$, then original dual problem is infeasible and the original primal problem is unbounded.*

4. *If $\tau^* = 0$, $c^T x^* \geq 0$, and $b^T y^* > 0$, then original primal problem is infeasible and the original dual problem is unbounded.*

5. *If $\tau^* = 0$, $c^T x^* < 0$, and $b^T y^* > 0$, then both original primal and dual problems are infeasible.*

The feasible Newton system for this formulation is

$$
\begin{array}{cccccccccl}
 & A\Delta x^k & -b\Delta\tau^k & +\bar{b}\Delta\theta^k & -\Delta u^k & & & & = 0, \\
-A^T\Delta y^k & & +c\Delta\tau^k & +\bar{c}\Delta\theta^k & & -\Delta s^k & & & = 0, \\
b^T\Delta y^k & -c^T\Delta x^k & & +\bar{o}\Delta\theta^k & & & -\Delta\phi^k & & = 0, \\
-\bar{b}^T\Delta y^k & -\bar{c}^T\Delta x^k & -\bar{o}\Delta\tau^k & & & & & -\Delta\rho^k & = 0,
\end{array}
$$

$$
\begin{aligned}
X^k\Delta s^k + S^k\Delta x^k &= \beta\mu^k e - X^k s^k, \\
Y^k\Delta u^k + U^k\Delta y^k &= \beta\mu^k e - Y^k u^k, \\
\tau^k\Delta\phi + \phi^k\Delta\tau^k &= \beta\mu^k - \tau^k\phi^k, \\
\theta^k\Delta\rho^k + \rho^k\Delta\theta^k &= \beta\mu^k - \theta^k\rho^k.
\end{aligned} \tag{19}
$$

To Drive the OSS system for Newton system (19), we define

$$
\begin{aligned}
\mathcal{A} &= \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -A & b & -\bar{b} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} & A^T & \mathbf{0} & -c & -\bar{c} \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & -b^T & c^T & \mathbf{0} & -\bar{o} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \bar{b}^T & \bar{c}^T & \bar{o}^T & \mathbf{0} \end{bmatrix}, \\
\mathcal{D} &= \begin{bmatrix} Y^k & \mathbf{0} & \mathbf{0} & \mathbf{0} & U^k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & X^k & \mathbf{0} & \mathbf{0} & \mathbf{0} & S^k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tau^k & \mathbf{0} & \mathbf{0} & \mathbf{0} & \phi^k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \theta^k & \mathbf{0} & \mathbf{0} & \mathbf{0} & \rho^k \end{bmatrix}, \\
\mathcal{R} &= \begin{bmatrix} \beta\mu^k e - Y^k u^k \\ \beta\mu^k e - X^k s^k \\ \beta\mu^k - \tau^k\phi^k \\ \beta\mu^k - \theta^k\rho^k \end{bmatrix}, \\
\Delta\mathcal{X} &= (\Delta u^k, \Delta s^k, \Delta\phi^k, \Delta\rho^k, \Delta y^k, \Delta x^k, \Delta\tau^k, \Delta\theta^k)^T,
\end{aligned} \tag{20}
$$

where $\mathbf{0}$ is the all-zero matrix. Then, the Newton system can be simplified as

$$
\begin{aligned}
\Delta\mathcal{X} &\in \text{Null}(\mathcal{A}), \\
\mathcal{D}\Delta\mathcal{X} &= \mathcal{R}.
\end{aligned} \tag{21}
$$

With a similar approach, we can find the basis for the null space of $\mathcal{A}$ by columns of

$$
\mathcal{V} = \begin{bmatrix} \mathbf{0} & -A & b & -\bar{b} \\ A^T & \mathbf{0} & -c & -\bar{c} \\ -b^T & c^T & \mathbf{0} & -\bar{o} \\ \bar{b}^T & \bar{c}^T & \bar{o}^T & \mathbf{0} \\ -I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -1 \end{bmatrix}.
$$

