Industrial and
Systems Engineering

On the Relationship Between the Value Function and the Efficient Frontier of a Mixed Integer Linear Optimization Problem

Samira Fallah[1], Ted K. Ralphs[1], and Natashia L. Boland[2]

[1]Department of Industrial and Systems Engineering, Lehigh University, USA
[2]School of Electrical Engineering, Computing and Mathematical Sciences, Curtin University, Perth, Western Australia

Technical Report 22T-005-R2

LEHIGH UNIVERSITY

COR@L
COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

On the Relationship Between the Value Function and the Efficient Frontier of a Mixed Integer Linear Optimization Problem

Samira Fallah*[1], Ted K. Ralphs†[1], and Natashia L. Boland‡[2]

[1]Department of Industrial and Systems Engineering, Lehigh University, USA
[2]School of Electrical Engineering, Computing and Mathematical Sciences, Curtin University, Perth, Western Australia

Original Publication: April 15, 2022
Last Revised: February 28, 2023

Abstract

In this paper, we investigate the connection between the efficient frontier (EF) of a general multiobjective mixed integer linear optimization problem (MILP) and the so-called *restricted value function* (RVF) of a closely related single-objective MILP. We demonstrate that the EF of the multiobjective MILP is comprised of points on the boundary of the epigraph of the RVF so that any description of the EF suffices to describe the RVF and vice versa. In the first part of the paper, we describe the mathematical structure of the RVF, including characterizing the set of points at which it is differentiable, the gradients at such points, and the subdifferential at all nondifferentiable points. Because of the close relationship of the RVF to the EF, we observe that methods for constructing so-called value functions and methods for constructing the EF of a multiobjective optimization problem, each of which have been developed in separate communities, are effectively interchangeable. By exploiting this relationship, we propose a generalized cutting plane algorithm for constructing the EF of a multiobjective MILP based on a generalization of an existing algorithm for constructing the classical value function. We prove that the algorithm is finite under a standard boundedness assumption and comes with a performance guarantee if terminated early.

1 Introduction

In this paper, we consider the relationship between the efficient frontier (EF) of a multiobjective mixed integer linear optimization problem (MILP) and a certain value function (VF), which we refer to as the *restricted value function* (RVF), associated with a closely related mixed integer linear

*saf418@lehigh.edu
†ted@lehigh.edu
‡natashia.boland@gmail.com

optimization problem (MILP). Informally, the main result states that the EF of the multiobjective MILP is comprised of points on the graph of the RVF so that any description of the EF suffices to describe the RVF and vice versa. More formally, we demonstrate that the EF is comprised of a subset of the points on the boundary of the epigraph of the RVF (also called the *graph* of the RVF) and characterize when the inclusion is strict (i.e., which points on the boundary of the epigraph are not part of the EF).

Although our results demonstrate that the EF and the RVF are closely related and indeed effectively interchangeable, their relationship has not been previously observed mainly because of the disparate ways in which these mathematical objects have been described in the separate literatures in which the concepts have been developed. Upon closer examination, the relationship between the RVF and the EF becomes clear and intuitive. While we believe this paper is the first to formally and explicitly establish the relationship, it can be seen implicitly in the results of several previous works, such as those by [Trapp et al., 2013], [Ralphs and Hassanzadeh, 2014], and [Bodur et al., 2016]. The so-called minimal tenders utilized in Trapp et al.'s algorithm for constructing the value function of a pure integer program can be seen as the points on the EF of a related multiobjective problem. Ralphs and Hassanzadeh [2014] generalized this concept in their work on the structure of the value function of a general MILP. More recently, Bodur et al. [2016] observed that in block-structured problems, the solution to the column generation subproblem can be viewed as equivalent to evaluating a certain value function, and solutions can thus be restricted only to so-called *nondominated points*.

The relationship described in the remainder of this paper has some apparently broad-ranging implications, including that those algorithms designed for the construction and/or approximation of the EF are effectively interchangeable with algorithms for the construction and/or approximation of the RVF and value functions in general. Because algorithms for these two tasks have so far been used in very different application domains and for very different purposes, there are likely many possibilities for the cross-pollination of ideas. To illustrate this, we propose a generalized cutting plane algorithm for constructing both the RVF and the EF. The approach we suggest is finite, exploits the discrete structure of the RVF, and provides a performance guarantee if terminated early. It is a modified version of an existing algorithm for constructing the full value function, and to the best of our knowledge, the approach is entirely different from existing algorithms for the construction of the EF. Additionally, our algorithm is one of few algorithms developed to date that addresses multiobjective MILPs in the presence of continuous variables with any number of objectives, and it yields improved bounds on the number and size of subproblems that need to be solved to determine the discrete structure of the EF. These bounds are also comparable to existing algorithms for the pure integer case (without continuous variables).

In the remainder of this section, we set the stage by formally defining the important terms and concepts. We first describe the terminology and basic properties related to multiobjective MILPs and their associated EFs before introducing the concept of the restricted value function. Although the RVF that we introduce is closely related to the classical value function of a single-objective MILP (it can be viewed as a generalization), we are not aware of any previous study of it. Its properties are much more difficult to characterize than those of the classical value function.

Multiobjective Optimization. The multiobjective MILP that serves as the focus of our study is defined as

$$\text{vinf}_{(x_I,x_C)\in X_{\text{MO}}} C_I x_I + C_C x_C, \qquad \text{(MO-MILP)}$$

where

$$X_{\text{MO}} = \left\{(x_I, x_C) \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \ : \ A_I x_I + A_C x_C = b\right\},$$

is the feasible region; $A \in \mathbb{Q}^{m\times n}$ is the coefficient matrix of the constraints; $b \in \mathbb{Q}^m$ is the right-hand side (RHS) of the constraints; and the rows of matrix $C \in \mathbb{Q}^{(l+1)\times n}$ are the multiple objectives of the problem. The vinf operator indicates that this is a vector minimization (multiobjective) problem, which means that there is not a single optimal value, but rather a set of nondominated vectors of objective values, as described below. A_I and C_I are the submatrices of A and C consisting of columns associated with the integer variables (indexed by set $I = \{0, \ldots, r-1\}$), while A_C and C_C are the submatrices corresponding to the columns associated with the continuous variables (indexed by set $C = \{r, \ldots, n-1\}$). We assume that the feasible region X_{MO} is bounded.

Multiobjective optimization aims to understand the trade-offs involved in optimizing multiple objectives simultaneously. This analysis is most naturally done in the $(l+1)$-dimensional space known as the *criterion space*, which contains the vectors of objective values associated with points in the n-dimensional *decision space*, which is the space containing the feasible region X_{MO}. While "solving" an MILP with a single objective means determining its unique optimal value, "solving" a multiobjective MILP means generating the set of all vectors in criterion space associated with the so-called *efficient solutions*, those for which there is no other solution for which the objective value is at least as good for every objective and strictly better for at least one objective.

We briefly review some concepts in multiobjective optimization, referring interested readers to [Ehrgott, 2005] for more details. An important concept in this context is that of dominance. The point $C_I x_I + C_C x_C \in \mathbb{R}^{l+1}$ in criterion space, associated with $(x_I, x_C) \in X_{\text{MO}}$, *dominates* $C_I x'_I + C_C x'_C \in \mathbb{R}^{l+1}$, associated with $(x'_I, x'_C) \in X_{\text{MO}}$, if $C_I x_I + C_C x_C \lneqq C_I x'_I + C_C x'_C$, i.e., $(C_I x_I + C_C x_C)_j \leq (C_I x'_I + C_C x'_C)_j$ for all $j = \{0, 1, \ldots, l\}$ and $(C_I x_I + C_C x_C)_j < (C_I x'_I + C_C x'_C)_j$ for at least one index $j \in \{0, 1, \ldots, l\}$. A point in criterion space that is not dominated by any other point is called a *nondominated point* (NDP). The vinf operator in (MO-MILP) indicates that the goal of the problem is to generate the set of all NDPs, known as the *EF*. A preimage of an NDP in the decision space is referred to as an *efficient solution*. A point $(x_I, x_C) \in X_{\text{MO}}$ that is not necessarily efficient but for which there does not exist $(x'_I, x'_C) \in X_{\text{MO}}$ such that $C_I x'_I + C_C x'_C < C_I x_I + C_C x_C$ is called *weakly efficient*, and the associated point $C_I x_I + C_C x_C$ in criterion space is referred to as a *weakly nondominated point* or a *weak NDP*. It is important to note that this terminology is a bit misleading since a weak NDP is in fact a *dominated* point and *not* an NDP.

The set of NDPs can be further divided into three subsets: supported, extreme supported, and unsupported. Supported NDPs are those that lie on the boundary of the convex hull of the EF. Extreme supported NDPs are those that are extreme points of the convex hull. Unsupported NDPs are those that lie within the interior of the convex hull. Figure 1b below illustrates these concepts.

Restricted Value Function. The so-called *restricted value function* (RVF) provides another way of analyzing the trade-offs in the multiobjective optimization problem. Specifically, we consider the following related MILP with a single objective obtained by imposing all but one of the objectives in (MO-MILP) as constraints, which we refer to as the *parametric constraints*. This MILP can be written as follows:

$$\inf_{(x_I, x_C) \in X} c_I^0 x_I + c_C^0 x_C, \tag{MILP}$$

where

$$X = \left\{ (x_I, x_C) \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} \ : \ C_I^{1:l} x_I + C_C^{1:l} x_C \le f, \ A_I x_I + A_C x_C = b \right\},$$

is the feasible region; c^0 is the first row of the matrix C; $C^{1:l}$ is the submatrix consisting of the remaining rows of C, and $f \in \mathbb{Q}^l$ is a fixed vector to be replaced shortly by a parameter to obtain the aforementioned RVF.

