An Inexact-Feasible Quantum Interior Point Method for Second-order Cone Optimization

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Quantum Interior Point Methods for Second Order Cone Optimization Problems

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Abstract

We present an inexact-feasible quantum interior point method for second-order conic optimization problems. Our analysis establishes the polynomial convergence for a short-step primal-dual algorithms based on the Monteiro-Zhang (MZ) family of directions. These results combine recent progress in quantum linear system solvers and on the block-encoding technique with a classical optimization tool known as iterative refinement. In the presence of quantum-accessible storage (also known as QRAM), we are able to obtain speedups in terms of the size of the problem, at the price of a linear dependence on the condition number.

1 Introduction

In Second-Order Cone Optimization (SOCO), one seeks to minimize a linear objective function subject to a feasible region defined by the intersection of an affine set with the Cartesian product of a finite number of second-order cones. Consequently, Linear Optimization Problems (LOPs) are a special case of SOCO. Interchangeably referred to as the *Lorentz* or "ice-cream" cone, a second-order cone can be defined as

$$\left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 - \sum_{i=2}^{n} x_i^2 \geq 0, x_1 \geq 0 \right\}.$$

Second-order cones admit a constructive matrix representation. For any $x \in \mathbb{R}^n$, consider the matrix

$$\mathrm{Arw}(x) = \begin{pmatrix} x_1 & x_{2:n} \\ x_{2:n}^\top & x_1 I_{n-1} \end{pmatrix}, \tag{1}$$

where $x_{2:n} \equiv (x_2, x_3, \ldots, x_n)$ and I_{n-1} denotes the identity matrix of order $n-1$. One can easily verify that $x \in \mathcal{K}$ if and only if $\mathrm{Arw}(x)$ is positive semidefinite. Accordingly, SOCOPs are a specific case of Semidefinite Optimization (SDO).

Given this relationship to LO and SDO, it should come as no surprise that there are polynomial-time algorithms for SOCO. Polynomial time algorithms for LOPs date back to the ellipsoid method [18]. Later, Karmarkar introduced his interior point method (IPM) for LOPs, which exhibited a favorable running time compared to the ellipsoid method [12]. Nesterov and Nemirovskii presented the first polynomial time IPM for general optimization problems over homogenous and self-dual (or, *symmetric)* cones [24], introducing the paradigm of self-concordant barrier functions. As discussed in [25], one approach to solving SOCOPs is to recast the SOCOP as an SDOP and apply an IPM to the resulting SDO reformulation. However, optimizing directly over the second order cone is preferred; IPMs applied to SOCO have an iteration bound

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that depends on the number of cones n, whereas SDO based approaches depend on the number of variables. In an SOCOP, the number of variables may be much larger than n, implying that the iteration bound for the direct approach is lower than that of classical IPMs for SDOPs. [25][1]

A significant effort was thus made to extend the search direction results for SDOPs to SOCOPs. This line of research was initiated by Alder and Alizadeh [1], who developed the SOCOP analog of the Alizadeh-Haeberly-Overton (AHO) search direction for SDOPs [2]. Likewise, Faybusovich provided analysis for the Nesterov-Todd method for SOCOPs [9]. Both Monteiro [23] and Tsuchiya [31] analyzed various search directions for SOCOPs using Jordan algebraic techniques. In our work, we choose the Nesterov-Todd (NT) [26, 30] direction, though our analysis generalizes for any member of the Monteiro-Zhang family of search directions for SOCOPs.

The second order cone has historically received a considerable amount of attention, as many optimization problems can be expressed as SOCOPs (see e.g., [11, 19, 35]). Notable applications include quadratically constrained convex quadratic optimization, robust linear optimization, portfolio optimization problems such as those presented by Markowitz [5, 20], and the support vector machine training problem (SVM) [8]. In fact, portfolio optimization and SVM have each been proposed as applications for quantum computing [15, 28, 29], efforts to develop a convergent quantum algorithm for directly solving SOCOPs have been unsuccessful.

1.1 Literature Review

To date, the only attempt at designing a quantum algorithm for SOCO was the Quantum Interior Point Method (QIPM) for SOCOPs presented by Kerenidis and Prakash [16]. At a high level, the idea is to use a quantum linear system algorithm (QLSA) to obtain a speed up for the bottleneck of the classical IPM; solving the Newton linear system. Specifically, in each iteration the Newton system coefficient matrix is block-encoded using quantum random access memory (QRAM) and subsequently solved with a QLSA developed by [7], which can be accomplished in time polylogarithmic in the dimension of the problem (though this excludes the time to extract a classical estimate of the solution). The claim in [16] is that an ϵ-approximate solution to an SOCOP can be found with their algorithm in time

$$\tilde{\mathcal{O}}\left(\sqrt{n}K\frac{\kappa\theta}{\delta}\log(1/\epsilon)\right)$$

where ϵ is the optimality gap, n is the number of second order cones, K the total number of variables, κ is an upper bound for the condition number of the Newton systems, $\theta \leq \sqrt{K}$ is a factor coming from quantum linear system solvers and δ is error parameter for the tomography algorithm. Now, to obtain a ξ-precise estimate of the solution (in ℓ_2-norm accuracy) using the tomography algorithm in [16], one needs to set $\delta = \xi^2/\kappa^2$, meaning the running time is more accurately written as

$$\tilde{\mathcal{O}}\left(\sqrt{n}K\frac{\kappa^3\theta}{\xi^2}\log(1/\epsilon)\right).$$

However, the algorithm presented in [16] is not convergent for SOCOPs due to reasons that we discuss below. Consequently, even with the noted clarifications, the above running time does not settle the complexity of QIPMs when applied to SOCOPs.

Similar to other early QIPMs (see e.g., [6, 14]), the QIPM in [16] attempts to directly quantize a classical IPM (in this case, the IPM of [23]). This leads to two critical issues which in turn prevent there from being any real guarantee their method will converge when applied to SOCOPs. The first major issue is that they neglect the issue of symmetrizing the Newton linear system to obtain the Monteiro-Zhang search directions for SOCO. Convergence results for the neighborhood they consider are only guaranteed when using these search directions, as the solutions to the Newton linear system must commute.

The next major issue is the way in which the tomography errors are handled. In their analysis, the authors in [16] consider a feasible neighborhood of the central path, which requires that the primal and dual

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[1]In the cotnext of SOCO n will be the number of cones, but for SDO n is the number of variables.

solutions remain in the interior of the primal and dual feasible sets until the algorithm terminates. This in turn only occurs when primal and dual feasibility are satisfied *exactly*, which cannot be guaranteed in the quantum setting without modifying the Newton linear system using and orthogonal subspace representation [4]. We discuss this issue from a technical standpoint in Section 3.3, but can summarize the issue as follows: the QIPM from [16] is in actuality an *infeasible* QIPM, but uses the analysis of a feasible IPM. One could prove convergence for their method by amending the issues of symmetrization, and conducting a new analysis which considers an *infeasible* central path and its neighborhood. However, we remark that infeasible QIPMs exhibit a number of disadvantages when compared to their feasible counterpart (see e.g., [4]), including a worse iteration complexity of $\mathcal{O}(n^2 \log(1/\epsilon))$, and would lead to a far less efficient algorithm compared the QIPM we present.

The first provably convergent QIPMs were developed by Augustino et al. [4] for SDO. To overcome the challenges overlooked by previous methods, they present two different algorithms. The first method quantizes the classical Inexact-Infeasible Interior Point Method (II-IPM) from Zhou and Toh [36], but its overall running time is found to be worse in every parameter when compared to classical IPMs. The second is a novel algorithm, which they call an Inexact-Feasible QIPM (IF-QIPM). By defining the primal and dual search directions as a linear combinations of bases for the null- and rowspaces of the constraint matrices, they devise a novel algorithm that guarantees feasibility of the iterates regardless of errors from tomography. The running time of their Inexact-Feasible QIPM (IF-QIPM) is given by

$$\tilde{\mathcal{O}}\left(\frac{n^{3.5}\kappa^2}{\epsilon}\right).$$

The IF-QIPM for SDO achieves an $\mathcal{O}(n^3)$ speedup in the dimension over the best known classical IPM, but at the cost of quadratic dependence on upper bound of condition number of the Newton linear system constraint matrix, κ.

This result was subsequently improved by Mohammadisaroudhi et al. [21], who developed an iterative refinement method for solving linear systems of equations with the use of a low precision quantum oracle, that is defined as the combined use of a QLSAs with a quantum state tomography algorithm (so that the output is a classical solution vector). In the context of linear systems, iterative refinement works in the following manner; after approximately solving the linear system, a series of *refining problems* are solved. The solution to the refining problem at a given iteration act as a correction to the error (for the original problem) of the previous iterate, and is used to update the current solution. By virtue of this framework, all of the calls to their oracle can be carried out in fixed precision ξ, and a $\hat{\xi}$-precise solution to the original problem, where $0 < \hat{\xi} \ll \xi$, can be obtained in $\mathcal{O}(\log(\hat{\xi})/\log(\xi))$ iterations, i.e. each run is $\tilde{\mathcal{O}}(1)$. The authors in [21] prove that utilizing their iterative refinement algorithm as a quantum oracle that returns a classical description of the solution to the Newton linear system improves the complexity of the IF-QIPM for SDO to

$$\tilde{\mathcal{O}}\left(n^{2.5}\theta_{NT}\kappa\right),$$

where θ_{NT} is an upper bound on the Frobenius norms of the Newton linear system coefficient matrices that arise over the course of the algorithm.[2] This $\tilde{\mathcal{O}}\left(\frac{n\kappa}{\epsilon}\right)$ improvement implies an exponential speedup for QIPMs applied to SDO, and an $\tilde{\mathcal{O}}\left(n^4\right)$ improvement with respect to dimension over classical IPMs.

1.2 Contributions

In this paper we present a short-step primal-dual IF-QIPM for SOCOPs that follows the general scheme of [16], but we take safeguards from the classical IPM literature in order to ensure the validity and convergence of the algorithm on SOCOPs. In particular, we *(i)* modify the Newton linear system using an orthogonal subspaces representation of the search directions as in [4]; *(ii)* we describe how to construct the corresponding linear system with the use of quantum-accessible data structures, making use of the block encoding techniques

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[2]In other words, θ is the largest observed normalization factor.

from [7, 10] along with the iterative refinement scheme of [21]; and *(iii)* provide a detailed analysis of the corresponding running time. The running time of the algorithm presented in this paper is

$$\tilde{\mathcal{O}}\left(\sqrt{n}K\theta\kappa\right).$$

To the best of our knowledge, this paper serves yields the first provably convergent quantum algorithm designed for optimization over the second order cone.

The rest of this paper is organized as follows. Section 2 presents the requisite quantum data structures, and quantum linear system solution techniques. In Section 3 we discuss the classical interior point method, in addition to important theoretical results from the IPM literature. Section 4 details important technical results that will play a crucial role in establishing polynomial convergence of our algorithm. In Section 5 we provide a Quantum Interior Point Method for SOCOPs and theoretical analysis of the running time for this Quantum Interior Point Method. Section 6 concludes the paper.

2 Preliminaries

For $a \in \mathbb{R}^n$, we denote its amplitude encoding by $|a\rangle$, defined as

$$|a\rangle = \frac{1}{\|a\|} \sum_{j \in [n]} a_j |j\rangle.$$

Notice that $|a\rangle$ is a $(\log n)$-qubit state; we assume that the sizes of all spaces are powers of 2 for simplicity. All logarithms are base 2. The smallest and largest singular values of a matrix A are denoted $\sigma_{\min}(A), \sigma_{\max}(A)$, and the smallest and largest eigenvalues are denoted $\lambda_{\min}(A), \lambda_{\max}(A)$. The condition number of A is $\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$.

2.1 Useful results on block-encoded matrices

We now provide formal definitions for the block-encoding of matrices on a quantum computer. The material in this section is mostly taken from [7, 10], which provide improvements over the framework discussed in [13, 14].

Definition 1 (Block encoding). *Let $A \in \mathbb{C}^{2^w \times 2^w}$ be a w-qubit operator. Then, a $(w + a)$-qubit unitary U is an (α, a, ξ)-block encoding of A if $U = \begin{pmatrix} \widetilde{A} & \cdot \\ \cdot & \cdot \end{pmatrix}$, such that*

$$\|\alpha\widetilde{A} - A\| \leq \xi.$$

An (α, a, ξ)-block encoding of A is said to be efficient if it can be implemented in time $T_U = \mathcal{O}(\mathrm{poly}(w))$.