The OSS for this formulation is

$$
\mathcal{D}\mathcal{V}\lambda^k = \mathcal{R}, \tag{22}
$$

where $\lambda^k \in \mathbb{R}^{n+m+2}$ and the size of the system will be $n+m+2$. We can calculate the Newton direction by $\Delta\mathcal{X} = \mathcal{V}\lambda^k$. If $\tilde{\lambda}^k$ is the inexact solution of the system 22, the inexact solution $\widetilde{\Delta\mathcal{X}}^k$ is a feasible direction since $\widetilde{\Delta\mathcal{X}}^k \in \text{Null}(\mathcal{A})$. To have a convergent IF-IPM, we again need that $\|r^k\| \leq \eta\mu^k$ where $r^k = \mathcal{D}\mathcal{V}\tilde{\lambda}^k - \mathcal{D}\mathcal{V}\lambda^k$. So, the error bound $\epsilon^k = \frac{\eta\mu^k}{\|\mathcal{D}\mathcal{V}\|}$ is needed.

Lemma 7. *Let* $(u^k, s^k, \phi^k, \rho^k, y^k, x^k, \tau^k, \theta^k) \in \mathcal{PD}_0$ *then the following statements hold.*

1. *Systems* (22) *and* (19) *are equivalent.*
2. *System* (22) *has a unique solution.*
3. *In the solution of System* (22), $(\Delta x^k)^T \Delta s^k + (\Delta y^k)^T \Delta u^k + \Delta\tau^k \Delta\phi^k + \Delta\theta\Delta\rho = 0$.

Proof. The first statement holds due to the construction of system (22). Since systems (22) and (19) are equivalent and system (19) has a unique solution (Roos et al., 2005), the system (22) has a unique solution.

To prove the third statement, let $\Delta\mathcal{X}^k$ be the solution of system (22). Since $\Delta\mathcal{X}^k \in \text{Null}(\mathcal{A})$, we have

$$\begin{aligned} \mathcal{A}\Delta\mathcal{X}^k &= 0, \\ (\Delta u^k, \Delta s^k, \Delta\phi^k, \Delta\rho^k, \Delta y^k, \Delta x^k, \Delta\tau^k, \Delta\theta^k)\mathcal{A}\Delta\mathcal{X}^k &= 0, \\ (\Delta x^k)^T \Delta s^k + (\Delta y^k)^T \Delta u^k + \Delta\tau^k \Delta\phi^k + \Delta\theta\Delta\rho &= 0. \end{aligned}$$

□

Now we can update IF-IPM of Algorithm 1 for self-dual embedding model as Algorithm 3.

Algorithm 3 Short-step IF-IPM for Self-dual Embedding Formulation

Choose $\zeta > 0$, $\eta = 0.1$, $\gamma = 0.3$ and $\beta = (1 - \frac{0.3}{\sqrt{n+m+2}})$.
$k \leftarrow 0$
$(y^0, x^0, \tau^0, \theta^0, u^0, s^0, \phi^0, \rho^0) = (e, e, 1, 1, e, e, 1, 1)$
while $\mathcal{X}^k \notin \mathcal{PD}_\zeta$ **do**
    $\mu^k \leftarrow \frac{(y^k)^T u^k + (x^k)^T s^k + \tau^k \phi^k + \rho^k \theta^k}{n+m+2}$
    $\epsilon^k \leftarrow \eta \frac{\mu^k}{\|\mathcal{D}^k \mathcal{V}\|_2}$
    $\lambda^k \leftarrow$ **solve** System (22) with error bound ϵ^k
    $\Delta\mathcal{X}^k = \mathcal{V}\lambda^k$
    $\mathcal{X}^{k+1} \leftarrow \mathcal{X}^k + \Delta\mathcal{X}^k$
    $k \leftarrow k+1$
if $\tau^k = 0$ **then**
    return Problem is infeasible.
else
    return $(\frac{x^k}{\tau^k}, \frac{y^k}{\tau^k}, \frac{s^k}{\tau^k}, \frac{u^k}{\tau^k})$