We now define the RVF $z : \mathbb{R}^l \to \mathbb{R} \cup \{\pm\infty\}$ associated with (MILP) to be the function

$$z(\zeta) = \inf_{(x_I, x_C) \in \mathcal{S}(\zeta)} c_I^0 x_I + c_C^0 x_C, \tag{RVF}$$

that returns the optimal solution value of (MILP) as a function of a RHS parameter $\zeta \in \mathbb{R}^l$, where

$$\mathcal{S}(\zeta) = \left\{ (x_I, x_C) \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} \ : \ C_I^{1:l} x_I + C_C^{1:l} x_C \le \zeta, \ A_I x_I + A_C x_C = b \right\}.$$

The function z is similar to the classical value function except that the RHS $b \in \mathbb{Q}^m$ of some constraints are fixed. As usual, we let $z(\zeta) = +\infty$ for $\zeta \notin \mathcal{C}$, where

$$\mathcal{C} = \left\{ \zeta \in \mathbb{R}^l \ : \ \mathcal{S}(\zeta) \neq \emptyset \right\}.$$

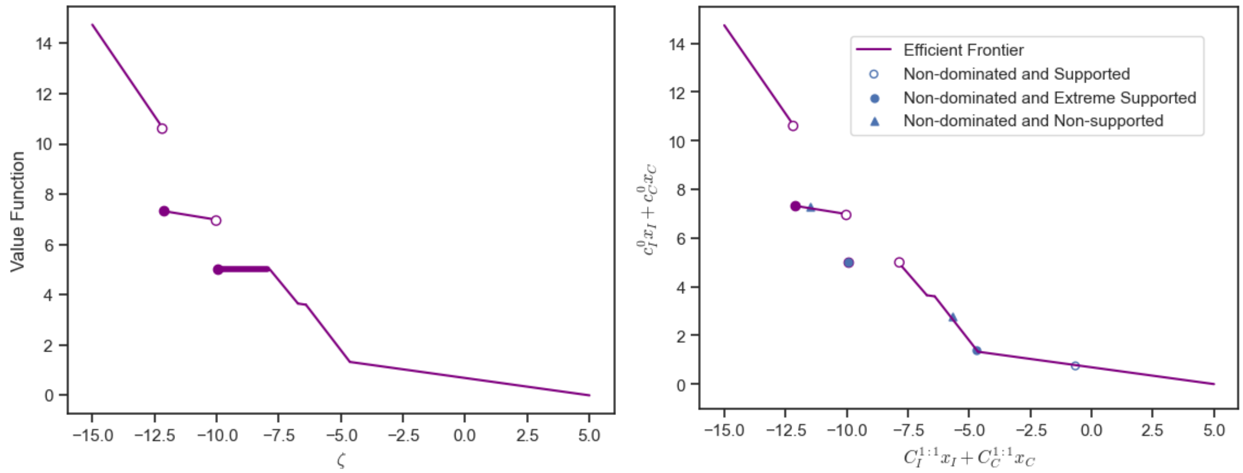
The function z is always bounded from below because of our assumption that the feasible region X_{MO} of the multiobjective MILP is bounded.

Example 1. *Here, we illustrate the concepts discussed so far. Consider an RVF defined by*

$$\begin{aligned} z(\zeta) = \min \quad & 2x_1 + 5x_2 + 7x_4 + 10x_5 + 2x_6 + 10x_7 \\ & -x_1 - 10x_2 + 10x_3 - 8x_4 + x_5 - 7x_6 + 6x_7 \le \zeta \\ & -x_1 + 4x_2 + 9x_3 + 3x_4 + 2x_5 + 6x_6 - 10x_7 = 4 \\ & x_4 + 5x_2 \le 5 \\ & x_7 + 5x_2 \le 5 \\ & x_j \in \{0, 1\}, \quad \forall j \in \{1, 2\} \\ & x_j \in \mathbb{R}_+, \quad \forall j \in \{3, 4, \ldots, 7\}, \end{aligned}$$

for all $\zeta \in \mathbb{R}$. This is not precisely in our standard form, since the third and fourth constraints are expressed as inequalities and upper bounds on the integer variables are embedded in their definition. However, it can easily be converted to our standard form with the addition of slack variables, which make no difference to the RVF or EF. Figure 1a below shows the value function for the MILP, while Figure 1b shows the EF for the associated multiobjective optimization problem. Note that the graph of the VF and the EF are identical except for the horizontal line segment between the points $(-10, 5)$ and $(-7\frac{5}{6}, 5)$, which has been thickened in the figure for emphasis.

In the remainder of the paper, we demonstrate that the RVF and the EF capture the same information and that algorithms for the construction of the two are effectively interchangeable. Throughout



(a) RVF (b) EF

Figure 1: The portion of the RVF and the EF associated with Example 1 with $\zeta \geq -15$. Note that the frontier extends to the left up to $\zeta = -57\frac{2}{3}$; the finite domain of z is $\mathcal{C} = [-57\frac{2}{3}, +\infty)$. The complete EF is shown in Figure 9.

the paper, we consider the given instance (MILP), with its associated RVF and the corresponding instance of (MO-MILP). The paper is organized as follows. We begin by reviewing related work in Section 2. In Section 3, we provide a characterization of the RVF in terms of a discrete set of integer parts of NDPs. In Section 4, we formalize the relationship between the RVF and the EF. Finally, in Section 5, we present our cutting plane algorithm for constructing both the EF and the value function. Both the value function representation and the cutting plane algorithm are finite under our assumption that X_{MO} is bounded. We analyze the theoretical performance of the algorithm compared with that of existing algorithms. Finally, we summarize our findings and concluding remarks and suggest directions for future work in Section 6.

2 Related Work

Methods both for constructing the EF of a multiobjective optimization problem and for constructing the value function of an MILP have been extensively studied in the open literature. We review some of the important and related literature from each area in the following subsections.

2.1 Multiobjective Optimization

Multiobjective optimization, which involves the analysis of trade-offs between multiple conflicting objective functions, has numerous applications across various fields, as most real-world problems arising in practice do have multiple objectives. In this study, we are interested in the algorithms that are capable of generating the exact EF for the MILP case.

The purpose of algorithms for multiobjective optimization is to generate the EF, i.e., in which there cannot be an improvement in one objective without sacrificing at least one other objective (see definitions in Section 1). The EF allows one to examine the trade-offs between the conflicting objectives. Algorithms in the multiobjective area can be classified into two categories: scalarization and non-scalarization methods. In the former one, many works have been done in the literature, but the latter one is not well-studied, although our proposed algorithm can be one of the latter ones. Scalarization techniques involve sampling the regions of the objective space to construct a piecewise description of the EF. Non-scalarization methods, on the other hand, use the outer approximations to bound the value function or the EF using convex or nonconvex approximations. The branch-and-bound framework is often used to implement these algorithms. In the remainder of this section, we will summarize the scalarization and non-scalarization methods proposed in the literature.

One straightforward way of solving a multiobjective problem is by a technique known as *scalarization* in which the problem is reformulated as a single-objective problem. A wide variety of scalarization methods have been proposed in the literature, including the *weighted sum* method [Zadeh, 1963], the *perpendicular search* method [Chalmet et al., 1986], the *weighted Tchebycheff* method [Bowman, 1976, Yu, 1973, Zeleny, 1973], the ϵ*-constraint* method [Haimes, 1971], the Hybrid method [Guddat et al., 1985], Benson's method [Benson, 1978], and the Pacoletti-Serafini method [Pascoletti and Serafini, 1984]. These methods typically involve an iterative process in which a list of NDPs (efficient solutions) and unexplored regions of the criterion space (decision space) are maintained. At each iteration, the algorithm searches for a new NDP (efficient solution) within the unexplored criterion space (decision space) region and expands the list of NDPs (efficient solutions). The process is repeated until there are no more unexplored regions. For a comprehensive review of scalarization methods in the multiobjective optimization field, the reader is referred to [Ehrgott, 2006].

The most straightforward scalarization method is the weighted sum method. The single objective created by this method is a weighted sum of the original objectives. When the weights are all positive, the solution to the weighted sum problem is guaranteed to be nondominated. On the other hand, not all nondominated solutions can be generated as a solution to some weighted sum problem—only the so-called *supported NDPs* can be generated in this way. The NDPs that cannot be found via weighted sum based scalarization are called *unsupported* NDPs. The perpendicular search method is similar to the weighted sum, but it is modified to ensure that the corner points of the criterion space (i.e., the solutions obtained using the lexicographic method for different combinations of the exploration region) will not be chosen as the next NDP.

Several methods have been developed to address unsupported NDPs, including the augmented weighted Tchebycheff method and the ϵ-constraint method. The augmented weighted Tchebycheff method [Bowman, 1976] seeks to find NDPs within the exploration region by minimizing the distance to the ideal point, which is defined as the point whose components are obtained by minimizing the objective functions. Ralphs et al. [2006] proposed a weighted Tchebycheff scalarization algorithm for constructing the EF of a biobjective integer programming problem. The ϵ-constraint method involves minimizing a primary objective while restricting the other objectives through inequality constraints. In other words, it can be thought of as a parametric search on the RHS values of the objectives that are restricted in the constraint set. This parametric search is straightforward for biobjective problems; hence biobjective discrete optimization problems are often solved using the ϵ-constraint method.

To the best of our knowledge, the GoNDEF algorithm developed by [Rasmi and Türkay, 2019] is currently the only algorithm that utilizes scalarization techniques specifically designed to address multiobjective MILPs. GoNDEF is able to find all nondominated (ND) points of the problem, including line segments or facets composed of ND points. In order to do so, GoNDEF first applies a subroutine to fix the integer values that result in ND points. This results in a multiobjective linear program (MOLP) with a set of ND points, but it is important to note that not all ND points of the MOLP are necessarily in the set of ND points of the original MILP. GoNDEF consists of four main steps: (1) finding a new efficient integer solution, (2) solving the resulting MOLP with fixed efficient integer solutions, (3) finding all ND segments of the MILP, and (4) finding all ND facets of the MILP. The algorithm terminates when there are no more new efficient integer solutions to be found. Additionally, GoNDEF employs no-good constraints [Hooker, 1994, 2011] to exclude newly found efficient integer solutions from consideration in subsequent iterations.

Another way of solving a multiobjective problem is by non-scalarization methods in which the outer approximation is formed and improved by bounding the function with convex or nonconvex approximations. De Santis et al. [2020] and Forget et al. [2022b] have developed branch-and-bound frameworks for multiobjective optimization that utilize different linear relaxations at each node and can handle an arbitrary number of objectives. De Santis et al. [2020] focused on mixed integer convex optimization problems, while Forget et al. [2022b] targeted integer linear multiobjective optimization.

Forget et al. [2022b] employed outer approximation for computing the linear relaxation to generate lower bound sets, and to do so, they used the Benson-like algorithm. They also introduced the technique of warm-starting, which involves using a lower bound set from the father node to accelerate computation. In their subsequent work, Forget et al. [2022a] introduced the concept of objective branching to the multiobjective setting, allowing for the efficient computation of upper and lower bound sets in integer optimization problems with an arbitrary number of objective functions. Finally, Forget and Parragh [2022] generalized the technique of probing, originally discussed in [Savelsbergh, 1994], to the multiobjective setting in their work on enhancing the efficiency of integer optimization algorithms.