Recall that this only possible for matrices with $\|A\|_2 \leq 1$, so we allow A to be scaled by some constant:

$$U \approx \begin{pmatrix} \frac{A}{\alpha} & \cdot \\ \cdot & \cdot \end{pmatrix},$$

yielding

$$\|A - \alpha(\langle 0|^{\otimes a} \otimes I_{2^w})U(|0\rangle^{\otimes a} \otimes I_{2^w})\| \leq \xi.$$

The next proposition formalizes the construction of block encodings for matrices stored in QRAM. We do not provide details on the necessary data structure, referring the reader to [7] for an extensive discussion on this topic.

Lemma 1 (Lemma 50 in [10]). *Let $A \in \mathbb{C}^{m \times m}$ with $m = 2^w$ and $\xi > 0$.*

(i) *Fix $q \in [0, 2]$ and define $\mu_q(A) = \sqrt{n_q(A)n_{(2-q)}(A^\top)}$ for $n_q(A) = \max_i \|a_i\|_q^q$ (the q-th power of the maximum q-norm of the rows of A). If $A^{(q)}$ and $(A^{(2-q)})^\dagger$ are both stored in QRAM data structures, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\text{poly}(w \log \frac{1}{\xi}))$ and such that $U_R^\dagger U_L$ is a $(\mu_q(A), w+2, \xi)$-block-encoding of A.*

(ii) *If A is stored in a QRAM data structure, then there exist unitaries U_R and U_L that can be implemented in time $\mathcal{O}(\text{poly}(w \log \frac{1}{\xi}))$ and such that $U_R^\dagger U_L$ is an $(\|A\|_F, w+2, \xi)$-block-encoding of A.*

Definition 2 (State preparation pair)**.** *Let $y \in \mathbb{C}^m$ and $\|y\|_1 \leq \beta$. The pair of unitaries (P_L, P_R) is called a (β, p, ξ)-state-preparation-pair if $P_L |0\rangle^{\otimes p} = \sum_{j=0}^{2^p-1} c_j |j\rangle$ and $P_R |0\rangle^{\otimes p} = \sum_{j=1}^{2^p-1} d_j |j\rangle$ such that $\sum_{j=0}^{m-1} |\beta(c_j^* d_j) - y_j| \leq \xi$ and for all $j \in m, \ldots, 2^p - 1$ we have $c_j^* d_j = 0$.*

Proposition 1 (Lemma 52 in [10])**.** *(Linear combination of block-encoded matrices, with weights given by a state preparation pair) Let $A = \sum_{j=1}^m y_j A^j$ be an w-qubit operator, where A^j are matrices. Suppose P_L, P_R is a (β, p, ξ_1)-state-preparation pair for y, $W = \sum_{j=0}^{m-1} |j\rangle \langle j| \otimes U_j + ((I - \sum_{j=0}^{m-1} |j\rangle \langle j|) \otimes I_a \otimes I_s)$ is an $(w + a + p)$-qubit unitary with the property that U_j is an (α, a, ξ_2)-block-encoding of A^j. Then we can implement a $(\alpha\beta, a + p, \alpha\xi_1 + \alpha\beta\xi_2)$-block-encoding of A with a single use of W, P_R and $P_L^\dagger$.*

The following proposition is critical to the efficiency of the QIPMs, and states that one can construct a block encoding as a product of two block encoded matrices, in time that is polylogarithmic in the size of the matrices.

Proposition 2 (Lemma 4 in [7])**.** *(Product of block-encoded matrices) If U_A is an (α, a, δ)-block-encoding of an s-qubit operator A, and U_B is a (β, p, ξ)-block-encoding of an s-qubit operator B, then $(I_p \otimes U_A)(I_a \otimes U_B)$ is an $(\alpha\beta, a + p, \alpha\xi + \beta\delta)$-block-encoding of AB.*

The following proposition is similar to the previous result, with the difference being that here the normalization factor of the block encoding of the product is fixed, rather than dependent on the input.

Proposition 3 (Lemma 5 in [7])**.** *(Product of preamplified block-encoded matrices) Given an (α, a, δ)-block-encoding U_A of an s-qubit operator A, and a (β, p, ξ)-block-encoding U_B of an w-qubit operator B, with $\alpha \geq 1, \beta \geq 1$, then we can implement a $(2, a+p+2, \sqrt{2}(\delta+\xi+\gamma))$-block-encoding of AB in time $\mathcal{O}((\alpha(T_{U_A} + a) + \beta(T_{U_B} + p)) \log \frac{1}{\gamma})$, where T_{U_A} and T_{U_B} are the implementation times for U_A and U_B, respectively, if $\|A\|, \|B\| \leq 1$.*

We can now provide a result from [7] which states how block encodings can be used to solve quantum linear systems problems.

Theorem 1 (Theorem 30 in [7])**.** *(Solution of linear system) Let $r \in (0, \infty)$, $\kappa \geq 2$ and H a Hermitian matrix such that its nonzero eigenvalues lie in $[-1, -1/\kappa] \cup [1/\kappa, 1]$. Suppose that*

$$\delta = o\left(\frac{\xi}{\kappa^2 \log^3 \frac{\kappa^2}{\xi}}\right)$$

and U is an (α, a, δ)-block-encoding of H, that can be implemented in time T_U. Suppose further that we can prepare a state $|v\rangle$ that is in the image of H in time T_v. Then, for any ξ, we can output a state that is ξ-close to $H^{-1} |v\rangle / \|H^{-1}v\|$ in time

$$\mathcal{O}\left(\kappa\left(\alpha(a + T_U) \log^2\left(\frac{\kappa}{\xi}\right) + T_v\right) \log \kappa\right).$$

5

Proposition 4 (Corollary 32 in [7]). *(Norm estimation) Let $p \in (0, \infty)$, $\kappa \geq 2$ and H a Hermitian matrix such that its nonzero eigenvalues lie in $[-1, -1/\kappa] \cup [1/\kappa, 1]$. Suppose that*

$$\delta = o\left(\frac{\xi}{\kappa^2 \log^3 \frac{\kappa^2}{\xi}}\right)$$

and U is an (α, a, δ)-block-encoding of H, that can be implemented in time T_U. Suppose further that we can prepare a state $|v\rangle$ that is in the image of H in time T_v. Then we can output $\tilde{e}$ such that

$$(1 - \xi)\|H^{-1}|v\rangle\| \leq \tilde{e} \leq (1 + \xi)\|H^{-1}|v\rangle\|$$

in time

$$\mathcal{O}\left(\frac{\kappa}{\xi}\left(\alpha(a + T_U)\log^2\left(\frac{\kappa}{\xi}\right) + T_v\right)\log^3 \kappa \log \frac{\log \kappa}{\delta}\right).$$

2.2 Quantum State Tomography

Given that we use QLSAs to solve the Newton linear system in every iteration, the solutions we obtain are encoded as the quantum state $|\Delta x \circ \Delta y \circ \Delta s\rangle$. As discussed in [4], we require a classical description to efficiently construct the Newton linear system for the subsequent iteration. In order to update the current solutions (x, y, s) to the primal and dual SDOPs, we require a procedure to map the solution of the Newton linear system to a classical solution pair $(\Delta x, \Delta y, \Delta s)$. There are many tomography algorithms capable of performing this task (e.g., [17, 27]); and we make the standard assumption that we have access to a unitary preparing the quantum state and its inverse.

As done in [4], we utilize an optimal tomography algorithm from [33] to obtain an ℓ_2-norm estimate of the quantum state.

Theorem 2 ([33]). *Let $|\psi\rangle = \sum_{j=0}^{d-1} y_j |j\rangle$ be a quantum state, $y \in \mathbb{C}^d$ the vector with elements y_j, and $U|0\rangle = |\psi\rangle$. There is a quantum algorithm that, with probability at least $1 - \delta$, outputs $\tilde{y} \in \mathbb{R}^d$ such that $\|\Re(y) - \tilde{y}\|_2 \leq \epsilon$ using $\mathcal{O}(\frac{d}{\epsilon}\ln\frac{d}{\delta})$ applications of U and $\mathcal{O}(\frac{d^{1.5}}{\epsilon}\log\frac{d}{\epsilon}\log\frac{1}{\delta})$ additional gates; the total gate complexity of the algorithm is therefore $\tilde{\mathcal{O}}(T_U\frac{d}{\epsilon} + \frac{d^2}{\epsilon})$.*

Within our QIPM framework we treat the algorithm [33] as a black box oracle that returns a classical estimate when given with the solution state. By Proposition 1 and Theorems 1, 2, we can prepare the linear system, solve it with a QLSA from [7], and obtain a classical estimate of this solution in time $\tilde{\mathcal{O}}\left(\frac{d}{\epsilon}\kappa\theta\right)$, where θ is the normalization factor used in block-encoding the coefficient matrix M, and we used the notion $\kappa = \Omega(d)^3$. We provide a proof of this fact in the next section.

2.3 Iterative Refinement for the LSP

Iterative refinement (IR) is a classical optimization technique that is used to obtain accurate solutions to linear systems of equations. Recent work by Mohammdisaroudhi et al. [21] showed how IR can be leveraged to provide speedups in iterative quantum algorithms that rely on QLSAs. We summarize the core idea at high level, before providing theoretical results for the algorithm in [21], which is presented in full detail in Algorithm 1.

The algorithm commences by using a quantum low-precision linear systems oracle O_{LS} to compute an approximate solution $z^0 \in \mathbb{R}^d$ satisfying

$$Mz^0 \approx \zeta.$$

The oracle O_{LS} is defined by the use of the QLSA from [7] together with the tomography algorithm of [33], and thus outputs a classical solution vector. Additionally O_{LS} constructs the linear systems from data that

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[3]It is well known, see e.g. [4], that κ is much larger than d in the context of (Q)IPMs

is stored in QRAM, so we include the time do so in the overall cost of the oracle. In iterations $k = 1, 2, \ldots$, O_{LS} is used to produce a refined solution $z^k \leftarrow z^{k-1} + \frac{1}{\vartheta} c^{k-1}$, where c^{k-1} acts as a correction for the error

$$\varepsilon^{k-1} = \zeta - M z^{k-1},$$

by solving the refining system

$$M c^{k-1} = \vartheta_{k-1} \varepsilon^{k-1}.$$

These operations can all be carried out using the same level of accuracy or *fixed precision*, which is the approach we take in this work. Once the norm of the error term has been reduced to or below a desired precision, i.e. $\|\varepsilon^k\| \leq \tilde{\xi}$ for some $0 < \tilde{\xi} \ll \xi < 1$, the algorithm terminates and reports z^k as the solution.

Algorithm 1 Iterative Refinement for the Linear Systems Problem from [21]

Input: Coefficient matrix $M \in \mathbb{R}^{2^d \times 2^d}$, right hand side vector $\zeta \in \mathbb{R}^d$.
Choose error tolerances $0 < \tilde{\xi} \ll \xi < 1$.
Choose incremental scaling limit $\rho \in \mathbb{N}$ such that $\rho > 1$, and $\vartheta_0 = 1$.
Choose starting point z^0.
for $k = 1, 2, \ldots$

    1. $\varepsilon^{k-1} \leftarrow \zeta - M z^{k-1}$

    2. **if** $\|\varepsilon^{k-1}\| \leq \tilde{\xi}$ **return** x^k

    3. $\vartheta \leftarrow \min\left\{ \rho\vartheta, \frac{1}{\|\varepsilon^{k-1}\|} \right\}$

    4. $c^{k-1} \leftarrow$ **solve** $(M, \vartheta\varepsilon^{k-1})$ using O_{LS} with precision $\xi = 10^{-2}$

    5. $z^k \leftarrow z^{k-1} + \frac{1}{\vartheta} c^{k-1}$

    6. $k \leftarrow k + 1$

end

2.3.1 Overall running time of IR for the LSP

We now state a series of results from [21], and provide proofs of these results for completeness. We begin by providing a result on the complexity of the quantum linear systems oracle O_{LS}

Proposition 5 (Proposition 1 in [21])**.** *Suppose that the coefficient matrix $M \in \mathbb{R}^{2^d} \times \mathbb{R}^{2^d}$ and right hand side vector ζ are stored in a QRAM data structure. Then, the low-precision quantum linear systems oracle, $O_{LS}(\xi)$, has complexity*

$$\mathcal{O}\left(\mathrm{polylog}\left(\frac{d}{\xi}\right) \frac{d\kappa\theta}{\xi} \right),$$

where θ is the normalization used to block-encode M.