Theorem 4. *For Algorithm 3, the following statements hold.*

1. *The sequence $\{\mu_k\}_{k\in\mathbb{N}}$ converges linearly to zero.*
2. *For any $k \in \mathbb{N}$, $\mathcal{X}^k \in \mathcal{N}(\gamma)$.*
3. *After $\mathcal{O}(\sqrt{n}\log(\frac{1}{\zeta}))$ iterations, $\mathcal{X}^k \in \mathcal{PD}_\zeta$.*
4. *Iteration complexity of finding exact optimal solution is $\mathcal{O}(\sqrt{n}L)$*

As we discussed in detail in previous sections, we can also solve the Self-dual Embedding model by IR-IF-IPM using CG method with $\mathcal{O}(Ln^{2.5}\kappa_{\mathcal{V}})$ complexity and IR-IF-QIPM using QLSAs with $\mathcal{O}(Ln^{2.5}\text{polylog}(n)\kappa_{\mathcal{V}})$ complexity. Since matrix $\mathcal{V}$ is bigger and denser than T, we expect larger time complexity for solving Self-dual Embedding model than solving original problem with a initial interior feasible solution.

6 Numerical Experiments

The proposed IF-IPM is implemented in Python using both classical solvers and QISKIT AQUA simulator of the HHL algorithm. All the methods are implemented in a Python package available for public at url:
https://github.com/Fakhimi/interior_point_methods.
This solver gives the option of choosing quantum or classical linear solver to solve LO problems. Also, it has option for using Iterative Refinement to improve the precision.

```
                   Objective                 Residual
Iter      Primal          Dual          Primal    Dual     Compl    Cond-Num  ||M||    ||RHS||     Time
   0   1.26892397e+01  1.01820100e+01  3.02e+00 1.16e+01  8.10e+00  6.03e+01 3.48e+01 1.50e+00      0s
   1   1.23549215e+01  1.10104263e+01  1.78e+00 6.81e+00  5.61e+00  5.37e+01 2.30e+01 7.90e-01      0s
   2   1.24693053e+01  1.17552609e+01  9.60e-01 3.67e+00  3.28e+00  5.68e+01 1.86e+01 3.98e-01      0s
   3   1.22853224e+01  1.19145703e+01  4.97e-01 1.90e+00  1.77e+00  8.07e+01 1.81e+01 1.95e-01      0s
   4   1.23057445e+01  1.21167648e+01  2.52e-01 9.62e-01  9.11e-01  1.62e+02 1.74e+01 9.60e-02      0s
   5   1.23115764e+01  1.22164635e+01  1.27e-01 4.85e-01  4.64e-01  2.90e+02 1.68e+01 4.77e-02      0s
   6   1.22897786e+01  1.22420315e+01  6.39e-02 2.44e-01  2.35e-01  4.55e+02 1.64e+01 2.40e-02      0s
   7   1.22784898e+01  1.22545458e+01  3.21e-02 1.22e-01  1.18e-01  8.58e+02 1.63e+01 1.18e-02      0s
   8   1.22731641e+01  1.22611741e+01  1.60e-02 6.13e-02  5.92e-02  1.71e+03 1.63e+01 5.87e-03      0s

The algorithm stopped after 9 iterations in 0.01 seconds.

Primal variables:   [+4.359 +0.002 +0.032 +0.005 +1.527 +0.003 +1.886 +0.008]
Dual slacks:        [+0.002 +3.574 +0.234 +1.497 +0.005 +2.530 +0.004 +0.882]
Dual variables:     [+0.915 +2.062 -4.638 +10.322]

Primal objective:   1.22731641e+01
Dual objective:     1.22611741e+01

Primal residual:    1.60e-02
Dual residual:      6.13e-02
Complementraty:     5.92e-02

Number of Iter:     9
Run time:           0.01
```