De Santis et al. [2020] present the MOMIX algorithm, a branch-and-bound method based on the linear outer approximations of the image set (i.e., criterion space), and characterize it as the first non-scalarization-based deterministic algorithm for constructing the EF. The MOMIX algorithm constructs the true value function or EF by a lower approximation that is based on relaxing the integrality constraints and updates the lower bound approximation in each specific node in the branch-and-bound by taking the local upper bounds into account (using Lagrange multipliers), and an upper approximation that is based on the potential NDPs. The local upper bound sets are updated based on the newly found upper bound and are used to prune the nodes in the branch-and-bound and approximate the dominated set. Comparing the MOMIX algorithm with the proposed RVF Algorithm, the RVF Algorithm does not use a lower approximation to construct the true value function or the EF; it relies on an upper approximation that is updated iteratively until it becomes exact. To update the upper approximation, the RVF algorithm solves a nonconvex problem at each iteration. In contrast, MOMIX updates the lower approximation using a lower convex envelope and the upper approximation using potential NDPs and local upper bounds at each node. The RVF algorithm's upper approximation procedure may result in tighter bounds compared to the MOMIX's upper approximation, as the RVF algorithm exploits dual information and updates the approximation based on the optimal dual (the approximation will be exact in the specific stability region), but MOMIX uses potential NDPs and local upper bounds of each node for the upper approximation. In subsequent work, De Santis and Eichfelder [2021] presented a branch-and-bound algorithm for multiobjective convex quadratic integer optimization problems. As a preprocessing step, the authors used the ideal points of the restricted objective functions to compute lower bounds quickly.

In particular, there are three algorithms designed for solving multiobjective mixed integer convex optimization problems with any number of objectives without using scalarization techniques, according to [De Santis et al., 2020, Eichfelder and Warnow, 2021a, Eichfelder et al., 2022]. These algorithms include MOMIX [De Santis et al., 2020] and HyPaD [Eichfelder and Warnow, 2021a] algorithms, which can solve multiobjective mixed integer convex optimization problems, and the algorithm proposed in [Eichfelder et al., 2022], which can address the convex and nonconvex case. Both the MOMIX [De Santis et al., 2020] and the algorithm in [Eichfelder et al., 2022] operate in the decision space, while the HyPaD algorithm [Eichfelder and Warnow, 2021a] works in the criterion space to compute enclosures of the NDPs. Eichfelder and Warnow [2021b] further discussed the insights on the implementation details and numerical experiments for the HyPaD algorithm.

For approaches to approximate the EF using outer approximations, we refer the interested reader to [Benson, 1998, Hamel et al., 2014, Csirmaz, 2016, Löhne and Weißing, 2015, Ruzika and Wiecek, 2005]. A comprehensive survey of branch-and-bound methods for multiobjective linear integer problems can be found in [Przybylski and Gandibleux, 2017].

For a detailed review of the literature on multiobjective optimization, we refer the interested reader to [Ehrgott and Gandibleux, 2000, Ehrgott and Wiecek, 2005, Ehrgott et al., 2016]. In Section 5.3, we discuss related and recent algorithms in depth and provide a review of the state-of-the-art, particularly for general mixed integer problems. For the most recent survey and comprehensive overview of algorithms for multiobjective MILPs, we recommend [Halffmann et al., 2022].

2.2 Value Function

The classical value function of an MILP is a well-studied concept, and understanding its structure is crucial for many applications due to its role as a core ingredient in optimality conditions used in a variety of algorithms for solving optimization problems. These optimality conditions are also employed in formulating and solving important classes of multistage and multilevel optimization problems, in which optimality conditions are embedded as constraints in a larger optimization problem. Additionally, optimality conditions are also the basis for techniques used for warm-starting and sensitivity analysis, which are the areas in which the connection to multiobjective optimization is most apparent.

There have been several studies investigating the structure of the VF in MILPs. Blair and Jeroslow [1977] and Blair and Jeroslow [1979] identified fundamental properties of the value function, including that it is comprised of a minimum of a finite number of polyhedral functions. Blair and Jeroslow [1982] showed that the VF of a pure integer linear optimization problem (PILP) is a *Gomory function*, which is the maximum of subadditive functions known as *Chvátal functions*. Blair and Jeroslow [1984] extended this result to general MILPs, demonstrating that they are the maximum of Gomory functions. Finally, Blair [1995] identified what was then referred to as a "closed-form" representation of the MILP VF, the so-called *Jeroslow formula*, though this did not lead to what could be considered a practical representation. Güzelsoy and Ralphs [2007] further studied the properties of the value function as it is related to methods of warm-starting and sensitivity analysis and also suggested a method of construction for the case of an MILP with a single constraint. Ralphs and Hassanzadeh [2014] extended this work by providing further details on the structure and properties of the VF for a general MILP and suggesting a practical representation.

Most methods for constructing the value function have focused on the case of pure integer programs, where the discrete structure is the most evident and finite representation is the easiest to achieve. Wolsey [1981] used a cutting-plane method to derive a sequence of Chvátal functions that leads to constructing the full value function for a PILP. Conti and Traverso [1991] employed reduced Gröbner bases and modified the classical Buchberger's algorithm to solve PILPs. Later, Schultz et al. [1998] used Gröbner basis methods to solve two-stage stochastic programs with complete integer recourse and different RHSs. The authors identified a countable set known as the candidate set of the first-stage variables in which the optimal solution is contained. Then Ahmed et al. [2004] developed a global optimization algorithm for solving general two-stage stochastic programs with integer recourse and discrete distributions by exploiting the structure of the second-stage integer problem VF. The authors demonstrated that their algorithm avoids enumerating the search space. Kong et al. [2006] considered a two-stage pure IP and presented a superadditive dual formulation that exploits the VF in both stages, solving that reformulation by a global branch-and-bound or level-set approach. Trapp and Prokopyev [2015] proposed a constraint aggregation based approach to alleviate the memory requirement for storing the VF. Zhang and Özaltın [2021] first generalized the complementary slackness theorem to bilevel IP (BIP) and showed that it can be an advantage for constructing the value functions of BIP. The authors also demonstrated that the value functions of BIPs can be constructed by bilevel minimal RHS vectors and presented a dynamic programming algorithm for constructing the BIP VF. Finally, Brown et al. [2021] used a Gilmore-Gomory approach to construct the IP VF.

There have been relatively few algorithmic advances in finding the VF of a general MILP. Bank et al. [1982] studied the qualitative and quantitative stability properties of mixed integer multiobjective optimization problems, which can also be considered an MILP VF. Guzelsoy and Ralphs [2006] proposed algorithms for constructing the value function of an MILP with a single constraint. The properties of the VF and a method for constructing the VF in the case of a general MILP were discussed in [Ralphs and Hassanzadeh, 2014]. In the current work, we generalize the work in [Ralphs and Hassanzadeh, 2014] to the multiobjective setting.

3 Structure of the RVF

Before presenting the main theoretical result of the paper regarding the relationship between the RVF and the EF, we describe some basic properties of the RVF. The first step in understanding its structure in the general MILP case is to examine the structure of the function that arises when the underlying problem is a (continuous) linear optimization problem (LP).

3.1 Structure of the Restricted LP Value Function

We refer to the special case of the RVF when there are no integer variables ($r = 0$) as the *restricted LP value function* (RLPVF). We define the RLPVF $z_{\text{LP}} : \mathbb{R}^l \to \mathbb{R} \cup \{\pm\infty\}$ by

$$
\begin{aligned}
z_{\text{LP}}(\zeta) = \inf \;\; & c_C^0 x_C && \text{(RLPVF-a)}\\
& C_C^{1:l} x_C \leq \zeta && \text{(RLPVF-b)}\\
& A_C x_C = \beta && \text{(RLPVF-c)}\\
& x_C \in \mathbb{R}_+^{n-r}. && \text{(RLPVF-d)}
\end{aligned}
$$

for all $\zeta \in \mathbb{R}^l$. As with the RVF, z_{LP} takes values in the extended reals. This has implications that we discuss below. In this section, we consider $\beta \in \mathbb{Q}^m$ to be a fixed vector throughout, while in Section 3.2, we consider a parametric class of functions of this form, with different values of β arising from fixing the integer part of the solution in (RVF). As previously, we assume that z_{LP} is bounded from below.

Blair and Jeroslow [1977] studied the structure of the classical LP value function in which the full right-hand side is parametric ($m = 0$). It is well established that in this special case, the function z_{LP} is a polyhedral function whose epigraph is a polyhedral cone whose facets are associated with feasible solutions to the dual of the LP. In the more general case of the RLPVF, the function is instead a *slice* of this full LP value function, and its epigraph is hence the intersection of a hyperplane with a polyhedral cone.

In order to analyze the structure of this function, we consider the dual of the LP that arises in the evaluation of $z_{\text{LP}}(\hat{\zeta})$ for $\hat{\zeta} \in \mathbb{R}^l$, which is

$$
\sup_{(u,v)\in\mathcal{P}_D} \hat{\zeta}^\top u + \beta^\top v, \tag{D-RLP}
$$

where u is the vector of dual variables associated with the parametric constraints (RLPVF-b), v is the vector of dual variables associated with the nonparametric constraints (RLPVF-c), and the feasible region is

$$
\mathcal{P}_D = \left\{ (u, v) \in \mathbb{R}_-^l \times \mathbb{R}^m \; : \; C_C^{1:l\top} u + A_C^\top v \leq c_C^0 \right\}.
$$

By assumption, $\mathcal{P}_D$ is nonempty, since z_{LP} is bounded below. Note that the feasible region $\mathcal{P}_D$ of this LP is independent of the value of $\hat{\zeta}$.

Next, let $\mathcal{E}$ and $\mathcal{R}$ be the sets of extreme points and extreme rays of $\mathcal{P}_D$, respectively. Recall that $\mathcal{R}$ represents the set of extreme elements of the recession cone

$$
W = \left\{ (e, h) \in \mathbb{R}_-^l \times \mathbb{R}^m \; : \; C_C^{1:l\top} e + A_C^\top h \leq 0 \right\}.
$$

By Farkas' lemma [Farkas, 1902], $z_{\text{LP}}(\zeta)$ is finite (the associated optimization problem (D-RLP) has an optimal solution) if and only if $\zeta \in \mathcal{C}_{\text{LP}}$, where

$$
\mathcal{C}_{\text{LP}} = \left\{ \zeta \in \mathbb{R}^l \; : \; \zeta^\top e + \beta^\top h \leq 0 \;\; \forall (e, h) \in \mathcal{R} \right\}.
$$

Otherwise, we have $z_{\text{LP}}(\zeta) = +\infty$. Note that $\mathcal{C}_{\text{LP}}$ is a polyhedron.

Putting all of this together, we have the following proposition characterizing z_{LP} and yielding a finite combinatorial description.