Proof. With M stored in a QRAM data structure, it follows from combining Lemma 1 a $(\theta, d+2, \xi_M)$-block encoding U of M can be implemented in time $T_U = \tilde{\mathcal{O}}(1)$. Applying Theorem 1 the QLS problem can be solved in time

$$T_{LS} = \mathcal{O}\left(\kappa \left(\theta(d + T_U) \log^2\left(\frac{\kappa}{\xi}\right) + T_\zeta \right) \log \kappa \right) = \tilde{\mathcal{O}}\left(\kappa \left(\theta + T_\zeta \right) \right),$$

where T_ζ is the time required to prepare a quantum state $|\zeta\rangle$ encoding the right hand side ζ:

$$|\zeta\rangle = \frac{1}{\|\zeta\|} \sum_i \zeta_i |i\rangle.$$

7

When ζ is stored in a QRAM data structure, we can construct its amplitude encoding in time $T_\zeta = \mathcal{O}\left(\text{polylog}(d/\xi)\right)$ (see [7, Theorem 34]). Thus, T_{LS} can be written as

$$T_{LS} = \tilde{\mathcal{O}}\left(\kappa\theta\right).$$

Applying Theorem 2 with $T_U = T_{LS}$, the quantum linear systems oracle O_{LS} has a worst-case running time of

$$\tilde{\mathcal{O}}\left(T_{LS}\frac{d}{\xi} + \frac{d^2}{\xi}\right) = \tilde{\mathcal{O}}\left((\kappa\left(\theta + T_\zeta\right))\frac{d}{\xi} + \frac{d^2}{\xi}\right) = \tilde{\mathcal{O}}\left(\kappa\theta\frac{d}{\xi} + \frac{d^2}{\xi}\right) = \tilde{\mathcal{O}}\left(d\kappa\theta\frac{1}{\xi}\right),$$

where we have used $\kappa = \Omega(d)$ in the worst case. $\square$

The next result from [21] establishes that Algorithm 1 generates a sequence of solutions which solve the linear system at hand with increasing accuracy.

Theorem 3 (Theorem 4 in [21]). *Let $\varepsilon^{k-1} = \zeta - Mz^{k-1}$ for $k \geq 0$, and c^{k-1} be an ξ-precise solution of the refining system $Mc^{k-1} = \vartheta_k\varepsilon^{k-1}$, then $z^k = z^{k-1} + \frac{1}{\vartheta_k}c^{k-1}$ is an $\frac{\xi}{\vartheta_k}$-precise solution of the original system $Mz = \zeta$ for $k \geq 1$.*

Proof. For any ξ-precise solution c^{k-1} of the refining system $Mc = \vartheta_k\varepsilon^{k-1}$, we have

$$\|\vartheta_k\varepsilon^{k-1} - Mc^{k-1}\| \leq \xi.$$

Hence,

$$\left\|\zeta - M\left(z^{k-1} + \frac{1}{\vartheta_k}c^{k-1}\right)\right\| = \left\|\zeta - Mz^{k-1} - M\frac{1}{\vartheta_k}c^{k-1}\right\| = \frac{1}{\vartheta_k}\left\|\vartheta_k\varepsilon^{k-1} - Mc^{k-1}\right\| \leq \frac{\xi}{\vartheta_k}.$$

Therefore, we conclude that $z^k = z^{k-1} + \frac{1}{\vartheta_k}c^{k-1}$ is an $\frac{\epsilon}{\vartheta_k}$-precise solution of the original system $Mz = \zeta$ for $k \geq 1$. $\square$

The final results of this section provide the polynomial convergence and overall complexity of Algorithm 1.

Theorem 4 (Theorem 5 in [21]). *Let $0 < \hat{\xi} \ll \xi$, $\rho > 1$ and $\vartheta_0 = 1$. Then, Algorithm 1 terminates in at most $\mathcal{O}\left(\frac{\log(\hat{\xi})}{\log(\xi)}\right)$ iterations.*

Proof. From Theorem 3, it follows that after k iterations we have a $\frac{\xi}{\vartheta_k}$-precise solution to our linear system. Now,

$$\vartheta_k = \min\left\{\rho\vartheta_{k-1}, \frac{1}{\|r^{k-1}\|}\right\} \geq \vartheta_{k-1}\min\left\{\rho, \frac{1}{\xi}\right\}.$$

Setting $\chi = \min\left\{\rho, \frac{1}{\xi}\right\} > 1$, it follows

$$\vartheta_k \geq \chi\vartheta_{k-1} \geq \chi^k.$$

Thus, we have $\frac{\xi}{\vartheta_k} \leq \frac{\epsilon}{\chi^k} \leq \hat{\xi}$ when k satisfies

$$k \geq \frac{\log(\xi/\hat{\xi})}{\log\chi} \geq \frac{\log(\xi/\hat{\xi})}{-\log\xi}.$$

As a consequence, $k = \mathcal{O}\left(\frac{\log(\hat{\xi})}{\log(\xi)}\right)$ iterations suffice to obtain $\hat{\xi}$-precise solution, and the proof is complete. $\square$

Theorem 5 (Theorem 6 in [21]). *Let $0 < \hat{\xi} \ll \xi$ and the problem data be stored in a quantum accessible data structure. Then, Algorithm 1 obtains a $\hat{\xi}$-precise solution of a linear system in time*

$$\tilde{\mathcal{O}}\left(d\theta\kappa\right),$$

where θ is the normalization factor of M.

8

Proof. By combining results of Proposition 5 and Theorem 4, Algorithm 1 converges to a ξ-precise solution (in ℓ_2-norm) to the LSP with worst case overall complexity

$$\mathcal{O}\left(\frac{d\theta\kappa}{\xi}\operatorname{polylog}\left(\frac{d}{\xi}\right)\frac{\log(\hat{\xi})}{\log(\xi)}\right).$$

Due to the fact every iteration calls O_{LS} with fixed precision, e.g. $\xi_k = 10^{-2}$ for all iterations $k = 1, 2, \ldots$, ξ can be treated as a constant which is neglected in big-$\mathcal{O}$ notation. The overall running time of Algorithm 1 is thus:

$$\tilde{\mathcal{O}}\left(d\theta\kappa\right).$$

$\square$

3   The Classical Interior Point Method for SOCOPs

In this section, we provide a detailed overview of SOCOPs, the central path and a small neighborhood of the central path.

3.1   The Primal and Dual SOCOPs

We consider the standard SOCOP which has the form

$$z_P = \min_x \left\{ \sum_{i=1}^n c_i^\top x_i : \sum_{i=1}^n A_i x_i = b, x_i \in \mathcal{K}_i, \ i \in [n] \right\} \tag{2}$$

where $x_i \in \mathbb{R}^{k_i}$, $i \in [n]$ are the variables, $b \in \mathbb{R}^m$, $A_i \in \mathbb{R}^{m \times k_i}$ and $c_i \in \mathbb{R}^{k_i}$, $i \in [n]$, are the data, and $\mathcal{K}_i$ is the second order cone of dimension k_i given by

$$\mathcal{K}_i = \left\{ x_i = (x_{i0}, x_{i1}) \in \mathbb{R} \times \mathbb{R}^{k_i - 1} : x_{i0} - \|x_{i1}\| \geq 0 \right\}, \quad \forall i \in [n].$$

It is a well-known fact that second order cones are self-dual; letting

$$\mathcal{K}_i^* = \left\{ s_i \in \mathbb{R}^{k_i} : s_i^\top x_i \geq 0, \ \forall x_i \in \mathcal{K}_i \right\}$$

denote the dual cone of $\mathcal{K}_i$, we have $\mathcal{K}_i = \mathcal{K}_i^*$ for every $i \in [n]$.[4] The dual problem associated with the primal problem is then

$$z_D = \max_y \left\{ b^\top y : A_i^\top y + s_i = c_i, s_i \in \mathcal{K}_i, \forall i \in [n] \right\}. \tag{3}$$

Before proceeding further, let us define the quantities

$$K = k_1 + \cdots + k_n, \qquad\qquad \mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_n,$$
$$A = (A_1 A_2 \ldots A_n) \in \mathbb{R}^{m \times K}, \qquad\qquad c = (c_1, \ldots, c_n) \in \mathbb{R}^K,$$
$$x = (x_1, \ldots, x_n) \in \mathbb{R}^K, \qquad\qquad s = (s_1, \ldots, s_n) \in \mathbb{R}^K,$$

and the primal and dual feasible sets:

$$\mathcal{P} = \{ x \in \mathcal{K} : Ax = b \}, \quad \mathcal{D} = \left\{ (y, s) \in \mathbb{R}^m \times \mathcal{K} : A^\top y + s = c \right\}.$$

Without loss of generality we assume that $m = \mathcal{O}(K)$. Using these quantities, the primal and dual SOCOPs (2) and (3) can be rewritten as

$$z_P = \min_x \left\{ c^\top x : Ax = b, x \in \mathcal{K} \right\}, \tag{P}$$
$$z_D = \max_y \left\{ b^\top y : A^\top y + s = c, s \in \mathcal{K} \right\}. \tag{D}$$

---

[4]For this reason, we simply write $\mathcal{K}_i$ when referencing the dual cone.

Continuing along this line, the sets of *interior feasible solutions* of (P) and (D) are:

$$\mathcal{P}^0 \equiv \{x : Ax = b, x \in \text{int}\,(\mathcal{K})\}, \quad \mathcal{D}^0 \equiv \{(y, s) : A^\top y + s = c,\ s \in \text{int}\,(\mathcal{K})\},$$

respectively, where $\text{int}\,(\mathcal{K})$ denotes the interior of $\mathcal{K}$.

Without loss of generality, we assume that there exists a strictly feasible point (x^0, y^0, s^0) such that $\mathcal{P}^0 \times \mathcal{D}^0 \neq \emptyset$, i.e. the Interior Point Condition (IPC) is satisfied [34]. This is mild assumption; one can easily cast the original SOCOP as a slightly larger one via the self-dual embedding model, for which a strictly feasible solution can be obtained in a straightfoward manner. See Andersen, Roos, and Terlaky [3] for more details. Additionally, we assume the rows of the matrix $A = (A_1, \ldots, A_n)$ are linearly independent such that A has full row rank. With the IPC satisfied, it follows that one can obtain an optimal solution of the SOCOP by solving the system

$$x^\top s = 0, \quad Ax = b, \quad A^\top y + s = c,$$

for $(x, s, y) \in \mathcal{K} \times \mathcal{K} \times \mathbb{R}^m$.

3.2 Euclidean Jordan algebra, the central path and Newton directions

From here we develop the primal-dual interior point method studied in this paper using the Euclidean Jordan algebra associated with second order cones following [23]. The Eucildean Jordan algebra for the second order cone $\mathcal{K}_i$ is defined by a bilinear $\circ : \mathbb{R}^{k_i} \times \mathbb{R}^{k_i} \to \mathbb{R}^{k_i}$:

$$x_i \circ s_i = \left(x_i^\top s_i,\ x_{i0} s_{i1} + s_{i0} x_{i1}\right), \ \forall x_i, s_i \in \mathbb{R}^{k_i},$$

where the unit element is given by $e_i = (1, 0, \ldots, 0)$. The natural extension of the algebra for $\mathcal{K}_i$ to $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_n$ is defined as

$$x \circ s = (x_1 \circ s_1, \ldots, x_n \circ s_n), \ \forall x, s \in \mathbb{R}^K,$$

with unit element $e \equiv (e_1, \ldots, e_n)$. Further, for $x \in \mathbb{R}^K$, we let $\textbf{mat}(x)$ denote the matrix $\textbf{diag}(X_1, \ldots X_n)$ where

$$X_i = \begin{pmatrix} x_{i0} & x_{i1}^\top \\ x_{i1} & x_{i0}I \end{pmatrix}, \quad i = 1, \ldots, n. \tag{4}$$

It follows trivially that

$$x \circ s = \textbf{mat}(x)s = \textbf{mat}(s)x.$$

Further, $\textbf{mat}(x)$ is a symmetric matrix, with its smallest and largest eigenvalues defined as

$$\lambda_{\min}(\textbf{mat}(x)) = \min \{x_{i0} - \|x_{i1}\| \ : \ i = 1, \ldots, n\} \tag{5a}$$
$$\lambda_{\max}(\textbf{mat}(x)) = \max \{x_{i0} + \|x_{i1}\| \ : \ i = 1, \ldots, n\}. \tag{5b}$$

respectively. For more details, see Lemma 2.13 of [32]. Observe that $\textbf{mat}(x)$ is symmetric positive semidefinite if and only if $x \in \mathcal{K}$, and symmetric positive definite if and only if $x \in \text{int}\,(\mathcal{K})$.