Figure 2: IF-IPM for Embedding formulation $\eta = 0$ and $\beta = 0.5$

```
                  Objective                   Residual
Iter      Primal            Dual          Primal    Dual     Compl    Cond-Num  ||M||    ||RHS||      Time
   0   1.26850540e+01  1.01511774e+01  3.02e+00 1.15e+01  8.10e+00  6.03e+01 3.48e+01 1.50e+00       0s
   1   1.23681653e+01  1.09980915e+01  1.78e+00 6.79e+00  5.62e+00  5.37e+01 2.30e+01 7.86e-01       0s
   2   1.24786167e+01  1.17359128e+01  9.58e-01 3.66e+00  3.30e+00  5.68e+01 1.87e+01 3.96e-01       0s
   3   1.22821177e+01  1.19138334e+01  4.97e-01 1.90e+00  1.77e+00  8.04e+01 1.81e+01 1.94e-01       0s
   4   1.23016005e+01  1.21218252e+01  2.53e-01 9.67e-01  9.05e-01  1.62e+02 1.74e+01 9.62e-02       0s
   5   1.23114501e+01  1.22164870e+01  1.28e-01 4.87e-01  4.65e-01  2.91e+02 1.68e+01 4.81e-02       0s
   6   1.22895339e+01  1.22424861e+01  6.44e-02 2.46e-01  2.36e-01  4.53e+02 1.64e+01 2.41e-02       0s
   7   1.22786590e+01  1.22545846e+01  3.23e-02 1.23e-01  1.19e-01  8.54e+02 1.63e+01 1.19e-02       0s
   8   1.22732242e+01  1.22612459e+01  1.62e-02 6.19e-02  5.97e-02  1.69e+03 1.63e+01 5.91e-03       0s

The algorithm stopped after 9 iterations in 0.01 seconds.

Primal variables:   [+4.359 +0.002 +0.033 +0.005 +1.527 +0.003 +1.886 +0.009]
Dual slacks:        [+0.002 +3.573 +0.234 +1.497 +0.005 +2.530 +0.004 +0.882]
Dual variables:     [+0.915 +2.062 -4.638 +10.322]

Primal objective:   1.22732242e+01
Dual objective:     1.22612459e+01

Primal residual:    1.62e-02
Dual residual:      6.19e-02
Complementraty:     5.97e-02

Number of Iter:     9
Run time:           0.01
```

Figure 3: IF-IPM for Embedding formulation $\eta = 0.5$ and $\beta = 0.5$

```
                  Objective                   Residual
Iter      Primal            Dual          Primal    Dual     Compl    Cond-Num  ||M||    ||RHS||      Time
   0   1.63476002e+01  8.18853258e+00  1.31e-15 1.17e-15  8.16e+00  2.14e+02 1.37e+02 5.00e+00       0s
   1   1.45836033e+01  1.05040696e+01  1.07e-15 1.26e-15  4.08e+00  1.71e+02 1.50e+02 1.65e+00       0s
   2   1.33178221e+01  1.12780552e+01  9.19e-16 1.74e-15  2.04e+00  2.59e+02 1.44e+02 7.75e-01       0s
   3   1.28540060e+01  1.18341225e+01  9.24e-16 9.98e-16  1.02e+00  6.40e+02 1.33e+02 3.78e-01       0s
   4   1.25884120e+01  1.20784703e+01  1.80e-15 1.15e-15  5.10e-01  1.07e+03 1.23e+02 1.85e-01       0s
   5   1.24276399e+01  1.21726691e+01  9.76e-16 1.10e-15  2.55e-01  1.67e+03 1.17e+02 9.32e-02       0s
   6   1.23474328e+01  1.22199474e+01  1.04e-15 1.54e-15  1.27e-01  3.18e+03 1.16e+02 4.54e-02       0s
   7   1.23076219e+01  1.22438792e+01  6.49e-16 1.96e-15  6.37e-02  6.35e+03 1.16e+02 2.26e-02       0s

The algorithm stopped after 8 iterations in 0.01 seconds.

Primal variables:   [+4.376 +0.002 +0.034 +0.005 +1.529 +0.003 +1.880 +0.009]
Dual slacks:        [+0.002 +3.563 +0.243 +1.490 +0.005 +2.603 +0.004 +0.843]
Dual variables:     [+0.908 +2.073 -4.685 +10.405]

Primal objective:   1.23076219e+01
Dual objective:     1.22438792e+01

Primal residual:    6.49e-16
Dual residual:      1.96e-15
Complementraty:     6.37e-02

Number of Iter:     8
Run time:           0.01
```