Proposition 3.1. *$z_{\rm LP}$ is a polyhedral function over $\mathcal{C}_{\rm LP}$ and we have that*

$$z_{\rm LP}(\zeta) := \max_{(u,v)\in\mathcal{P}_D} \zeta^\top u + \beta^\top v = \max_{(u,v)\in\mathcal{E}} (\zeta^\top u + \beta^\top v), \qquad \forall \zeta \in \mathcal{C}_{\rm LP}. \tag{D-RLPVF}$$

Proof. By (D-RLPVF), the epigraph of RLPVF over $\mathcal{C}_{\rm LP}$ is

$$\left\{(\zeta, w) \in \mathcal{C}_{\rm LP} \times \mathbb{R} \ : \ \zeta^\top u + \beta^\top v \leq w, \ \forall (u,v) \in \mathcal{E}\right\},$$

which is a polyhedron. Therefore, $z_{\rm LP}$ is a polyhedral function over $\mathcal{C}_{\rm LP}$. The characterization (D-RLPVF) follows from strong duality and the well-known fact that when an LP has a finite optimum, that optimum is achieved at an extreme point. ■

Example 2. *In Figure 2, we plot an instance of (RLPVF) obtained from Example 1 by setting $(x_1, x_2) = (0, 0)$. In the plot, the affine function associated with each extreme point of $\mathcal{P}_D$ are shown for a part of the finite domain. For this example, the dual problem (D-RLPVF) can be expressed as*

$$\begin{aligned} z_{\rm LP}(\zeta) = \quad & \max \ \zeta u + 4v_1 + 5v_2 + 5v_3 \\ & \text{s.t.} \ 10u + 9v_1 \leq 0 \\ & \quad -8u + 3v_1 + v_2 \leq 7 \\ & \quad u + 2v_1 \leq 10 \\ & \quad -7u + 6v_1 \leq 2 \\ & \quad 6u - 10v_1 + v_3 \leq 10 \\ & \quad u \leq 0 \\ & \quad v_2 \leq 0 \\ & \quad v_3 \leq 0, \end{aligned} \tag{Ex1-D}$$

and we have that the set $\mathcal{E}$ of extreme points and the set $\mathcal{R}$ of extreme rays of $\mathcal{P}_D$ are

$$\begin{aligned} \mathcal{E} = \{&(-0.15, 0.16, 0, 0), (0, 0, 0, 0), (-2.35, -2.41, -4.59, 0), \\ &(-1.33, -1.22, 0, 0), (-1.61, -1.97, 0, 0), (0, -1, 0, 0)\}. \\ \mathcal{R} = \{&(0, 0, 0, -1), (0, -1, 0, -10), (-1, -2.66, 0, -20.66), \\ &(-1, -1.166, -4.5, -5.66), (0, 0, -1, 0)\}. \end{aligned}$$

As expected, the function defined by (RLPVF) is a polyhedral function with the nondifferentiable points being those at which the optimal basis changes. Not all affine pieces are needed to describe the function and a minimal description is as follows. The finite domain is $\mathcal{C}_{\rm LP} = [-55.464, +\infty]$ and for $\zeta \in \mathcal{C}_{\rm LP}$, we have

$$\begin{aligned} z_{\rm LP}(\zeta) = \max(0, &-2.35\zeta - 2.41 \cdot 4 - 4.59 \cdot 5, \\ &- 1.61\zeta - 1.97 \cdot 4, \\ &- 1.33\zeta - 1.22 \cdot 4, \\ &- 0.15\zeta + 0.16 \cdot 4). \end{aligned}$$

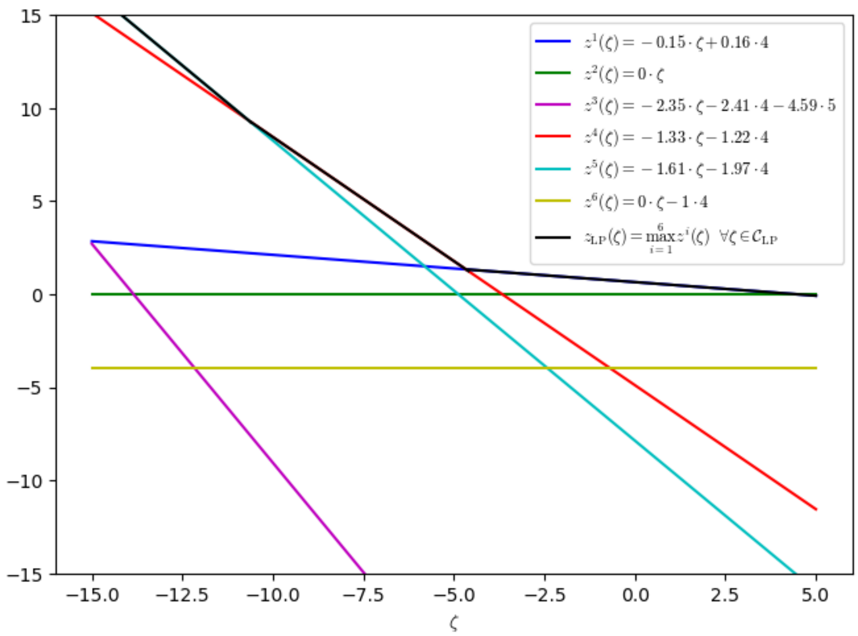


Figure 2: The RLPVF associated with Example 1 when $(x_1, x_2) = (0, 0)$

Observe that the gradients of the RLPVF are precisely the optimal dual solutions corresponding to the parametric constraints. At points of nondifferentiability (the breakpoints), the directional derivatives in direction d (in this case, we have $d \in \{1, -1\}$) of the RLPVF are given by $d^\top u$, where u is one of the alternative optimal dual solutions corresponding to the parametric constraints.

Since all polyhedral functions are convex, it follows that z_{LP} is convex over $\mathcal{C}_{\text{LP}}$. The differentiability of a convex function is characterized by its subdifferentials. It is well known that for the full value function of an LP, the subdifferential at a given right-hand side vector within the finite domain of the value function is the set of all optimal solutions to the dual of the LP associated with that given right-hand side.

A similar result can be obtained for the RLPVF by projecting the set of optimal dual solutions onto the subspace of the dual variables associated with the parametric constraints (RLPVF-b). For $\zeta \in \mathcal{C}_{\text{LP}}$, the face of all optimal solutions of the LP dual (D-RLP) is a polyhedron, which can be described as the sum of the convex hull of its extreme points (a subset of $\mathcal{E}$) and the conic hull of its extreme rays (a subset of $\mathcal{R}$), as

$$\operatorname{conv}(\{(u, v) \in \mathcal{E} \;:\; \zeta^\top u + \beta^\top v = z_{\text{LP}}(\zeta)\}) \quad + \quad \operatorname{cone}\{(e, h) \in \mathcal{R} \;:\; \zeta^\top e + \beta^\top h = 0\},$$

where the two sets are added in the usual way—any vector in the sum is the sum of one vector from each set respectively. For $\zeta \in \mathcal{C}_{\text{LP}}$, we define sets $\mu(\zeta)$ and $\nu(\zeta)$ to be the projection of each of these two sets onto the subspace of the dual variables associated with the parametric constraints so that we have

$$\begin{aligned} \mu(\zeta) &= \{\hat{u} \in \mathbb{R}^l \;:\; (\hat{u}, \hat{v}) \in \operatorname{conv}(\{(u, v) \in \mathcal{E} \;:\; \zeta^\top u + \beta^\top v = z_{\text{LP}}(\zeta)\})\} \\ &= \operatorname{conv}(\{u \in \mathbb{R}^l \;:\; \zeta^\top u + \beta^\top v = z_{\text{LP}}(\zeta),\; (u, v) \in \mathcal{E}\}), \end{aligned} \tag{1}$$

and

$$\begin{aligned}\nu(\zeta) &= \{\hat{e} \in \mathbb{R}^l \ : \ (\hat{e}, \hat{h}) \in \text{cone}(\{(e,h) \in \mathcal{R} \ : \ \zeta^\top e + \beta^\top h = 0\})\} \ \cup \ \{\mathbf{0}\} \\ &= \text{cone}(\{e \in \mathbb{R}^l \ : \ \zeta^\top e + \beta^\top h = 0, \ (e,h) \in \mathcal{R}\}) \ \cup \ \{\mathbf{0}\}.\end{aligned}$$

Here the first definition of $\mu(\zeta)$ is the direct projection of the convex hull of optimal extreme points. This is easily seen to be equivalent to the second form, which is more convenient to work with. Note that we require $\nu(\zeta)$ to contain $\{\mathbf{0}\}$ to avoid technicalities in the proofs to follow since there are cases where it would otherwise be empty. Later in this section, we show that the subdifferential of the RLPVF at $\zeta \in \mathcal{C}_{\text{LP}}$ is precisely the set $\mu(\zeta) + \nu(\zeta)$.

Before fully characterizing the subdifferentials, we first examine the properties of the directional derivatives to highlight some technicalities that arise at the boundaries of the finite domain $\mathcal{C}_{\text{LP}}$. From the definition of $\mathcal{C}_{\text{LP}}$, the following characterization of its interior and boundary can be easily derived.

- $\zeta \in \text{int}\, \mathcal{C}_{\text{LP}}$ if and only if $\zeta^\top e + \beta^\top h < 0 \quad \forall (e,h) \in \mathcal{R}$.
- $\zeta \in \mathcal{C}_{\text{LP}}$ is on the boundary of $\mathcal{C}_{\text{LP}}$ if and only if $\zeta^\top e + \beta^\top h \leq 0 \quad \forall (e,h) \in \mathcal{R}$ and there exists $(\hat{e}, \hat{h}) \in \mathcal{R}$ such that $\zeta^\top \hat{e} + \beta^\top \hat{h} = 0$.
- $\zeta \notin \mathcal{C}_{\text{LP}}$ if and only if there exists $(\hat{e}, \hat{h}) \in \mathcal{R}$ such that $\zeta^\top \hat{e} + \beta^\top \hat{h} > 0$.

Observe that for $\zeta \in \text{int}\, \mathcal{C}_{\text{LP}}$, we have $\nu(\zeta) = \{\mathbf{0}\}$.

We now consider a given $\hat{\zeta}$ on the boundary of $\mathcal{C}_{\text{LP}}$. By the characterization above, there exists $(\hat{e}, \hat{h}) \in \mathcal{R}$ such that $\hat{\zeta}^\top \hat{e} + \beta^\top \hat{h} = 0$ and $\hat{e} \in \nu(\hat{\zeta}) \setminus \{\mathbf{0}\}$. As an aside, it is interesting to observe that this means that specifically for points on the boundary of $\mathcal{C}_{\text{LP}}$, we have $z_{\text{LP}}(\hat{\zeta}) < +\infty$ while $\nu(\hat{\zeta})$ contains nonzero elements. Therefore, the set of optimal solutions to the LP (D-RLP), which is $\arg\max_{(u,v) \in \mathcal{E}} (\hat{\zeta}^\top u + \beta^\top v)$, is unbounded. However, since the rays of this optimal face of $\mathcal{P}_D$ have an objective value of zero, the optimal value itself remains bounded, whereas, for points outside of $\mathcal{C}_{\text{LP}}$, this optimal value would become unbounded. Now let $d \in \mathbb{R}^l$ be such that $d^\top \hat{e} > 0$. Then we have that $\hat{\zeta} + \epsilon d \notin \mathcal{C}_{\text{LP}}$ for all $\epsilon > 0$, since $(\hat{\zeta} + \epsilon d)^\top \hat{e} + \beta^\top \hat{h} > 0$. Thus, we can interpret d as a direction pointing out of $\mathcal{C}_{\text{LP}}$.