We can define the central path of our primal-dual SOCO pair (P)-(D) as the set of solutions $(x, s, y) \in \mathcal{K} \times \mathcal{K} \times \mathbb{R}^m$ that solve

$$x \circ s = \nu e \quad Ax = b, \quad A^\top y + s = c, \tag{6}$$

for all $\nu > 0$. When combined with our assumption that A has full row rank, the IPC implies that (6) has a unique solution $(x, s, y) = (x_\nu, s_\nu, y_\nu) \in \text{int}\,(\mathcal{K}) \times \text{int}\,(\mathcal{K}) \times \mathbb{R}^m$. Furthermore, this solution (x_ν, s_ν, y_ν) depends on $\nu > 0$ both continuously and analytically, and as $\nu \to 0$, we obtain an optimal solution to the primal-dual SOCOP [23].

We outline a classical IPM for SOCOPs in Algorithm 2. The algorithm begins with an initial solution pair (x^0, y^0, s^0) which is strictly feasible for the SOCO primal and dual problems (P) and (D), i.e., $(x^0, y^0, s^0) \in \mathcal{P}^0 \times \mathcal{D}^0$. Moreover, this initial solution has a duality gap of

$$x^{0^\top} s^0 = \mu^0 n,$$

10

and a distance to the central path of

$$d(x, s) \leq \gamma \mu^0$$

for some $\gamma \leq (0, 1)$. Later in this section, we present suitable choices for $d(x, s)$, which in turn allow us to define various neighborhoods of the central path.

In each iteration of the classical (exact-feasible) IPM for SOCOPs presented in [23], one seeks to solve the Newton linear system

$$S\Delta x + X\Delta s = \sigma \mu e - Xs, \quad A^\top \Delta y + \Delta s = c - s - A^\top y, \quad A\Delta x = b - Ax, \tag{7}$$

for $(\Delta x, \Delta s, \Delta y)$, where $X \equiv \mathbf{mat}(x)$, $S \equiv \mathbf{mat}(s)$, and $\mu \equiv \mu(x, s) = x^\top s / n$. Further, a solution $(\Delta x, \Delta s, \Delta y)$ to the system (7) is the so called AHO direction.

Finally, letting $\mu' = \mu(x', s')$, we update the solutions for the subsequent iterate using the rule: $(x, y, s) = (x+\Delta x, y+\Delta y, s+\Delta s)$ and update μ accordingly, setting $(x+\Delta x)^\top (s+\Delta s) = \mu' n$. Once μ has been reduced to or below a desired optimality tolerance $\epsilon > 0$, we can conclude that the central path parameter has been sufficiently reduced, at which point Algorithm 2 terminates and reports the optimal solution (x^*, s^*, y^*).

Algorithm 2 Classical interior point method for SOCOPs

Choose constants $\gamma \in (0, 1/3)$, $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$;
Set $\sigma = 1 - \delta/\sqrt{2n}$;
Choose initial point $(x^0, y^0, s^0) \in \mathcal{N}_2(\gamma)$, store in QRAM
Set $\mu_0 = \mu(x^0, s^0)$
while $\mu > \epsilon$:

1. Solve the Newton linear system to obtain $(\Delta x, \Delta y, \Delta x)$

2. Update solution

$$x \leftarrow x + \Delta x \; y \leftarrow y + \Delta y \text{ and } s \leftarrow s + \Delta s$$

3. $\mu = \mu(\alpha)$

3.3 The Nullspace Representation of the Newton Linear System

In [16], it is assumed that the Newton linear system

$$S\Delta x + X\Delta s = \sigma \mu e - Xs, \quad A^\top \Delta y + \Delta s = 0, \quad A\Delta x = 0,$$

can be solved exactly using a quantum computer for $(\Delta x, \Delta s, \Delta y)$. Applying quantum state tomography to extract a classical estimate of the solution obtained via QLSA introduces noise to the classical representation, which will only satisfy

$$S\Delta x + X\Delta s = \sigma \mu e - Xs + \xi_c, \quad A^\top \Delta y + \Delta s = \xi_d, \quad A\Delta x = \xi_p,$$

where (ξ_c, ξ_d, ξ_p) are the quantum errors. Therefore there is no guarantee that the classical estimates obtained over the course of the algorithm satisfy primal or dual feasibility. This of course invalidates the primal-dual IPM framework they employ. Rather, consider the following system in which there is some error in the complementary equation but primal and dual feasibility are satisfied exactly:

$$S\Delta x + X\Delta s = \sigma \mu e - Xs + r, \quad A^\top \Delta y + \Delta s = 0, \quad A\Delta x = 0,$$

where $r \in \mathbb{R}^K$ is a residual term.

As it turns out, the constraints stipulating primal and dual feasibility have a useful interpretation, which the authors in [4] showed can be leveraged to overcome the challenges that arise in quantizing QIPMs on noisy quantum devices. For primal-dual feasible solution we have $x \in \mathcal{N}(A)$ and $s \in \mathcal{R}(A)$, where $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the nullspace and rowspace of $A \in \mathbb{R}^{m \times K}$, respectively. Given our assumption that A has full row rank, we can choose the matrices $\{A_1^\top, \ldots, A_n^\top\}$ as our basis for the rowspace of A. To compute a basis for $\mathcal{N}(A)$ we can employ Gaussian elimination or a QR-factorization of A; defined by:

$$A^\top = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix}.$$

It follows that the columns of $Q_2 \in \mathbb{R}^{K \times (K-m)}$ form a basis for $\mathcal{N}(A)$, so we set

$$\Delta s = -A^\top \Delta y, \tag{8a}$$

$$\Delta x = Q_2 \Delta z. \tag{8b}$$

Hence, if we are using the AHO direction, substituting these expressions for Δx and Δs into the complementrarity equation, at each iteration of our IF-QIPM for SOCOP, we can simply solve:

$$\begin{bmatrix} SQ_2 & -XA \end{bmatrix} \begin{bmatrix} \Delta z \\ \Delta y \end{bmatrix} = \sigma \mu e - Xs. \tag{9}$$

The next result formalizes how solving (9) guarantees that primal and dual feasibility to be satisfied.

Proposition 6. *Let $(x, s, y) \in \mathcal{P}^0 \times \mathcal{D}^0$ be given, and let $(\overline{\Delta z}, \overline{\Delta y})$ be a classical estimate of the solution to (9) where $(\overline{\Delta z}, \overline{\Delta y}) = (\Delta z + \xi_z, \Delta y + \xi_y)$, and ξ_z and ξ_y are the error terms of the system (9). Letting $\overline{\Delta x}$ and $\overline{\Delta s}$ be given by (8a) and (8b), for any stepsize α we have*

$$A(x + \alpha \overline{\Delta x}) = b, \ and \ A^\top(y + \alpha \overline{\Delta y}) + (s + \alpha \overline{\Delta s}) = c.$$

Proof. For any $(x, s, y) \in \mathcal{P}^0 \times \mathcal{D}^0$, $Ax = b$ and $A^\top y + s = c$ hold. Therefore,

$$A(x + \alpha \overline{\Delta x}) = Ax + \alpha A \overline{\Delta x} = b + \alpha A Q_2 \Delta z = b,$$

and similarly,

$$A^\top(y + \alpha \overline{\Delta y}) + (s + \alpha \overline{\Delta s}) = \left[A^\top y + s \right] + \alpha \left[A^\top \overline{\Delta y} + \overline{\Delta s} \right]$$
$$= c + \alpha \left[A^\top \overline{\Delta y} + (-A^\top \overline{\Delta y}) \right] = c.$$

$\square$

3.4 Scaling, eigenvalues and neighborhoods of the central path

From here, we follow Monteiro and Tsuchiya [23] and introduce a group of scaling automorphims which map the cone $\mathcal{K}$ onto itself. We then use this group to characterize the scale-invariant neighborhoods of the central path.

Let

$$\mathcal{G}_i = \left\{ \lambda \widetilde{T}_i : \lambda > 0, \widetilde{T}_i \in \mathbb{R}^{k_i \times k_i}, \widetilde{T}_i^\top J_{k_i} \widetilde{T}_i = J_{k_i}, (\widetilde{T}_i)_{00} > 0 \right\}, \tag{10}$$

where

$$J_{k_i} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} \in \mathbb{R}^{k_i \times k_i}.$$

It follows that $\mathcal{G}_i$ is defined as the auto-morphism group of the cone $\mathcal{K}_i$; the set of all nonsingular matrices T_i satisfying $\mathcal{K}_i = T_i(\mathcal{K}_i) \equiv \{T_i x_i : x_i \in \mathcal{K}_i\}$.[5] Likewise, choosing

$$\mathcal{G} \equiv \{T = \mathbf{diag}(T_1, \ldots, T_n) : T_i \in \mathcal{G}_i, i = 1, \ldots, n\},$$

we have that $\mathcal{G}$ is a subgroup of the auto-morphism group of the cone $\mathcal{K}$.

The next result from [32] provides an explicit definition of the unique $\mathcal{G} \succ 0$ that maps e to x.

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[5]A proof of this fact can be found in the appendix of [23].

Proposition 7 (Proposition 2.1 [32])**.** *For any $x \in \text{int}(\mathcal{K})$, there exists a unique symmetric matrix in $\mathcal{G}$ which maps e to x given by $T_x \equiv \boldsymbol{diag}(T_{x_1}, \ldots, T_{x_n})$ where for all $i = 1, \ldots, n$,*

$$T_{x_i} = \begin{pmatrix} x_{i0} & x_{i1}^\top \\ x_{i1} & \beta_{x_i} I + \dfrac{x_{i1} x_{i1}^\top}{\beta_{x_i} + x_{i0}} \end{pmatrix}, \tag{11}$$

and

$$\beta_{x_i} = \sqrt{x_{i0}^2 - \|x_{i1}\|^2}. \tag{12}$$

Moreover, T_x is positive definite.

Following [23, 32], we define the set of $2n$ eigenvalues $\left\{ \lambda_i^j : i = 1, \ldots, n, j = 0, 1 \right\}$ corresponding to $v = (v_1, \ldots, v_n) \in \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}$ of the Cartesian algebra to be

$$\lambda_i^0 = \lambda_i^0(v) \equiv v_{i0} - \|v_{i1}\|, \quad \lambda_i^1 = \lambda_i^1(v) \equiv v_{i0} + \|v_{i1}\|,$$

for $i = 1, \ldots, n$. Hence, we can conclude $v \in \mathcal{K}$ if and only if $\lambda_i^0 \geq 0$ for all i and similarly, that $v \in \text{int}(\mathcal{K})$ if and only if $\lambda_i^0 > 0$. For a more detailed explanation of eigenvalues in the context of SOCOPs, we refer the reader to Section 2.4 of [32].

We now use these definitions to express the neighborhoods of the central path. For any $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$, we have the following distance metrics $d(x, s)$:

$$d_2(x, s) \equiv \sqrt{\sum_{i=1,\ldots,n, j=0,1} \left(\lambda_i^j(w_{xs} - \mu) \right)^2} = \sqrt{2} \|w_{xs} - \mu e\|.$$

$$d_\infty(x, s) \equiv \max_{\substack{i=1,\ldots,n, \\ j=0,1}} \left| \lambda_i^j(w_{xs}) - \mu \right| = \max_{i=1,n} \left\{ |w_{i0} + \|w_{i1}\| - \mu|, |w_{i0} - \|w_{i1}\|, -\mu| \right\},$$

where $\mu = \mu(x, s)$ and

$$w_{xs} = (w_1, \ldots, w_n) \equiv T_x s. \tag{13}$$

Then, for $\gamma \in (0, 1)$, the short-step (or Euclidean), and infinity neighborhoods of the central path are respectively given by

$$\mathcal{N}_2(\gamma) = \left\{ (x, s, y) \in \mathcal{P}^0 \times \mathcal{D}^0 : d_2(x, s) \leq \gamma\mu(x, s) \right\},$$
$$\mathcal{N}_\infty(\gamma) = \left\{ (x, s, y) \in \mathcal{P}^0 \times \mathcal{D}^0 : d_\infty(x, s) \leq \gamma\mu(x, s) \right\}.$$

One can observe that $d_\infty(x, s) \leq d_2(x, s)$ for all $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$, and therefore $\mathcal{N}_2(\gamma) \subset \mathcal{N}_\infty(\gamma)$ [23]. We point out that the product $w_{xs} = T_x s$ is the SOCO analog of the quantity $X^{1/2} S X^{1/2}$ that arises in the context of SDO. This quantity constitutes the scaled dual variable when the primal variable X is scaled to the identity matrix I. For our purposes, $w_{xs} = T_x s$ is the scaled dual variable when we scale x to e, which is why we replace the distance measure $\|X^{1/2} S X^{1/2} - \mu I\|_F$ that arises in the context of SDO with $\sqrt{2}\|w_{xs} - \mu e\|$.