Figure 4: IF-IPM with initial interior solution $\eta = 0$ and $\beta = 0.5$

```
                 Objective                  Residual
Iter      Primal            Dual          Primal     Dual      Compl     Cond-Num  ||M||    ||RHS||      Time
   0   1.63555631e+01  8.18309123e+00  1.70e-15 1.53e-15  8.17e+00  2.14e+02 1.37e+02 5.00e+00      0s
   1   1.45903099e+01  1.05005190e+01  2.25e-15 2.90e-15  4.09e+00  1.71e+02 1.50e+02 1.65e+00      0s
   2   1.33127848e+01  1.12759246e+01  2.22e-15 1.93e-15  2.04e+00  2.59e+02 1.44e+02 7.77e-01      0s
   3   1.28512490e+01  1.18349397e+01  2.87e-15 2.98e-15  1.02e+00  6.38e+02 1.33e+02 3.79e-01      0s
   4   1.25855542e+01  1.20797167e+01  2.42e-15 1.96e-15  5.06e-01  1.07e+03 1.23e+02 1.85e-01      0s
   5   1.24266121e+01  1.21737239e+01  3.06e-15 1.62e-15  2.53e-01  1.67e+03 1.17e+02 9.29e-02      0s
   6   1.23469118e+01  1.22202614e+01  2.74e-15 3.33e-15  1.27e-01  3.21e+03 1.16e+02 4.51e-02      0s
   7   1.23073316e+01  1.22440421e+01  3.44e-15 3.01e-15  6.33e-02  6.39e+03 1.16e+02 2.24e-02      0s

The algorithm stopped after 8 iterations in 0.00 seconds.

Primal variables:   [+4.376 +0.002 +0.033 +0.005 +1.529 +0.003 +1.881 +0.009]
Dual slacks:        [+0.002 +3.563 +0.242 +1.490 +0.005 +2.603 +0.004 +0.844]
Dual variables:     [+0.908 +2.073 -4.685 +10.405]

Primal objective:   1.23073316e+01
Dual objective:     1.22440421e+01

Primal residual:    3.44e-15
Dual residual:      3.01e-15
Complementraty:     6.33e-02

Number of Iter:     8
Run time:           0.00
```

Figure 5: IF-IPM with initial interior solution $\eta = 0.5$ and $\beta = 0.5$

7 Conclusion

Motivated by efficient use of QLSA in IPMs, an Inexact Feasible IPM is developed with $\sqrt{n}L$ iteration complexity. The improvement in complexity comes from taking feasible steps using fast but inexact quantum or classical linear solvers. To reach this goal, a novel system called Orthogonal Spaces System is proposed to produce inexact but feasible Newton steps. In consequence, an Inexact Feasible Quantum IPM is developed to solve LO problems almost as fast as the best classical IPMs. We analyzed both IF-IPM with the CG method and IF-QIPM theoretically and empirically. It is necessary to use either an iterative refinement scheme or a preconditioning technique to avoid exponential complexity for finding an exact optimal solution using IF-IPM with both QLSAs and CG method. Besides of condition number of coefficient matrix and error of QLSA, the norm of the right-hand side vector is also important for IF-QIPM because of scaling.

For further research, the IF-IPM can be developed for other optimization problems such as conic and nonlinear optimization problems. In addition, other optimization methods such as Quasi-Newton or Trust-region methods can be adapted for using QLSA. Since all current QIPMs are hybrid algorithms, there are some barriers avoiding quantum speed up such as the cost of building systems in classical computers and the cost of Quantum Tomography algorithms. To overcome these barriers, a pure quantum optimization method is needed.

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