Per the above discussion, we, therefore, define, for arbitrary $\zeta \in \mathcal{C}_{\text{LP}}$, the set

$$\delta^-(\zeta) = \text{cone}(\{d \in \nu(\zeta) \ : \ \exists (e,h) \in \mathcal{R} \text{ such that } \zeta^\top e + \beta^\top h = 0, \ d^\top e > 0\}) \setminus \{\mathbf{0}\},$$

which is the set of all directions pointing out of $\mathcal{C}_{\text{LP}}$ at ζ. Note that with this definition, we have $\delta^-(\zeta) = \emptyset$ for $\zeta \in \text{int}\, \mathcal{C}_{\text{LP}}$, as expected. This set will be used in the proofs below.

In the remainder of the paper, we consider several results characterizing the directional derivatives of both the RLPVF and the RVF. The notion of directional derivative we consider is the one in terms of limits, as follows. For general $f : \mathbb{R}^n \to \mathbb{R}$, we take the directional derivative of f at $\bar{x}$ in direction d to be

$$\nabla_d f(\bar{x}) = \lim_{t \searrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}.$$

For both the RLPVF and the RVF considered in the next section, this limit may go to $+\infty$ at points of discontinuity and we take the directional derivative to have the value $+\infty$ in such cases. For z_{LP}, we have continuity over the finite domain $\mathcal{C}_{\mathrm{LP}}$, but discontinuities at points on the boundary of $\mathcal{C}_{\mathrm{LP}}$, since we define z_{LP} over the extended reals. Then the directional derivative $\nabla_d z_{\mathrm{LP}}(\zeta)$ of z_{LP} at $\zeta \in \mathcal{C}_{\mathrm{LP}}$ in direction d is finite if and only if $d \notin \delta^-(\zeta)$. For $\zeta \in \operatorname{int} \mathcal{C}_{\mathrm{LP}}$, $\delta^-(\zeta) = \emptyset$ and the directional derivative is finite in all directions.

From the properties of convex functions and subdifferentials, we can alternatively characterize the directional derivative as

$$\nabla_d z_{\mathrm{LP}}(\zeta) = \max_{u \in \partial z_{\mathrm{LP}}(\zeta)} u^\top d.$$

where $\partial z_{\mathrm{LP}}(\zeta)$ denotes the subdifferential of z_{LP} at $\zeta \in \mathbb{R}^l$. The proposition presented next follows from this characterization. Note that this can be seen as a corollary of our later characterization of the subdifferential as well, but we prove it now from first principles before moving to the more general result.

Proposition 3.2. *For $\zeta \in \mathcal{C}_{\mathrm{LP}}$, we have that*

$$\nabla_d z_{\mathrm{LP}}(\zeta) = \begin{cases} \max_{u \in \mu(\zeta)} u^\top d, & \text{if } d \notin \delta^-(\zeta), \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $d \in \mathbb{R}^l$.

Proof. We let $\zeta \in \mathcal{C}_{\mathrm{LP}}$ and $d \in \mathbb{R}^l$ be given and consider $\nabla_d z_{\mathrm{LP}}(\zeta)$. There are two cases.

(i) If $d \in \delta^-(\zeta)$, then we have already seen that $\nabla_d z_{\mathrm{LP}}(\zeta) = +\infty$, since $z_{\mathrm{LP}}(\zeta + \epsilon d) = +\infty$ for all $\epsilon > 0$. Equivalently, we have that $\mu(\zeta + \epsilon d) = \emptyset$ and thus $\max_{u \in \mu(\zeta)} u^\top d = +\infty$, proving the result in this first case.

(ii) If $d \notin \delta^-(\zeta)$, by (D-RLPVF), there $\exists \epsilon > 0$ such that $z_{\mathrm{LP}}(\zeta + \epsilon d) < +\infty$. As such, there exists $(\hat{u}, \hat{v}) \in \mathcal{E}$ such that

$$z_{\mathrm{LP}}(\zeta + td) = (\zeta + td)^\top \hat{u} + \beta^\top \hat{v}, \qquad \forall t \in [0, \epsilon]. \tag{2}$$

Thus $\nabla_d z_{\mathrm{LP}}(\zeta) = \hat{u}^\top d$. Note that by taking $t = 0$ we have that $\hat{u} \in \mu(\zeta)$. Now suppose, for the sake of contradiction, that $\overline{u}^\top d > \hat{u}^\top d$ for some $\overline{u} \in \mu(\zeta)$. Then by (1), there must exist $(u, v) \in \mathcal{E}$ with $u \in \mu(\zeta)$ and $u^\top d > \hat{u}^\top d$. But then for any $t \in (0, \epsilon]$, we have

$$\begin{aligned}(\zeta + td)^\top u + \beta^\top v = z_{\mathrm{LP}}(\zeta) + td^\top u > z_{\mathrm{LP}}(\zeta) + td^\top \hat{u} &= \zeta^\top \hat{u} + \beta^\top \hat{v} + td^\top \hat{u} \\ &= z_{\mathrm{LP}}(\zeta + td),\end{aligned}$$

where the final equality follows from (2), which contradicts (D-RLPVF) at $\zeta + td$. The result follows.

■

We now characterize the subdifferentials of the restricted LP value function. Applying the standard definition for the subdifferential of a convex function [Rockafellar, 1997], we have that the subdifferential $\partial z_{\rm LP}(\hat{\zeta})$ of $z_{\rm LP}$ at $\hat{\zeta} \in \mathcal{C}_{\rm LP}$ is[1]

$$\partial z_{\rm LP}(\hat{\zeta}) = \left\{ g \in \mathbb{R}^l \ : \ z_{\rm LP}(\zeta) \geq g^\top(\zeta - \hat{\zeta}) + z_{\rm LP}(\hat{\zeta}), \ \forall \zeta \in \mathcal{C}_{\rm LP} \right\}.$$

By the following two lemmas, we show that for $\hat{\zeta} \in \mathcal{C}_{\rm LP}$, $\partial z_{\rm LP}(\hat{\zeta}) = \mu(\hat{\zeta}) + \nu(\hat{\zeta})$.

Lemma 3.3. *For all $\zeta \in \mathcal{C}_{\rm LP}$, we have that*

$$\mu(\zeta) + \nu(\zeta) \subseteq \partial z_{\rm LP}(\zeta).$$

Proof. Let $\hat{\zeta} \in \mathcal{C}_{\rm LP}$, $\hat{u} \in \mu(\hat{\zeta})$, and $\hat{e} \in \nu(\hat{\zeta})$ be given. We show that $g = \hat{u} + \hat{e} \in \partial z_{\rm LP}(\hat{\zeta})$. Let $\hat{v}$ be such that $(\hat{u}, \hat{v}) \in \mathcal{P}_D$ and $\hat{\zeta}^\top \hat{u} + \beta^\top \hat{v} = z_{\rm LP}(\hat{\zeta})$; and let $\hat{h}$ be such that $(\hat{e}, \hat{h}) \in W$ and $\hat{\zeta}^\top \hat{e} + \beta^\top \hat{h} = 0$. Such $\hat{v}$ and $\hat{h}$ must exist by the definitions of $\mu(\hat{\zeta})$ and $\nu(\hat{\zeta})$, respectively. Note, however, that we have $(\hat{e}, \hat{h}) = (0, 0)$ when $\zeta \in \operatorname{int} \mathcal{C}_{\rm LP}$. Then

$$g^\top \hat{\zeta} = \hat{\zeta}^\top \hat{u} + \hat{\zeta}^\top \hat{e} = z_{\rm LP}(\hat{\zeta}) - \beta^\top \hat{v} - \beta^\top \hat{h}. \tag{3}$$

Now for any $\zeta \in \mathcal{C}_{\rm LP}$, we have that

$$\begin{aligned} g^\top \zeta = \zeta^\top \hat{u} + \zeta^\top \hat{e} = \ & \zeta^\top \hat{u} + \zeta^\top \hat{e} + \beta^\top \hat{v} - \beta^\top \hat{v} + \beta^\top \hat{h} - \beta^\top \hat{h} \leq \\ & \zeta^\top \hat{u} + \beta^\top \hat{v} - \beta^\top \hat{v} - \beta^\top \hat{h}, \end{aligned} \tag{4}$$

which follows from the fact that $(\hat{e}, \hat{h}) \in W$ implies that $\zeta^\top \hat{e} + \beta^\top \hat{h} \leq 0$. Since $(\hat{u}, \hat{v}) \in \mathcal{P}_D$, and $z_{\rm LP}(\zeta)$ is finite (as $\zeta \in \mathcal{C}_{\rm LP}$), we must have

$$\zeta^\top \hat{u} + \beta^\top \hat{v} \leq \max \left\{ \zeta^\top u + \beta^\top v \ : \ (u, v) \in \mathcal{P}_D \right\} \leq z_{\rm LP}(\zeta), \tag{5}$$

by weak duality. Thus, by (5), (4) and (3), respectively, we have

$$\begin{aligned} z_{\rm LP}(\zeta) \geq \ & \zeta^\top \hat{u} + \beta^\top \hat{v} \geq g^\top \zeta + \beta^\top \hat{v} + \beta^\top \hat{h} = \\ & g^\top \zeta + \beta^\top \hat{v} + \beta^\top \hat{h} + z_{\rm LP}(\hat{\zeta}) - \beta^\top \hat{v} - \beta^\top \hat{h} - g^\top \hat{\zeta} = g^\top(\zeta - \hat{\zeta}) + z_{\rm LP}(\hat{\zeta}). \end{aligned}$$

Since ζ was chosen arbitrarily from $\mathcal{C}_{\rm LP}$, it must be that $g \in \partial z_{\rm LP}(\hat{\zeta})$, as required. ∎