The following result from [32] establishes the invariance properties of the eigenvalues of w_{xs}.

Proposition 8 (Proposition 2.4 in [32])**.** *Suppose that $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$ and $G \in \mathcal{G}$. Let $(\widetilde{x}, \widetilde{s}) \equiv (G^\top x, G^{-1} s)$, $w \equiv w_{xs}$ and $\widetilde{w} = w_{\widetilde{x}\widetilde{s}}$. Then:*

  (a) $\widetilde{w}_{i0} = w_{i0}$ and $\|\widetilde{w}_{i1}\| = \|w_{i1}\|$ for every $i = 1, \ldots, n$.

  (b) $\lambda_i^j(\widetilde{w}) = \lambda_i^j(w)$ for every $i = 1, \ldots, n$ and $j = 0, 1$.

  (c) $d_2(\widetilde{x}, \widetilde{s}) = d_2(x, s)$ and $d_\infty(\widetilde{x}, \widetilde{s}) = d_\infty(x, s)$.

4   Technical Results

In this section, we establish results that are necessary for proving polynomial convergence of the IF-QIPM for SOCOPs. The results are adapted from [22] to account for the inexactness in our solution to the complementarity equation. We first prove that the AHO direction (15) is well defined in the neighborhood $\mathcal{N}_\infty$ for $\gamma \in (0, 1/3)$, regarldess of errors introduced to the complementarity of the solution by quantum tomography. We then show that each member of the Monterio-Zhang (MZ) family of directions for SDO in the context of SOCO reduces to the AHO direction in the scaled space. In this manner, our results for the AHO direction extend to each member of the MZ family of directions.

Given $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$, we follow [23] in defining the quantities

$$X \equiv \mathbf{mat}(x),$$
$$S \equiv \mathbf{mat}(s),$$
$$R_{xs} \equiv T_x X^{-1} S T_x, \tag{14a}$$
$$W_{xs} \equiv \mathbf{mat}(w_{xs}). \tag{14b}$$

Further, note that the triple (x, s, y) is the solution which serves as the classical counterpart to the solution of our quantum Newton linear system $|z, y\rangle$. In particular, (x, s, y) solves the system

$$S\Delta x + X\Delta s = \sigma\mu e - Xs + r \tag{15a}$$
$$A^\top \Delta y + \Delta s = 0 \tag{15b}$$
$$A\Delta x = 0, \tag{15c}$$

where r is a residual term which captures the errors introduced by our use of inexact tomography. We assume that r satisfies

$$\|r\| \leq \eta\|\sigma\mu e - Xs\| \leq \eta\gamma\sigma\mu/\sqrt{2} \tag{AR1}$$

for $\eta \in (0, 1)$.

Lemma 2 (Lemma 1 in [23]). *For any $x \in \text{int}(\mathcal{K})$, the matrices X and T_x satisfy:*

(a) $X - T_x = U_x = \mathbf{diag}(U_{x_i} : i = 1, \ldots, n)$, *where*

$$U_{x_i} = \begin{bmatrix} 0 & 0 \\ 0 & (x_{i0} - \beta_{x_i})P_{x_i} \end{bmatrix}$$

and P_{x_i} is the orthogonal projection matrix onto the subspace orthogonal to x_{il}, namely

$$P_{x_i} \equiv I - \frac{x_{i1}x_{i1}}{\|x_{i1}\|^2} \tag{16}$$

(b) $T_x X^{-1} = X^{-1} T_x = \mathbf{diag}(I - x_{i0}^{-1} U_{x_i} : i = 1, \ldots, n)$; *as a consequence $T_x X^{-1} e = e$;*

(c) X *and* T_x *commute and* $X \succeq T_x$.

The following two lemmas from [23] play a crucial role in the analysis of our algorithm.

Lemma 3 (Lemma 2 in [23]). *We have $R_{xs} = \mathbf{diag}(R_i : i = 1, \ldots, n)$, where*

$$R_i = \begin{bmatrix} w_{i0} & w_{i1}^\top \\ w_{i1} & \widetilde{R}_i \end{bmatrix} \tag{17}$$

with $(w_{i0}, w_{i1}) = T_{x_i} s_i \in \mathbb{R} \times \mathbb{R}^{k_i - 1}$ and

$$\widetilde{R}_i \equiv \frac{1}{x_{i0}}\left[w_{i1}x_{i1}^\top + \beta_{x_i}^2 s_{i0}I\right] = \frac{w_{i1}x_{i1}^\top}{x_{i0}} + \left(w_{i0} - \frac{w_{i1}^\top x_{i1}}{x_{i0}}\right)I. \tag{18}$$

Lemma 4 (Lemma 3 in [23]). *Let $(x, s, y) \in \text{int}\,(\mathcal{K}) \times \text{int}\,(\mathcal{K}) \times \mathbb{R}^m$ be a triple such that*

$$\max_{i,j} \left| \lambda_i^j(w_{xs}) - v \right| \leq \gamma\nu,$$

for some scalars $\gamma > 0$ and $\nu > 0$. Then,

$$\|R_{xs} - W_{xs}\| \leq 2\gamma\nu, \tag{19a}$$
$$\|W_{xs} - \nu I\| \leq \gamma\nu. \tag{19b}$$

As a consequence,

$$\|R_{xs} - \nu I\| \leq 3\gamma\nu.$$

Lemma 5 (Lemma 4 in [23]). *Let $(x, s, y) \in \text{int}\,(\mathcal{K}) \times \text{int}\,(\mathcal{K}) \times \mathbb{R}^m$ be a triple such that*

$$\|R_{xs} - \nu I\| \leq \tau\nu, \tag{20}$$

for some scalars $\tau \in (0, 1)$ and $\nu > 0$. Assume that $(u, v) \in \mathbb{R}^K \times \mathbb{R}^K$ and $h \in \mathbb{R}^K$ satisfy

$$Su + Xv = h, \quad u^T v \geq 0, \tag{21}$$

and define $\delta_u \equiv \|T_x^{-1}u\|$ and $\delta_v \equiv \|T_x v\|$. Then,

$$\delta_u \leq \frac{\|T_x X^{-1}h\|}{(1-\tau)\nu}, \quad \delta_v \leq \frac{2\|T_x X^{-1}h\|}{1-\tau}. \tag{22}$$

The following result establishes the validity of using a classical estimate of the solution to our quantum solution to obtain a classical solution to the system (15). Moreover, this solution is unique.

Theorem 6. *Let $(x, s, y) \in \text{int}\,(\mathcal{K}) \times \text{int}\,(\mathcal{K}) \times \mathbb{R}^m$ and $(\Delta z, \Delta y)$ be a classical estimate of the solution to the quantum Newton linear system (9) after using tomography. Then,*

$$(\Delta x, \Delta s, \Delta y) = \left(Q_2 \Delta z, -A^\top y, \Delta y \right)$$

is a solution to the classical Newton linear system (15).

Proof. As in [4], the first part of the statement can be verified trivially via substitution. $\square$

Now we are in a position to state the result from [23] which establishes the well-definedness of the AHO direction.

Theorem 7 (Theorem 1 in [23]). *Let $(x, s, y) \in \text{int}\,(\mathcal{K}) \times \text{int}\,(\mathcal{K}) \times \mathbb{R}^m$ be a point such that*

$$\max_{i,j} \left| \lambda_i^j(w_{xs}) - v \right| \leq \gamma\nu,$$

for some scalars $\gamma \in (0, 1/3)$ and $\nu > 0$. Then, the system (15) has a unique solution. In particular, the AHO direction is well-defined at every point $(x, s, y) \in \text{int}\,(\mathcal{K}) \times \text{int}\,(\mathcal{K}) \times \mathbb{R}^m$ such that $d_\infty(x, s) < \mu(x, s)/3$.

Corollary 1. *If the system (15) has a unique solution, then the quantum Newton linear system (9) has a unique solution.*

For our work here, we define

$$x(\alpha) \equiv x + \alpha\Delta x, \qquad\qquad s(\alpha) \equiv s + \alpha\Delta s,$$
$$y(\alpha) \equiv y + \alpha\Delta y, \qquad\qquad \mu(\alpha) \equiv \frac{x(\alpha)^\top s(\alpha)}{n}.$$

In the lemmas that follow, we seek to provide a bound on

$$\sqrt{2} \left\| T_x^{-1}x(\alpha) \circ T_x s(\alpha) - \mu(\alpha)e \right\|,$$

which majorizes our centrality measure $d_2\left(x(\alpha), s(\alpha) \right)$. We make one additional assumption; that the residual term r satisfies

$$\|T_x X^{-1}r\| \leq \eta \|T_x X^{-1}(\sigma\mu e - Xs)\|. \tag{AR2}$$

Lemma 6. *Let* $(x, s, y) \in \mathcal{P}^0 \times \mathcal{D}^0$ *and let* $(\Delta x, \Delta s, \Delta y)$ *be a solution to the system* (15) *for some* $\sigma \in \mathbb{R}$. *Then for every* $\alpha \in \mathbb{R}$ *we have:*

$$\mu(\alpha) = (1 - \alpha + \sigma\alpha)\mu + \alpha\frac{e^\top r}{n}, \tag{23a}$$

$$T_x^{-1}x(\alpha) \circ T_x s(\alpha) - \mu(\alpha)e = (1 - \alpha)(w_{xs} - \mu e) + \alpha(W_{xs} - R_{xs})\widetilde{\Delta x}]$$
$$+ \alpha\left[T_x X^{-1}r - \frac{e^\top r}{n}e\right] + \alpha^2\widetilde{\Delta x} \circ \widetilde{\Delta s}, \tag{23b}$$

where $\mu = \mu(x, s)$ *and*

$$\widetilde{\Delta x} \equiv T_x^{-1}\Delta x, \qquad \widetilde{\Delta s} \equiv T_x \Delta s. \tag{24}$$

Proof. Since $(x, s, y) \in \mathcal{P}^0 \times \mathcal{D}^0$ by assumption, applying (15b) and (15c), one has $\Delta x^\top \Delta s = 0$. Next, multiplying (15a) on the left by $e^\top$ yields

$$e^\top S\Delta x + e^\top X\Delta s = s^\top \Delta x + x^\top \Delta s = \sigma n\mu - x^\top s + e^\top r = -(1 - \sigma)n\mu + e^\top r.$$

Hence,

$$x(\alpha)^\top s(\alpha) = (x + \alpha\Delta x)^\top (s + \alpha\Delta s) = x^\top s + \alpha\left(s^\top \Delta x + x^\top \Delta s\right)$$
$$= n\mu + \alpha[-(1 - \sigma)n\mu + e^\top r] = n\mu\left[1 - \alpha(1 - \sigma)\right] + \alpha e^\top r.$$

Thus, (23a) follows from dividing both sides of this equality by n. On the other hand, multiplying (15a) on the left by $T_x X^{-1}$, as well as applying (13), (14a), (24) and Lemma 2(b), yields

$$\widetilde{\Delta s} = T_x \Delta s = T_x X^{-1}\left(\sigma\mu e - Xs + r - S\Delta x\right) = \sigma\mu e - w_{xs} + T_x X^{-1}r - R_{xs}\widetilde{\Delta x}.$$

We then apply this relation, along with (13), (14b), (24) and the fact that $u \circ v = \mathbf{mat}(u)v$ for all $u, v \in \mathbb{R}^K$, from which it follows

$$T_x^{-1}x(\alpha) \circ T_x s(\alpha) = T_x^{-1}(x + \alpha\Delta x) \circ T_x (s + \alpha\Delta s)$$
$$= \left(e + \alpha\widetilde{\Delta x}\right) \circ \left(w_{xs} + \alpha\widetilde{\Delta s}\right)$$
$$= w_{xs} + \alpha\left(w_{xs} \circ \widetilde{\Delta x} + \widetilde{\Delta s}\right) + \alpha^2\widetilde{\Delta x} \circ \widetilde{\Delta s}$$
$$= w_{xs} + \alpha\left[\sigma\mu e - w_{xs} + T_x X^{-1}r + (W_{xs} - R_{xs})\widetilde{\Delta x}\right] + \alpha^2\widetilde{\Delta x} \circ \widetilde{\Delta s}.$$

Combining the above result with (23a) yields (23b) and the proof is complete. □

The following lemma establishes bounds on the norms involving the residual terms that appear in the rest of our analysis.