The reverse set containment makes use of the epigraph of the value function. By Proposition 3.1, $z_{\rm LP}$ is a polyhedral function, so its epigraph, given by

$$\operatorname{epi} z_{\rm LP} = \left\{ (\zeta, w) \in \mathbb{R}^l \times \mathbb{R} \ : \ \zeta \in \mathcal{C}_{\rm LP}, \ w \geq z_{\rm LP}(\zeta) \right\},$$

is a polyhedron. By the definition of the subdifferential, for any $\hat{\zeta} \in \mathcal{C}_{\rm LP}$ and subgradient $g \in \partial z_{\rm LP}(\hat{\zeta})$, the hyperplane

$$\left\{ (\zeta, w) \in \mathbb{R}^l \times \mathbb{R} : (-g, 1)^\top \begin{pmatrix} \zeta \\ w \end{pmatrix} = z_{\rm LP}(\hat{\zeta}) - g^\top \hat{\zeta} \right\},$$

[1] This is obviously equivalent to $\partial z_{\rm LP}(\hat{\zeta}) = \{ g \in \mathbb{R}^l \ : \ z_{\rm LP}(\zeta) \geq g^\top(\zeta - \hat{\zeta}) + z_{\rm LP}(\hat{\zeta}), \ \forall \zeta \in \mathbb{R}^l \}$, since $z_{\rm LP}(\zeta) = +\infty$ for $\zeta \notin \mathcal{C}_{\rm LP}$.

is a hyperplane that supports epi z_{LP} at $\zeta = \hat{\zeta}$ and the inequality

$$(-g, 1)^\top \begin{pmatrix} \zeta \\ w \end{pmatrix} \geq z_{\text{LP}}(\hat{\zeta}) - g^\top \hat{\zeta}, \tag{6}$$

is satisfied by all $(\zeta, w) \in \text{epi}\, z_{\text{LP}}$, with equality attained at $(\hat{\zeta}, z_{\text{LP}}(\hat{\zeta}))$.

The proof also makes use of the extreme points and extreme rays of the dual LP feasible set. Recall $\mathcal{E}$ and $\mathcal{R}$ are the (finite) sets of extreme points and extreme rays of $\mathcal{P}_D$, respectively. Recall that for $\zeta \in \mathcal{C}_{\text{LP}}$, we have that

$$\zeta^\top e + \beta^\top h \leq 0, \qquad \forall (e, h) \in \mathcal{R},$$

and that by LP duality, we have

$$z_{\text{LP}}(\zeta) = \max \left\{ \zeta^\top u + \beta^\top v \;:\; (u, v) \in \mathcal{E} \right\}.$$

Consequently, as is also shown in [Ralphs and Hassanzadeh, 2014],

$$\begin{aligned} \text{epi}\, z_{\text{LP}} = \big\{ (\zeta, w) \;:\; & \zeta^\top e + \beta^\top h \leq 0,\ \forall (e, h) \in \mathcal{R} \text{ and} \\ & w \geq \zeta^\top u + \beta^\top v,\ \forall (u, v) \in \mathcal{E} \big\}. \end{aligned} \tag{7}$$

We now prove the second lemma.

Lemma 3.4. *For all $\zeta \in \mathcal{C}_{\text{LP}}$,*

$$\partial z_{\text{LP}}(\zeta) \subseteq \mu(\zeta) + \nu(\zeta).$$

Proof. Let $\hat{\zeta} \in \mathcal{C}_{\text{LP}}$ and $g \in \partial z_{\text{LP}}(\hat{\zeta})$ be given. Then by (6) and (7), it must be that $(\hat{\zeta}, z_{\text{LP}}(\hat{\zeta}))$ is an optimal solution of the LP

$$\left\{ \begin{array}{ll} \min & (-g, 1)^\top \begin{pmatrix} \zeta \\ w \end{pmatrix} \\ \text{s.t.} & (\zeta, w) \in \text{epi}\, z_{\text{LP}} \end{array} \right. = \left\{ \begin{array}{lll} \min_{\zeta, w} & -g^\top \zeta + w & \\ \text{s.t.} & \zeta^\top e + \beta^\top h \leq 0, & \forall (e, h) \in \mathcal{R}, \\ & w \geq \zeta^\top u + \beta^\top v, & \forall (u, v) \in \mathcal{E}, \end{array} \right. \tag{8}$$

which has LP dual

$$\begin{array}{ll} \max_{\lambda, \gamma} & \displaystyle\sum_{e := (u,v) \in \mathcal{E}} \lambda_e \beta^\top v + \sum_{r := (e,h) \in \mathcal{R}} \gamma_r \beta^\top h \\ \text{s.t.} & \displaystyle\sum_{e := (u,v) \in \mathcal{E}} \lambda_e u + \sum_{r := (e,h) \in \mathcal{R}} \gamma_r e = g \\ & \displaystyle\sum_{e \in \mathcal{E}} \lambda_e = 1 \\ & \lambda \in \mathbb{R}^{\mathcal{E}}_+, \quad \gamma \in \mathbb{R}^{\mathcal{R}}_+, \end{array} \tag{9}$$

where the γ is the vector of dual variables associated with the first set of constraints, and λ is the vector of dual variables associated with the second set of constraints. Recall that $(\hat{\zeta}, z_{\text{LP}}(\hat{\zeta}))$ is an optimal solution of (8) and let (λ^*, γ^*) be an optimal solution of (9). By complementary slackness, it must be that for $r = (e, h) \in \mathcal{R}$, if $\gamma^*_r > 0$ then $-\hat{\zeta}^\top e = \beta^\top h$ so $\hat{\zeta}^\top e + \beta^\top h = 0$, and hence $e \in \nu(\hat{\zeta})$. Furthermore, for $p = (u, v) \in \mathcal{E}$, if $\lambda^*_p > 0$ then $z_{\text{LP}}(\hat{\zeta}) = \hat{\zeta}^\top u + \beta^\top v$, and hence $u \in \mu(\hat{\zeta})$. Thus, by the constraints of (9), g can be written as a convex combination of elements of $\mu(\hat{\zeta})$, which must also be an element of $\mu(\hat{\zeta})$, since it is a polyhedron, plus a non-negative combination of elements of $\nu(\hat{\zeta})$, which must also be an element of $\nu(\hat{\zeta})$ since it is a cone. The result follows. ∎

Proposition 3.5. *For all $\zeta \in \mathcal{C}_{\text{LP}}$,*

$$\partial z_{\text{LP}}(\zeta) = \mu(\zeta) + \nu(\zeta).$$

Proof. The result follows from Lemma 3.3 and Lemma 3.4. ■

As a consequence, we can see that differentiability of $z_{\text{LP}}(\zeta)$ occurs at points in the interior of its domain at which the dual LP has a unique solution.

Corollary 3.6. *$z_{\text{LP}}(\zeta)$ is differentiable at ζ if and only if $\zeta \in \operatorname{int} \mathcal{C}_{\text{LP}}$ and the dual problem (D-RLP) has a unique optimal solution.*

Proof. $\zeta \in \operatorname{int} \mathcal{C}_{\text{LP}}$ if and only if $\nu(\zeta) = \{\mathbf{0}\}$. Furthermore, for $\zeta \in \operatorname{int} \mathcal{C}_{\text{LP}}$, $\mu(\zeta)$ consists of a singleton if and only if (D-RLP) has a unique optimum. So by Proposition 3.5, the subdifferential $\partial z_{\text{LP}}(\zeta)$ is also a singleton if and only if the dual problem given in (D-RLP) has a unique optimum. The result follows. ■

Example 3. *We illustrate the relationship between the directional derivatives and the optimal dual solutions of the corresponding optimization problem* (D-RLP)*. The gradients of the RLPVF for Example 1 when $(x_1, x_2) = (0, 0)$, depicted in Figure 2, at ζ where the function is differentiable, are precisely the optimal dual solutions corresponding to the constraints with the parametric RHSs, denoted by u, of the associated optimization problem* (D-RLP)*. At points where the function is not differentiable (i.e., over the breakpoints), the directional derivatives of the RLPVF at ζ in direction $d \in \mathbb{R}^l$ (if they exist) are given by $d^\top u$, where u can be an optimal dual solution corresponding to the constraints with the parametric RHSs of the associated optimization problems* (D-RLP)*.*

3.2 Structure of the Restricted Value Function

We now characterize the structure of the RVF by observing that the RVF is the minimum of a finite number of translations of functions of the form (RLPVF) for different values of the (previously) non-parametric RHS β. Each of these translated functions defines a *stability region* over which the integer part of all solutions defining points on the graph of the associated RLPVF is fixed.

To further develop our characterization of the value function, we define the following sets of integer parts of solutions by projecting $\mathcal{S}(\zeta)$ onto the space of the integer variables:

$$\mathcal{S}_I(\zeta) = \operatorname{proj}_I \mathcal{S}(\zeta) = \left\{ x_I \in \mathbb{Z}^r_+ \ : \ (x_I, x_C) \in \mathcal{S}(\zeta) \right\}, \text{ and}$$

$$\mathcal{S}_I = \bigcup_{\zeta \in \mathcal{C}} \mathcal{S}_I(\zeta).$$

Thus, $\mathcal{S}_I$ is the set of all integer parts of points in $\mathcal{S}(\zeta)$ for some $\zeta \in \mathcal{C}$. For a given $\hat{x}_I \in \mathcal{S}_I$, the *continuous restriction* (CR) with respect to $\hat{x}_I$ is the function[2]

$$\bar{z}(\zeta;\ \hat{x}_I) = c_I^0 \hat{x}_I + z_{\text{LP}}(\zeta - C_I^{1:l} \hat{x}_I;\ b - A_I \hat{x}_I), \quad \forall \zeta \in \mathcal{C}, \tag{CR}$$

[2] In the multiobjective optimization literature, the corresponding multiobjective LP is sometimes referred to as a *slice problem* [Belotti et al., 2013].

where we now add a secondary parameter to the previously defined function z_{LP}. In this section, $z_{\text{LP}}(\cdot;\ \beta)$ is similar to the previously defined z_{LP} except that we now wish to allow for a parametric family of such functions with different vectors for the non-parametric RHS β in (RLPVF). The form shown above then denotes precisely a translation of a function of the form (RLPVF) for $\beta = b - A_I\hat{x}_I$. In the remainder of the paper, we refer to functions $\bar{z}(\cdot;\ x_I)$ for $x_I \in \mathcal{S}_I$ as *bounding functions*, since they bound the RVF from above, as demonstrated in Proposition 3.7.