Lemma 7. *Let* r *be a residual term satisfying the assumptions* (AR1)-(AR2). *Then,*

$$\sqrt{2}\left\|T_x X^{-1}r\right\| \leq \eta\gamma\sigma\mu \tag{25a}$$

$$\sqrt{2}\left\|T_x X^{-1}r - \frac{e^\top r}{n}e\right\| \leq \eta\gamma\sigma\mu. \tag{25b}$$

Proof. Recall from from Lemma 2(b) that $T_x X^{-1}e = e$. By our assumption (AR2) we have:

$$\left\|T_x X^{-1}r\right\| \leq \eta\left\|T_x X^{-1}(\sigma\mu e - Xs)\right\| = \eta\left\|(\sigma\mu e - w_{xs})\right\| \leq \eta\gamma\sigma\mu/\sqrt{2}.$$

and therefore (25a) holds.

For ease of notation, we define $\widetilde{r} \equiv T_x X^{-1} r$. Then, expanding the square and using the fact that $e^\top e = n$ along with $T_x X^{-1} e = e$, we observe

$$
\begin{aligned}
\left\| \widetilde{r} - \frac{e^\top r}{n} e \right\|^2 &= \left(\widetilde{r} - \frac{e^\top r}{n} e \right)^\top \left(\widetilde{r} - \frac{e^\top r}{n} e \right) \\
&= \|\widetilde{r}\|^2 - \frac{2}{n} r^\top \left(T_x X^{-1} \right)^\top \left(e^\top r e \right) + \frac{1}{n} (e^\top r)^2 \\
&= \|\widetilde{r}\|^2 - \frac{2}{n} r^\top T_x X^{-1} \left(e^\top r e \right) + \frac{1}{n} (e^\top r)^2 \\
&= \|\widetilde{r}\|^2 - \frac{2}{n} r^\top \underbrace{\left(T_x X^{-1} e \right)}_{= \, e} r^\top e + \frac{1}{n} (e^\top r)^2 \\
&= \|\widetilde{r}\|^2 - \underbrace{\frac{1}{n} (e^\top r)^2}_{\geq \, 0} \leq \|\widetilde{r}\|^2.
\end{aligned}
$$

That is, (25b) holds. $\qquad\square$

The following result provides bounds on the norms of the scaled search directions $\|\widetilde{\Delta x}\|$ and $\|\widetilde{\Delta s}\|$ that account for the inexactness in solving the complementarity equation, and therefore corresponds to the inexact analogue of Lemma 6 in [23].

Lemma 8. *Suppose that $(x, s, y) \in \mathcal{N}_2(\gamma)$ for some constant $\gamma \in (0, 1/3)$ and let $(\Delta x, \Delta s, \Delta y)$ be the unique solution of the system* (15), *which is a classical estimate of the solution to the quantum Newton linear system* (9). *Then, the directions $\widetilde{\Delta x}$ and $\widetilde{\Delta s}$ as defined in* (24) *satisfy:*

$$
\left\| \widetilde{\Delta x} \right\| \leq \frac{(1+\eta)\Theta}{2}, \quad \left\| \widetilde{\Delta s} \right\| \leq (1+\eta)\Theta\mu, \tag{26}
$$

where $\eta \in (0, 1)$, $\mu \equiv \mu(x, s)$ and

$$
\Theta = \frac{2 \left[\gamma^2/2 + (1-\sigma)^2 n \right]^{1/2}}{1 - 3\gamma}.
$$

Proof. First we note that $w_{xs}^\top e = s^\top T_x e = s^\top x = n\mu$ and $\|w_{xs} - \mu e\| \leq \gamma\mu/\sqrt{2}$. Hence,

$$
\begin{aligned}
\|w_{xs} - \sigma\mu e\|^2 &= \|w_{xs} - \mu e\|^2 + \|\mu e - \sigma\mu e\|^2 + 2(1-\sigma)\mu(w_{xs} - \mu e)^\top e \\
&\leq \left[\frac{\gamma^2}{2} + (1-\sigma)^2 n \right] \mu^2. \tag{27}
\end{aligned}
$$

Due to the fact that $d_\infty(x, s) \leq d_2(x, s) \leq \gamma\mu$, applying Lemma 4 with $\nu = \mu$ one can observe that (20) holds for $\tau = 3\gamma < 1$, and $\nu = \mu$.

Thus, we apply (13), (27), Lemma 2(b) and Lemma 5 with $\nu = \mu$, $(u, v) = (\Delta x, \Delta s)$, $h = \sigma\mu e - Xs + r$ and $\tau = 3\gamma$ such that

$$
\begin{aligned}
\left\| \widetilde{\Delta x} \right\| &\leq \frac{\left\| T_x X^{-1}(\sigma\mu e - Xs + r) \right\| \mu^{-1}}{1 - 3\gamma} \leq \frac{\left(\left\| T_x X^{-1}(\sigma\mu e - Xs) \right\| + \left\| T_x X^{-1} r \right\| \right) \mu^{-1}}{1 - 3\gamma} \\
&\leq \frac{(1+\eta) \left\| T_x X^{-1}(\sigma\mu e - Xs) \right\| \mu^{-1}}{1 - 3\gamma} = \frac{(1+\eta)\Theta}{2}, \\
\left\| \widetilde{\Delta s} \right\| &\leq \frac{2 \left\| T_x X^{-1}(\sigma\mu e - Xs + r) \right\|}{1 - 3\gamma} \leq \frac{2 \left(\left\| T_x X^{-1}(\sigma\mu e - Xs) \right\| + \left\| T_x X^{-1} r \right\| \right)}{1 - 3\gamma} \\
&\leq \frac{2(1+\eta) \left\| T_x X^{-1}(\sigma\mu e - Xs) \right\|}{1 - 3\gamma} = (1+\eta)\Theta\mu.
\end{aligned}
$$

$\qquad\square$

Lemma 9 (Lemma 7 in [23])**.** *Let $u_i, v_i \in \mathbb{R}^{k_i}$ for $i = 1, \ldots, n$ and define $u \equiv (u_1, \ldots, u_n)$ and $v \equiv (v_1, \ldots, v_n)$. Then,*

$$\|u \circ v\| \leq \sqrt{2}\|u\|\|v\|.$$

In the next result, we establish an upper bound on a quantity which majorizes our distance metric $d_2(x(\alpha), s(\alpha))$. This result can be viewed as the inexact analogue of Lemma 8 from [23].

Lemma 10. *Suppose that $(x, s, y) \in \mathcal{N}_2(\gamma)$ for some constant $\gamma \in (0, 1/3)$ and let $(\Delta x, \Delta s, \Delta y)$ be the unique solution of the system (15) for some $\sigma \in \mathbb{R}$. Further, $(\Delta x, \Delta s, \Delta y)$ is a classical estimate of the solution $(\Delta z, \Delta y)$ to the quantum Newton linear system (9). Then, for any $\alpha \in [0, 1]$ we have*

$$\sqrt{2} \left\| T_x^{-1} x(\alpha) \circ T_x s(\alpha) - \mu(\alpha)e \right\|$$
$$\leq \quad \left[(1-\alpha)\gamma + \sqrt{2}\alpha\gamma(1+\eta)\Theta + \alpha^2(1+3\eta)\Theta^2 + \alpha\eta\gamma \right]\mu. \tag{28}$$

Proof. Noting the fact that $d_\infty(x, s) \leq d_2(x, s)$, it follows that the assumption made in Lemma 4 holds with $\nu = \mu$. We therefore can apply (19a) with $\nu = \mu$, (23b), (26) and Lemma 9, which for all $\alpha \in [0, 1]$ yields:

$$\left\| T_x^{-1} x(\alpha) \circ T_x s(\alpha) - \mu(\alpha)e \right\|$$
$$= \sqrt{2} \left\| (1-\alpha)(w_{xs} - \mu e) + \alpha(W_{xs} - R_{xs})\widetilde{\Delta x} + \alpha \left[T_x X^{-1} r - \frac{e^\top r}{n} e \right] \right\|$$
$$\leq (1-\alpha)\sqrt{2}\|w_{xs} - \mu e\| + \alpha\sqrt{2}\|R_{xs} - W_{xs}\|\|\widetilde{\Delta x}\| + \alpha^2\sqrt{2}\|\widetilde{\Delta x} \circ \widetilde{\Delta s}\|$$
$$\quad + \alpha\sqrt{2}\left\| T_x X^{-1} r - \frac{e^\top r}{n} e \right\|$$
$$\leq (1-\alpha)\sqrt{2}\|w_{xs} - \mu e\| + \alpha\sqrt{2}\|R_{xs} - W_{xs}\|\|\widetilde{\Delta x}\| + \alpha^2\sqrt{2}\|\widetilde{\Delta x} \circ \widetilde{\Delta s}\|$$
$$\quad + \alpha\sqrt{2}\left\| T_x X^{-1} r \right\|$$
$$\leq (1-\alpha)\gamma\mu + \sqrt{2}\alpha\gamma(1+\eta)\Theta\mu + 2\alpha^2\|\widetilde{\Delta x}\|\|\widetilde{\Delta s}\| + \alpha\eta\gamma\mu$$
$$\leq (1-\alpha)\gamma\mu + \sqrt{2}\alpha\gamma(1+\eta)\Theta\mu + 2\alpha^2 \left[\frac{1}{2}(1+\eta)\Theta\right][(1+\eta)\Theta\mu] + \alpha\eta\gamma\mu$$
$$= (1-\alpha)\gamma\mu + \sqrt{2}\alpha\gamma(1+\eta)\Theta\mu + \alpha^2(1+\eta)^2\Theta^2\mu + \alpha\eta\gamma\mu$$
$$= (1-\alpha)\gamma\mu + \sqrt{2}\alpha\gamma(1+\eta)\Theta\mu + \alpha^2(1+2\eta+\eta^2)\Theta^2\mu + \alpha\eta\gamma\mu$$
$$\leq (1-\alpha)\gamma\mu + \sqrt{2}\alpha\gamma(1+\eta)\Theta\mu + \alpha^2(1+3\eta)\Theta^2\mu + \alpha\eta\gamma\mu$$
$$= \left[(1-\alpha)\gamma + \sqrt{2}\alpha\gamma(1+\eta)\Theta + \alpha^2(1+3\eta)\Theta^2 + \alpha\eta\gamma \right]\mu,$$

where the final inequality follows from the fact that $\eta^2 < \eta$ as $\eta \in (0, 1)$. $\qquad\square$

We are now able to state the following lemma from [23], which shows that the left hand side of (28) majorizes $d(x(\alpha), s(\alpha))$.

Lemma 11 (Lemma 9 in [23])**.** *Suppose $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$ and let $\mu = \mu(x, s)$. Then,*

$$d_2(x, s) \equiv \sqrt{2}\|w_{xs} - \mu\| = \min_{G \in \mathcal{G}} \|x_G \circ s_G - \mu e\|,$$

where $x_G \equiv G^\top x$ and $s_G \equiv G^{-1} s$ for every $G \in \mathcal{G}$.

Proof. The result follows directly from Lemma 2.10 in [32], and is provided in full detail in [23]. $\qquad\square$

In the final lemma of this section, we provide a result which serves to establish the feasibility of the sequence of iterates.

Lemma 12 (Lemma 10 in [23])**.** *Let $(x, s) \in \mathcal{K} \times \mathcal{K}$ be given. If $x \circ s \in \text{int}(\mathcal{K})$, then $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$. In particular, if $\sqrt{2}\|x \circ s - \nu e\| \leq \gamma\nu$ for some $\gamma \in (0, 1)$ and $\nu > 0$, then $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$.*

4.1 The MZ family of directions

In this section we discuss the MZ family directions for SOCOPs. Given that these directions arise as a natural extension of the Monteiro and Zhang family of directions for SDOPs, this family also emerges from computing the AHO direction (15) associated with a scaled problem, and subsequently mapping the resulting direction to the original (unscaled) space.

In particular, for a given matrix $G \in \mathcal{G}$, we consider the change of variables

$$\widetilde{x} \equiv G^\top x, \quad (\widetilde{s}, \widetilde{y}) \equiv \left(G^{-1}s, y\right). \tag{29}$$

Upon scaling the data:

$$\widetilde{c} \equiv G^{-1}c, \quad \widetilde{A} \equiv AG^{-\top}, \quad \widetilde{b} = b,$$

one can observe that the primal and dual SOCOPs (P) and (D) can be rewritten according to this scaling as:

$$(\widetilde{P}) \min \left\{\widetilde{c}^\top \widetilde{x} : \widetilde{A}\widetilde{x} = \widetilde{b}, \ \widetilde{x} \in \mathcal{K}\right\}, \tag{30}$$

$$(\widetilde{D}) \max \left\{\widetilde{b}^\top \widetilde{y} : \widetilde{A}^\top \widetilde{y} + \widetilde{s} = \widetilde{c}, \ \widetilde{s} \in \mathcal{K}\right\}. \tag{31}$$

Now, one can observe that by Proposition 8, along with the relation $\mu(x,s) = \mu(\widetilde{x}, \widetilde{s})$, it follows

$$(\widetilde{x}, \widetilde{s}, \widetilde{y}) \in \widetilde{\mathcal{N}}_2(\gamma) \iff (x, s, y) \in \mathcal{N}_2(\gamma), \quad (\widetilde{x}, \widetilde{s}, \widetilde{y}) \in \widetilde{\mathcal{N}}_\infty(\gamma) \iff (x, s, y) \in \mathcal{N}_\infty(\gamma),$$

where $\widetilde{\mathcal{N}}_2$ and $\widetilde{\mathcal{N}}_\infty$ denote the 2-norm and ∞-norm neighborhoods of the central path for the scaled primal-dual SOCO pair (30)-(31). Further, for $\nu > 0$ denote a point on the central path of (30)-(31) by $(\widetilde{x}_\nu, \widetilde{s}_\nu, \widetilde{y}_\nu)$ where $(\widetilde{x}_\nu, \widetilde{s}_\nu, \widetilde{y}_\nu) = (Gx_\nu, G^{-1}s_\nu, y_\nu)$.

Now to see how $G \in \mathcal{G}$ induces scaled Newton direction, note that a strictly feasible point $(x, s, y) \in \mathcal{P}^0 \times \mathcal{D}^0$ maps to an interior feasible point $(\widetilde{x}, \widetilde{s}, \widetilde{y})$ as given by (29). At this scaled point, $(\widetilde{x}, \widetilde{s}, \widetilde{y})$ we computed the corresponding AHO direction $(\widetilde{\Delta x}, \widetilde{\Delta s}, \widetilde{\Delta y})$, which we can map back to the original space; providing either the AHO direction, or another direction from the MZ family with scaling G. Hence,

$$(\Delta x, \Delta s, \Delta y) = (\Delta x_G, \Delta s_G, \Delta y_G)$$

where

$$\Delta x = G^{-\top} \widetilde{\Delta x}, \quad (\Delta s, \Delta y) = (G\widetilde{\Delta s}, \widetilde{\Delta y}).$$

That is; $(\Delta x, \Delta s, \Delta y) = (\Delta x_G, \Delta s_G, \Delta y_G)$ solves the system

$$\widetilde{S}G^\top \Delta x + \widetilde{X}G^{-1}\Delta s = \sigma\mu e - \widetilde{X}\widetilde{s} + r, \quad A^\top \Delta y + \Delta s = 0, \quad A\Delta x = 0, \tag{32}$$

with $\widetilde{X} = \mathbf{mat}(\widetilde{x})$ and $\widetilde{S} = \mathbf{mat}(\widetilde{s})$. Then, for any member of the MZ family of directions for SOCO with associated scaling matrix G, we can write the scaled quantum Newton linear system as

$$\begin{bmatrix} \widetilde{S}G^\top Q_2 & -\widetilde{X}G^{-1}A \end{bmatrix} \begin{bmatrix} \Delta z \\ \Delta y \end{bmatrix} = \sigma\mu e - \widetilde{X}\widetilde{s} + r. \tag{33}$$

Now, the choice of G determines the scaled AHO direction $(\Delta x_G, \Delta s_G, \Delta y_G)$ at the point $(\Delta x, \Delta s, \Delta y)$. As such, there are many possible choices of G, each endowed with its own properties. For example, setting $G = I$ yields the AHO direction, whereas $G = T_s$ yields the HKM direction, and $G = T_x^{-1}$ for the dual counterpart of the HKM direction. Further, the MZ family of directions includes the Nesterov-Todd (NT) direction $G = G_{xs}$, where G_{xs} is the unique positive semidefinite matrix satisfying $G^2x = s$.

Therefore, all of the results presented in this section for the unscaled AHO direction apply to every direction in the MZ family of directions, as we have demonstrated that every such direction reduces to the AHO direction in the scaled space, along with the fact that the centrality measures and duality gap are scale invariant. These results are summarized in the following result from [23] and one additional result for the quantum Newton linear system.

Corollary 2 (Corollary 2 in [23])**.** *Let $G \in \mathcal{G}$ and $(x, s, y) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K}) \times \mathbb{R}^m$ be a point such that*

$$\max_{i,j} \left| \lambda_i^j(w_{xs}) - v \right| \le \gamma\nu,$$

for some scalars $\gamma \in (0, 1/3)$ and $\nu > 0$. Then, the system (32) has a unique solution.

Corollary 3. *If the system (32) has a unique solution, then the quantum Newton linear system (33) has a unique solution.*

Proof. The proof follows exactly from the proof of Corollary 1. $\square$

5 Analysis of the Inexact-Feasible Quantum Interior Point Method for SOCOPs

In this section, we present our Inexact-Feasible Quantum Interior Point Method for SOCOPs. We establish our algorithm's polynomial convergence, as well as results on implementing block encodings of the quantum Newton linear system, concluding with the overall running time of the SOCO-IF-QIPM.

5.1 A quantum interior point method for SOCOPs

We quantize an IF-IPM for SOCO which is presented in Algorithm 3. To initialize the algorithm, we calculate the bases for the nullspace and rowspace of A, and store these matrices in QRAM. Calculating a basis for the nullspace $\mathcal{N}(A)$ can be accomplished via QR-factorization or Gaussian elimination. Yet, as this operation need only be performed once (as $\mathcal{R}(A)$ and $\mathcal{N}(A)$ do not change over the course of the algorithm), we can ignore the cost of this step as it pertains to the overall running time of the algorithm.

Each iteration of our IF-QIPM proceeds as follows. In step (2), we compute any matrix products or inverses required for the quantum Newton linear system such that we can implement a block encoding of the quantum Newton linear system in time $\tilde{\mathcal{O}}(1)$. This step requires time $\mathcal{O}(n^\omega)$, where ω is the matrix multiplication constant. In theory $\omega \approx 2.3728$, yet in practice such steps are typically conducted using Gauss elimination, which requires time $\mathcal{O}(n^3)$.

In step (8) we solve the quantum Newton linear system via block encodings, and extract a classical estimate of the resulting quantum state using an Algorithm 1. A detail analysis for step (8) can be found in section 2.3, which can be accomplished in time $\tilde{\mathcal{O}}(K\theta\kappa)$.

From here, we map our classical estimate $(\Delta z, \Delta y)$ of the solution to the quantum Newton linear system to $(\Delta x, \Delta s, \Delta y)$. Finally, we classically update the current solutions to the SOCOP x, s, and y, and the central path parameter μ.

Algorithm 3 Short-step Quantum interior point method for SOCOPs

---

Input: Choose constants $\gamma \in (0, 1/3)$ and $\delta \in (0, 1)$
Set $\sigma = 1 - \delta/\sqrt{2n}$.
Let $\epsilon \in (0, 1)$ and $(x^0, y^0, s^0, \nu^0) \in \mathcal{N}_2(\gamma)$
Set $\mu_0 = \mu(x^0, s^0)$.
Compute bases for $\mathcal{N}(A)$ and $\mathcal{R}(A)$ and store these matrices in QRAM.
while $\mu > \epsilon$:

1. $\mu = \mu(\alpha)$

2. Compute any matrix products, and inverses necessary for the quantum Newton linear system classically, and store in QRAM

3. Use Algorithm 1 with $\xi_k = 10^{-2}$ to solve quantum Newton linear system, and extract classical estimate of solution

4. Map classical estimate $(\Delta z, \Delta y)$ of $|\Delta z \circ \Delta y\rangle$ to $(\Delta x, \Delta s, \Delta y)$ according to (8)

5. Update solution

$$x \leftarrow x + \Delta x \ y \leftarrow y + \Delta y \text{ and } s \leftarrow s + \Delta s$$
$$k \leftarrow k + 1$$

5.2 Polynomial convergence

In order to have a convergent algorithm, we need to ensure that the errors ξ_k introduced from QLSA are properly chosen. Yet, by virtue of our use of the iterative refinement scheme in Algorithm 1, we can always choose $\xi_k = 10^{-2}$, a fact that yields significant improvements in performance.

In the following result we procure the analysis of one iteration of the IF-QIPM for SOCOPs for chosen constants γ, δ and η.

Theorem 8. *Let* $\gamma \in (0, 1/3)$, $\delta \in (0, 1)$ *and* $\eta \in (0, 1)$ *be constants satisfying*

$$(2 + 4\eta)\frac{2(\gamma^2 + \delta^2)}{(1 - 3\gamma)^2} + \eta\gamma \leq \left[\left(1 - \frac{\delta}{2n}\right) - \frac{\sqrt{2}}{\sqrt{n}}\eta\right]\gamma. \tag{34}$$

Suppose that $(x, s, y) \in \mathcal{N}_2(\gamma)$ *and let* $(\Delta x, \Delta s, \Delta y)$ *denote the solution to the system* (15) *obtained from a classical estimate of the solution to our quantum Newton linear system* (9) *with* $\sigma \equiv 1 - \delta/\sqrt{2n}$. *Then,*

(a) $(\widehat{x}, \widehat{s}, \widehat{y}) \equiv (x + \Delta x, s + \Delta s, y + \Delta y) \in \mathcal{N}_2(\gamma)$

(b) $\mu(\widehat{x}, \widehat{s}) = (1 - \delta/\sqrt{2n})\mu(x, s)$.

Proof. In the following proof, we seek to establish that

$$\left\|T_x^{-1}x(\alpha) \circ T_x s(\alpha) - \mu(\alpha)e\right\| \leq \gamma\mu(\alpha).$$

Given that we make no assumption regarding the sign of r, we can use the fact that $|e^\top r| \leq \|e\| \|r\| \leq \sqrt{2n}\eta\gamma\nu$ such that

$$\left|\frac{e^\top r}{n}\right| \leq \frac{\sqrt{2n}}{n}\eta\gamma\mu = \frac{\sqrt{2}}{\sqrt{n}}\eta\gamma\mu.$$

Then, applying (34), Lemma 10 and using the fact that $\Theta \geq \sqrt{2}\gamma$, for all $\alpha \in [0,1]$ we have

$$\sqrt{2}\left\|T_x^{-1}x(\alpha) \circ T_x s(\alpha) - \mu(\alpha)e\right\| \leq \left[(1-\alpha)\gamma + \alpha(2+4\eta)\Theta^2 + \alpha\eta\gamma\right]\mu$$

$$= \left((1-\alpha)\gamma + \alpha(2+4\eta)\frac{2(\gamma^2+\delta^2)}{(1-3\gamma)^2} + \alpha\eta\gamma\right)\mu$$

$$\leq \left((1-\alpha)\gamma + \left[\left(1-\frac{\delta}{2n}\right) - \frac{\sqrt{2}}{\sqrt{n}}\eta\right]\gamma\right)\mu$$

$$\leq \left((1-\alpha)\gamma + \left(1-\frac{\delta}{2n}\right)\gamma\right)\mu + \gamma\frac{e^\top r}{n}.$$

Therefore, from (23a), it follows:

$$\sqrt{2}\left\|T_x^{-1}x(\alpha) \circ T_x s(\alpha) - \mu(\alpha)e\right\| \leq \gamma(1-\alpha+\alpha\sigma)\mu + \gamma\frac{e^\top r}{n} = \gamma\mu(\alpha) < \mu(\alpha).$$

Now, combining the above with Lemma 12, one can observe that we obviously have $T_x^{-1}x(\alpha) \in \text{int}\,(\mathcal{K})$ and $T_x s(\alpha) \in \text{int}\,(\mathcal{K})$ for each $\alpha \in [0,1]$. Thus, in view of Proposition 7, this means that for every $\alpha \in [0,1]$, it follows that $(x(\alpha), s(\alpha)) \in \text{int}\,(\mathcal{K}) \times \text{int}\,(\mathcal{K})$. Further, we can note that

$$Ax = b, \qquad\qquad\qquad A\Delta x = 0,$$
$$A^\top y + s = c, \qquad\qquad\qquad A^\top \Delta y + \Delta s = 0,$$

and thus we have $Ax(\alpha) = b$ and $A^\top y(\alpha) + s(\alpha) = c$. Therefore, we have shown that

$$(x(\alpha), s(\alpha), y(\alpha)) \in \mathcal{P}^0 \times \mathcal{D}^0$$

for every $\alpha \in [0,1]$.