Proposition 3.7. *For any $\hat{x}_I \in \mathcal{S}_I$, $\bar{z}(\cdot;\ \hat{x}_I)$ bounds z from above.*

Proof. For $\hat{x}_I \in \mathcal{S}_I$, we have

$$\begin{aligned}\bar{z}(\zeta;\ \hat{x}_I) = c_I^0\hat{x}_I + \ & z_{\text{LP}}(\zeta - C_I^{1:l}\hat{x}_I;\ b - A_I\hat{x}_I) \geq \\ & \min_{x_I \in \mathcal{S}_I} \left(c_I^0 x_I + z_{\text{LP}}(\zeta - C_I^{1:l}x_I;\ b - A_I x_I)\right) = z(\zeta), \quad \forall \zeta \in \mathcal{C}.\end{aligned}$$

■

We can now provide our first characterization of the RVF as the minimum of a set of such translations. We have that

$$z(\zeta) = \min_{x_I \in \mathcal{S}_I} \bar{z}(\zeta;\ x_I) = \min_{x_I \in \mathcal{S}_I} \left(c_I^0 x_I + z_{\text{LP}}(\zeta - C_I^{1:l}x_I;\ b - A_I x_I)\right) \quad \forall \zeta \in \mathcal{C}. \tag{RVF-eq}$$

When X_{MO} is bounded, S_I is finite, so the number of such functions required is finite.

Proposition 3.7 shows that any collection of points from $\mathcal{S}_I$ yields an upper approximation of z simply by taking the minimum of the associated set of bounding functions, previously defined as $\bar{z}(\cdot;\ x_I)$ for $x_I \in \mathcal{S}_I$. The algorithm described in Section 5 constructs a subset of $\mathcal{S}_I$ that fully describes the RVF by iteratively approximating it from above. In what follows, we make extensive use of the concept of a *stability region* associated with a given $\hat{x}_I \in \mathcal{S}_I$, denoted by $\mathcal{C}(\hat{x}_I)$, and defined to be

$$\mathcal{C}(\hat{x}_I) = \{\zeta \in \mathcal{C}\ :\ z(\zeta) = \bar{z}(\zeta;\ \hat{x}_I)\},$$

the subset of $\mathcal{C}$ for which the bounding function associated with $\hat{x}_I$ agrees with the RVF. We discuss the properties of stability regions shortly, but we first focus on characterizing the RVF. Clearly, elements of $\mathcal{S}_I$ that have an empty stability region do not contribute to the description of z. As such, the main result of this section is the following discrete characterization, in which we define a minimal subset of $\mathcal{S}_I$ required to fully describe z.

Theorem 3.8. *Let $\mathcal{S}_{\min}$ be a minimal subset of $\mathcal{S}_I$ with the property that for any $\zeta \in \mathcal{C}$, there exist $x_I \in \mathcal{S}_{\min}$ and $x_C \in \mathbb{R}_+^{n-r}$ such that $C_I^{1:l}x_I + C_C^{1:l}x_C \leq \zeta$, $A_I x_I + A_C x_C = b$, and $c_I^0 x_I + c_C^0 x_C = z(\zeta)$. Then for any $\zeta \in \mathcal{C}$ we have*

$$z(\zeta) = \min_{x_I \in \mathcal{S}_I} \bar{z}(\zeta;\ x_I) = \min_{x_I \in \mathcal{S}_{\min}} \bar{z}(\zeta;\ x_I).$$

Proof. Let $\zeta \in \mathcal{C}$. From (RVF-eq) and since $\mathcal{S}_{\min} \subseteq \mathcal{S}$, we have that

$$z(\zeta) = \min_{x_I \in \mathcal{S}_I} \bar{z}(\zeta;\ x_I) \leq \min_{x_I \in \mathcal{S}_{\min}} \bar{z}(\zeta;\ x_I).$$

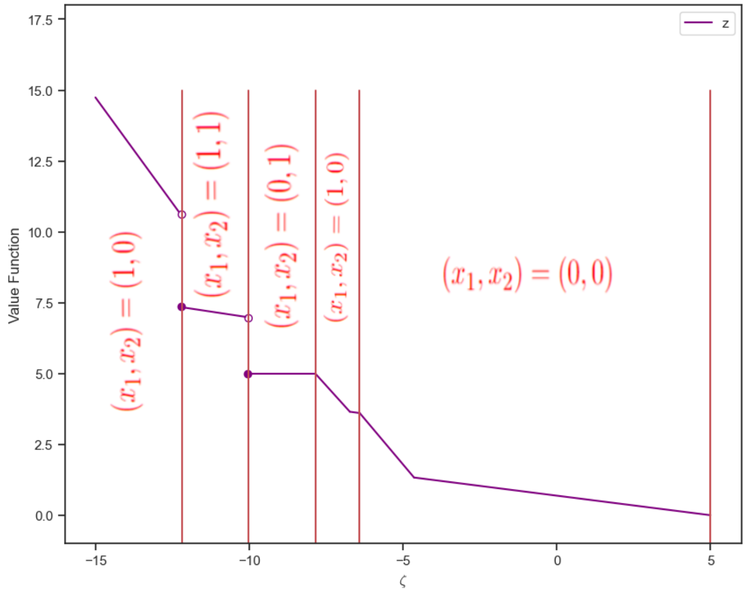


Figure 3: Stability regions and corresponding members of $\mathcal{S}_I$ in Example 1, for the region $\zeta \geq -15$

Now by definition of $\mathcal{S}_{\min}$, there exists $\hat{x}_I \in \mathcal{S}_{\min}$ and $\hat{x}_C \in \mathbb{R}^{n-r}_+$ such that $C^{1:l}_I \hat{x}_I + C^{1:l}_C \hat{x}_C \leq \zeta$, $A_I \hat{x}_I + A_C \hat{x}_C = b$, and $c^0_I \hat{x}_I + c^0_C \hat{x}_C = z(\zeta)$. So

$$\begin{aligned} \min_{x_I \in \mathcal{S}_{\min}} \bar{z}(\zeta;\ x_I) &= \min_{x_I \in \mathcal{S}_{\min}} \left(c^0_I x_I + z_{\text{LP}}(\zeta - C^{1:l}_I x_I;\ b - A_I x_I) \right) \\ &\leq c^0_I \hat{x}_I + z_{\text{LP}}(\zeta - C^{1:l}_I \hat{x}_I;\ b - A_I \hat{x}_I) \\ &\leq c^0_I \hat{x}_I + c^0_C \hat{x}_C = z(\zeta), \end{aligned}$$

where the first equation follows from (CR), the subsequent inequality follows since $\hat{x}_I \in \mathcal{S}_{\min}$, the next since, by its definition, $\hat{x}_C$ is feasible for $z_{\text{LP}}(\zeta - C^{1:l}_I \hat{x}_I;\ b - A_I \hat{x}_I)$, and the final equation follows from the definition of $\hat{x}_I$ and $\hat{x}_C$. We thus have both that $z(\zeta)$ is no greater and no less than $\min_{x_I \in \mathcal{S}_{\min}} \bar{z}(\zeta;\ x_I)$, and the result follows. ■

We call any set $\mathcal{S}_{\min}$ that satisfies the conditions of Theorem 3.8 a *(minimal) description* of the RVF. It is important to note that the stability region for a particular element of $\mathcal{S}_I$ does not necessarily have to be closed or connected.

Example 4. *Consider the stability regions of Example 1 for each member of $\mathcal{S}_I$, depicted in Figure 3 (for the right-hand part of the RVF). The stability region for a given member of $\mathcal{S}_I$ can be seen as the projection of the subset of the boundary of the epigraph of the RVF that agrees with the bounding function associated with x_I onto the domain of the VF (the subspace containing $\mathcal{C}$). For instance, the stability region for $x_I = (1,1)$ is the projection onto the ζ-axis of the isolated half-open line segment connecting $(\zeta, z(\zeta)) = (-12\frac{1}{6}, 7\frac{1}{3})$ to $(-10, 7\frac{2}{123})$, yielding the half-open interval $[-12\frac{1}{6}, -10)$. Additionally, the stability region for $x_I = (1,0)$ shown in Figure 3 illustrates that a stability region need not be connected, as it includes some values of $\zeta < -12\frac{1}{6}$ and some values of $\zeta > -7\frac{5}{6}$, but excludes the interval $[-12\frac{1}{6}, -7\frac{5}{6})$. This shows that the stability region for $x_I = (1,0)$*

consists of two disjoint intervals. The stability region for $x_I = (0,0)$ is a single interval, closed at both ends. The stability region for $x_I = (0,1)$ is a single closed interval, $[-10, -7\frac{5}{6}]$, on which the RVF is a horizontal line segment, exhibiting a flat piece. Further insights can be obtained by examining the bounding functions associated with each element of $\mathcal{S}_I$, provided in Appendix A.

The stability regions for Example 1 exhibit disjoint intervals, which demonstrate that a stability region need not be convex by virtue of not being connected. In addition, it is possible for a connected component of a stability region to be nonconvex, and non-closedness is also found in higher-dimensional instances. We illustrate this in Example 5.

Example 5. *Consider the RVF instance given by*

$$\begin{aligned} z(\zeta) = \min \quad & x_2 \\ & x_3 \le \zeta_1 \\ & x_4 \le \zeta_2 \\ & 2(1-x_1) \le x_j \le 2(1-x_1) + 5x_1, && j = 2,3,4 \\ & x_2 + x_3 \ge 5x_1 \\ & x_1 \le 1, \quad x_1 \in \mathbb{Z}_+ \\ & x_j \le 5, \quad x_j \in \mathbb{R}_+, && j = 2,3,4. \end{aligned}$$

As in previous examples, although this is not in the standard form as in (RVF), it can easily be made so with the addition of slack and surplus variables. Since these do not change the RVF or stability regions, we keep the instance in its natural form. The value function for this instance can be written explicitly as

$$z(\zeta) = \begin{cases} 2, & \zeta_1 \in [2,3] \text{ and } \zeta_2 \ge 2, \\ 5 - \zeta_1, & \zeta_1 \le 5 \text{ and } \zeta_2 < 2 \text{ or } \zeta_1 \in [0,2) \cup (3,5], \\ 0, & \text{otherwise}, \end{cases}$$

for $\zeta = (\zeta_1, \zeta_2) \in \mathcal{C} = \mathbb{R}^2_+$. The stability region for $x_I = (x_1) = (1)$ is

$$\{\zeta \in \mathbb{R}^2_+ \;:\; \zeta_1 < 2 \text{ or } \zeta_1 \ge 3 \text{ or } \zeta_2 < 2\},$$

which is shown in Figure 4, which displays both stability regions for this example. The stability region for $x_I = (x_1) = (1)$ is connected but is neither convex nor closed.