From here, we apply Lemma 11 choosing $\widetilde{x} = T_x^{-1}x(\alpha)$ and $\widetilde{s} = T_x s(\alpha)$, we can conclude that for any $\alpha \in [0,1]$,

$$d(x(\alpha), s(\alpha)) \leq \sqrt{2}\left\|T_x^{-1}x(\alpha) \circ T_x s(\alpha) - \mu(\alpha)e\right\| \leq \gamma\mu(\alpha).$$

Thus, (a) follows as $(\widehat{x}, \widehat{s}, \widehat{y}) = (x(1), s(1), y(1)) \in \mathcal{N}_2(\gamma)$. The statement (b) follows as a consequence of (23a). $\qquad\qquad\qquad\square$

Corollary 4. *Let $\gamma \in (0, 1/3)$, $\delta \in (0,1)$ and $\eta \in (0,1)$ be constants satisfying*

$$(2+4\eta)\frac{2(\gamma^2+\delta^2)}{(1-3\gamma)^2} + \eta\gamma \leq \left[\left(1-\frac{\delta}{2n}\right) - \frac{\sqrt{2}}{\sqrt{n}}\eta\right]\gamma. \tag{35}$$

Then, each iterate (x^k, s^k, y^k) generated by Algorithm 3 is an element of the neighborhood $\mathcal{N}_2(\gamma)$ and satisfies $x^{k^\top}s^k = (1-\delta/\sqrt{2n})^k x^{0^\top}s^0$. Further, Algorithm 3 terminates in at most $\mathcal{O}(\sqrt{n}\log(1/\epsilon))$ iterations.

Examples of constants that satisfy the condition in Corollary 4 are $\gamma = \delta = 1/50$ and $\eta = 1/3$.

5.3   The quantum newton linear system

We now discuss how to efficiently implement and solve the quantum Newton linear system using block encodings. Our work is left in general terms of $G \in \mathcal{G}$, such that it extends for every member of the MZ family of directions for SOCOP.

In order to take full advantage of the block encoding framework for QSLA, we pre-compute the scaling matrix G, and its inverse, classically, and store these matrices in QRAM. Then, we utilize a factorization of the Newton system so that we can construct the entire system in time $\tilde{\mathcal{O}}(1)$. The following two results establish the factors for the quantum Newton linear system (33).

22

Proposition 9. *Let*

$$M_1 = \begin{bmatrix} \widetilde{S}G^\top Q_2 & 0 \end{bmatrix}.$$

Let $\widetilde{S}$, $G^\top$ and Q_2 be stored in QRAM. Then, a $(\|M_1\|_F, O(\log n), \xi_{M_1})$-block encoding of M_1 can be constructed in time $\tilde{\mathcal{O}}(1)$.

Proof. Noting that $\widetilde{S}$ is stored in QRAM, using Proposition 1 we can construct a $(\|\widetilde{S}\|_F, \mathcal{O}(\log n), \xi_{\widetilde{S}})$-block encoding of $\widetilde{S}$ in time $\tilde{\mathcal{O}}(1)$. Next, with $G^\top$ stored in QRAM, we can again apply Proposition 1 and construct a $(\|G\|_F, \mathcal{O}(\log n), \xi_G)$-block encoding of $G^\top$ in time $\tilde{\mathcal{O}}(1)$.

From here we apply Proposition 2 with

$$\xi_{\widetilde{S}} = \frac{\xi_1}{2\|G^\top\|_F} \text{ and } \xi_G = \frac{\xi_1}{2\|\widetilde{S}\|_F},$$

yielding a $(\|\widetilde{S}G^\top\|_F, \mathcal{O}(\log n), \xi_1)$-block encoding of $\widetilde{S}G^\top$.

Given that Q_2 is stored in QRAM, we implement a $(\|Q_2\|_F, \mathcal{O}(\log n), \xi_Q)$-block encoding of Q_2 in time $\tilde{\mathcal{O}}(1)$. We once more apply Proposition 2, choosing

$$\xi_2 = \frac{\xi_{M_1}}{2\|Q_2\|_F} \text{ and } \xi_Q = \frac{\xi_{M_1}}{2\|\widetilde{S}G^\top\|_F},$$

which yields a $(\|M_1\|_F, O(\log n), \xi_{M_1})$-block encoding of M_1 can be constructed in time $\tilde{\mathcal{O}}(1)$, as desired. □

Proposition 10. *Let*

$$M_2 = \begin{bmatrix} 0 & -\widetilde{X}G^{-1}A \end{bmatrix}.$$

Let $\widetilde{X}$, G^{-1} and A be stored in QRAM. Then, a

$$(\|M_2\|_F, O(\log n), \xi/(\|M_2\|_F\kappa^2\log^2\frac{\kappa}{\xi}))$$

block encoding of M_2 can be constructed in time $\tilde{\mathcal{O}}(1)$.

Proof. First, given that $\widetilde{X}$ is stored in QRAM, using Proposition 1 we can construct a $(\|\widetilde{X}\|_F, \mathcal{O}(\log n), \xi_{\widetilde{X}})$-block encoding of $\widetilde{X}$ in time $\tilde{\mathcal{O}}(1)$. Next, with G^{-1} stored in QRAM, we can again apply Proposition 1 and construct a

$$(\|G^{-1}\|_F, \mathcal{O}(\log n), \xi_G)$$

block encoding of G^{-1} in time $\tilde{\mathcal{O}}(1)$.

From here we apply Proposition 2 with

$$\xi_{\widetilde{X}} = \frac{\xi_1}{2\|G^{-1}\|_F} \text{ and } \xi_G = \frac{\xi_1}{2\|\widetilde{X}\|_F},$$

yielding a $(\|\widetilde{X}G^{-1}\|_F, \mathcal{O}(\log n), \xi_1)$-block encoding of $-XG^{-1}$ in time $\tilde{\mathcal{O}}(1)$..

Noting that A is stored in QRAM, we implement a $(\|A\|_F, \mathcal{O}(\log n), \xi_A)$-block encoding of A in time $\tilde{\mathcal{O}}(1)$. We once more apply Proposition 2, choosing

$$\xi_2 = \frac{\xi_{M_2}}{2\|A\|_F} \text{ and } \xi_A = \frac{\xi_{M_2}}{2\|-XG^{-1}\|_F},$$

which yields a $(\|M_2\|_F, O(\log n), \xi/(\|M_2\|_F\kappa^2\log^2\frac{\kappa}{\xi}))$-block encoding of M_2 can be constructed in time $\tilde{\mathcal{O}}(1)$, as desired. □

The above construction reflects only one possible way to block encode factors of our Newton system constraint matrix[6], but proceeding as we have minimizes the amount of classical computation needed.

Proposition 11. *The quantum Newton linear system matrix (33) can be written compactly as:*

$$M_G = \begin{bmatrix} \widetilde{S}G^\top Q_2 & -\widetilde{X}G^{-1}A \end{bmatrix}, \tag{36}$$

Let $-\widetilde{X}, \widetilde{S}, G^{-1}, G^\top, A$ *and* Q_2 *be stored in QRAM. Then, a*

$$(\|M_G\|_F, \mathcal{O}(\log n), \xi/(\kappa^2 \log^2 \frac{\kappa}{\xi}))$$

block encoding of (36), can be constructed in time $\tilde{\mathcal{O}}(1)$.

Proof. Carrying out the calculations shows that (36) corresponds to the Newton linear system. We construct the two following block encodings:

$$M_1 = \begin{bmatrix} \widetilde{S}G^\top Q_2 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & -\widetilde{X}G^{-1}A \end{bmatrix},$$

using Prop.s 9-10. We choose the precision of this step so that we obtain

$$(\|M_1\|_F, \mathcal{O}(\log n), \xi/(\|M_1\|_F \kappa^2 \log^2 \frac{\kappa}{\xi}))$$

and $(\|M_2\|_F, \mathcal{O}(\log n), \xi/(\|M_2\|_F \kappa^2 \log^2 \frac{\kappa}{\xi}))$-block encodings, respectively, where κ refers to the condition number of (36), here and below. We add these two block encodings together using Proposition 1, obtaining a

$$(\max\{\|M_1\|_F, \|M_2\|_F\}, \mathcal{O}(\log n), \xi/(\kappa^2 \log^2 \frac{\kappa}{\xi}))$$

-block encoding of (36), in time $\tilde{\mathcal{O}}(1)$. Since $\max\{\|M_1\|_F, \|M_2\|_F\} = \mathcal{O}(\|M_G\|_F)$, we obtain the claimed result. $\qquad\square$

For completeness, we now use the factorization to solve the Newton system.

Theorem 9. *There is a quantum algorithm that given*

$$\left| \sigma\mu e - \widetilde{X}\widetilde{s} \right\rangle$$

and access to QRAM data structures encoding $\widetilde{S}$, $\widetilde{X}$, G^{-1} $G^\top$, Q_2, *outputs a state* ξ-*close to* $|\Delta z \circ \Delta y\rangle$ *in time* $\tilde{\mathcal{O}}(\kappa)$, *using the an appropriate direction from the MZ family of directions. We can also output an estimate of* $\|\Delta z \circ \Delta y\|$ *with relative error* δ *by increasing the running time by a factor* $\frac{1}{\delta}$.

Proof. This is a direct consequence of Prop. 11 and Thm. 1. $\qquad\square$

5.4 Overall running time

From Corollary 4, it follows that the IF-QIPM requires at most $\mathcal{O}(\sqrt{n}\log(1/\epsilon))$ iterations. In each iteration, we must prepare and solve the quantum Newton linear system; and obtain a classical description of the result quantum state encoding the solution of the Newton linear system. Utilizing Algorithm 1, we can accomplish this task in time $\tilde{\mathcal{O}}(K\theta\kappa)$. The total running time of the algorithm is therefore $\tilde{\mathcal{O}}(\sum_{k=1}^{\beta\sqrt{n}\log(1/\epsilon)} K\theta\kappa)$, which simplifies to

$$\tilde{\mathcal{O}}\left(\sqrt{n}K\theta\kappa\right).$$

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[6]One alternative would be to compute the products $\widetilde{S}G^\top Q_2$ and $-\widetilde{X}G^{-1}A$ classically and store them in QRAM, the block encodings of M_1 and M_2 could be trivially implemented in accordance with Proposition 1.

6 Conclusion

In this work we provide the first quantum interior point method that is valid for SOCOPs. Further, we provide explicit definitions for the Newton linear systems corresponding to the Monterio-Zhang family of search directions, as well as factorizations and results on constructing these matrices in a quantum setting. Our results demonstrate that by making use of QRAM and block encoding techniques, we are able to efficiently compute the scaling matrices needed for primal-dual symmetry and solve our Newton linear system efficiently using quantum linear solvers. We find that our algorithm provides a quadratic speed up in terms of the problem dimension nK, at the cost of linear dependence on the condition number. An open question remains as to whether or not our algorithm's dependence on the condition number of the Newton linear system constraint matrix can be improved to polylogarithmic.

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