The set $\mathcal{S}_{\min}$ in Theorem 3.8 is not necessarily unique, as it is possible for multiple integer solutions $x_I^1, x_I^2 \in \mathcal{S}_I$ with $x_I^1 \ne x_I^2$ to satisfy $\bar{z}(\cdot;\ x_I^1) = \bar{z}(\cdot;\ x_I^2)$, e.g., in the trivial case of duplicate variables. Furthermore, for any point $(\zeta, z(\zeta))$ on the graph of the RVF, there may be more than one integer part x_I with this point lying on the RVF for its associated bounding function. In other words, it is possible that stability regions may overlap, even for integer parts in $\mathcal{S}_{\min}$. Such a case occurs in Example 1: points in the purple colored line segment seen in Figure 4 are in the stability regions of two different integer parts, both of which are in $\mathcal{S}_{\min}$. Of course, for each integer part $x_I \in \mathcal{S}_{\min}$, there must exist at least *one* point on the boundary of the epigraph of the associated bounding function that is on the boundary of the epigraph of the RVF itself and is not on the boundary of the epigraph of any other bounding function; otherwise, $\mathcal{S}_{\min}$ would not be minimal.

We now turn our attention from the structure of stability regions and properties of $\mathcal{S}_{\min}$ to studying properties of the RVF itself arising from the finite description of Theorem 3.8. Figure 1a illustrates

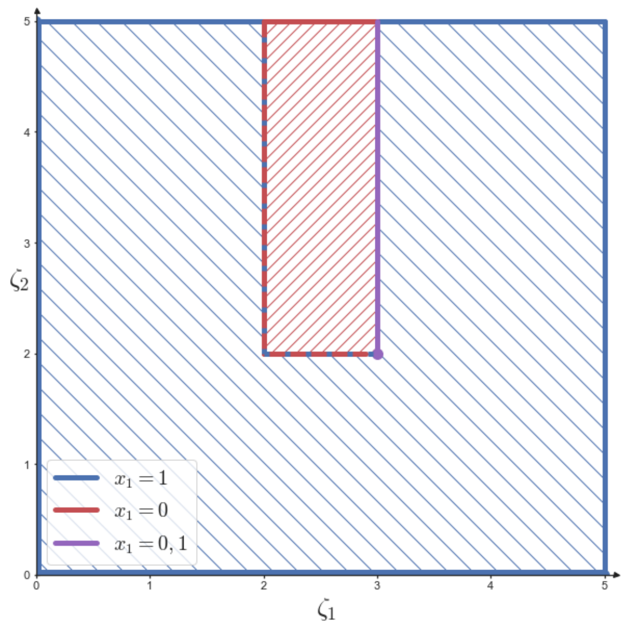


Figure 4: Stability regions of the instance in Example 5. The area shaded in red, including the red boundary lines, is not in the stability region for $x_1 = 1$, which is thus neither convex nor closed. The line segment shaded in purple is in the stability region of both $x_1 = 0$ and $x_1 = 1$.

the RVF of the MILP described in Example 1. The RVF is piecewise polyhedral, nonincreasing, and lower semi-continuous (here, nonincreasing means that $\zeta_1 \geq \zeta_2,\ \zeta_1, \zeta_2 \in \mathbb{R}^l \Rightarrow z(\zeta_1) \leq z(\zeta_2)$. We formalize this in Proposition 3.9, which can be viewed as a generalization of the previous results by [Nemhauser and Wolsey, 1988] and [Bank et al., 1982] for the full MILP VF.

Proposition 3.9. *The restricted value function* (RVF) *z is a lower semi-continuous, nonincreasing function that is composed of a minimum of a finite number of polyhedral functions.*

Proof. By Equation (RVF-eq), the RVF is the minimum of a finite number of functions, all of which are polyhedral by Proposition 3.1. The lower semi-continuity property can be established using a proof similar to that presented in [Nemhauser and Wolsey, 1988]. The nonincreasing property of the value function follows from the fact that it is the minimum of a set of nonincreasing functions and must therefore be nonincreasing itself. ■

The RVF may be discontinuous due to the fact that individual bounding functions themselves are discontinuous at the boundaries of their finite domains. When the RVF is discontinuous, the discontinuities occur at (some of) the boundaries between stability regions. At such points, as well as at points on the boundary of $\mathcal{C}$, there may be directions in which the directional derivative of z is infinite, exactly as can occur at points on the boundary of $\mathcal{C}_{\text{LP}}$ in the RLPVF.

Next, we describe the directional derivative of z at a point $\zeta \in \mathcal{C}$ in terms of the faces of optimal solutions to the duals of the LPs associated with the bounding functions in directions for which the directional derivative is finite. To characterize these directions, we need the analogues of the sets of optimal extreme points and extreme rays characterizing the optimal faces of solutions to the LP dual, as we had in the RLPVF case, but with the integer part of the solution fixed. For $x_I \in \mathcal{S}_I$ and $\zeta \in \mathbb{R}^l$, we define

$$\begin{aligned}\mu(\zeta;\ x_I) =& \{\hat{u} \in \mathbb{R}^l\ :\ (\hat{u}, \hat{v}) \in \operatorname{conv}(\{(u,v) \in \mathcal{E}\ :\ (\zeta - C_I^{1:l} x_I)^\top u + (b - A_I x_I)^\top v = \bar{z}(\zeta;\ x_I)\})\} \\ =& \operatorname{conv}(\{u \in \mathbb{R}^l\ :\ (\zeta - C_I^{1:l} x_I)^\top u + (b - A_I x_I)^\top v = \bar{z}(\zeta;\ x_I),\ (u,v) \in \mathcal{E}\}),\end{aligned}$$

and

$$\begin{aligned}\nu(\zeta;\ x_I) &= \{\hat{e} \in \mathbb{R}^l\ :\ (\hat{e}, \hat{h}) \in \operatorname{cone}(\{(e,h) \in \mathcal{R}\ :\ (\zeta - C_I^{1:l} x_I)^\top e + (b - A_I x_I)^\top h = 0\})\}\ \cup\ \{\mathbf{0}\} \\ &= \operatorname{cone}(\{e \in \mathbb{R}^l\ :\ (\zeta - C_I^{1:l} x_I)^\top e + (b - A_I x_I)^\top h = 0,\ (e,h) \in \mathcal{R}\})\ \cup\ \{\mathbf{0}\}.\end{aligned}$$

We further need to know the set of integer vectors associated with the bounding functions that agree with the RVF at a given point, which we denote as

$$\begin{aligned}S_I^*(\zeta) &= \left\{x_I \in S_I(\zeta)\ :\ c_I^0 x_I + c_C^0 x_C = z(\zeta),\ \exists x_C \text{ s.t. } (x_I, x_C) \in S(\zeta)\right\} \\ &= \left\{x_I \in S_I(\zeta)\ :\ \bar{z}(\zeta;\ x_I) = z(\zeta)\right\}.\end{aligned}$$

With all of this notation established, let us now consider a point $\hat{\zeta} \in \mathbb{R}^l$ at which the RVF is discontinuous. As with the RLPVF, the directional derivative of a bounding function associated with $x_I \in \mathcal{S}_I$ is infinite in direction $d \in \mathbb{R}^l$ at $\hat{\zeta}$ if and only if d is contained in the set

$$\begin{aligned}\delta^-(\hat{\zeta};\ x_I) = \operatorname{cone}(\{d \in \nu(\hat{\zeta};\ x_I)\ :\ &\exists (e,h) \in \mathcal{R} \text{ such that} \\ &(\hat{\zeta} - C_I^{1:l} x_I)^\top e + (b - A_I x_I)^\top h = 0,\ d^\top e > 0\}) \setminus \{\mathbf{0}\}.\end{aligned}$$

Finally, for the directional derivative of z itself to be infinite in direction $d \in \mathbb{R}^l$ at $\hat{\zeta} \in \mathbb{R}^l$, the derivatives of *all* bounding functions that agree with z at ζ must also be infinite, so we have the following proposition.

Proposition 3.10. *For $\zeta \in \mathcal{C}$ and $d \in \mathbb{R}^l$,*

$$\begin{aligned}\nabla_d z(\zeta) &= \min_{x_I \in \mathcal{S}_I^*(\zeta) \cap \mathcal{S}_{\min}} \nabla_d \bar{z}(\zeta;\ x_I) \\ &= \begin{cases} \min\limits_{x_I \in \mathcal{S}_I^*(\zeta) \cap \mathcal{S}_{\min}} (\max_{u \in \mu(\zeta;\ x_I)} u^\top d), & \text{if } d \notin \cap_{x_I \in \mathcal{S}_I^*(\zeta) \cap \mathcal{S}_{\min}} \delta^-(\zeta;\ x_I), \\ \infty, & \text{otherwise,} \end{cases}\end{aligned}$$

where $\mathcal{S}_{\min}$ is any minimal description of the RVF.

Proof. Let $d \in \mathbb{R}^l$, and a minimal description $\mathcal{S}_{\min}$ of z be given. By Theorem 3.8, z is the minimum of a finite set of functions, $\bar{z}(\cdot;\ x_I)$, for each $x_I \in \mathcal{S}_{\min}$, which is a finite set. Furthermore, by Proposition 3.1, each of these functions is polyhedral. Thus, Proposition B.1 (in Appendix B), which is a general result about functions that are the minimum of a finite number of polyhedral functions, applies and yields the first part.

The second part follows by substituting the formula for the directional derivative of the bounding function from Proposition 3.2.

■

Even though the RVF is not convex, nor expected to be continuous, it is still possible to define a notion of subdifferential by considering the local structure of the value function at a given point, which is inherited from the bounding functions that are active at the point and necessary to a minimal description of the value function.

We first observe that since (by Proposition 3.9) z is comprised of a minimum of a finite number of polyhedral functions, indicated by some minimal description $\mathcal{S}_{\min}$, as per in Theorem 3.8, z must satisfy the following property. We prove this below.

Property 3.11. *For any $\zeta \in \mathcal{C}$ and any minimal description $\mathcal{S}_{\min}$, there exists $\epsilon > 0$ such that*

$$z(\zeta') \geq \min_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \overline{z}(\zeta';\ x_I), \qquad \forall \zeta' \in B(\zeta;\ \epsilon), \tag{10}$$

where $B(\zeta;\ \epsilon)$ denotes the open ball of radius ϵ centered on ζ.

Note that because we may allow $z(\zeta)$ to take values in the extended real numbers, we do not need to restrict ζ' to $\mathcal{C}$ since $z(\zeta') = +\infty$ automatically satisfies the inequality for $\zeta' \notin \mathcal{C}$. Here the bounding functions $\overline{z}(\cdot;\ x_I)$ for $x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}$ are those that are active at ζ and necessary for the minimal description of the RVF provided by $\mathcal{S}_{\min}$.

Lemma 3.12. *The restricted value function z satisfies Property 3.11.*

Proof. Let $\mathcal{S}_{\min}$ be given as per Theorem 3.8 and let $\zeta \in \mathcal{C}$. Recall that by Theorem 3.8,

$$z(\zeta') = \min_{x_I \in \mathcal{S}_{\min}} \overline{z}(\zeta';\ x_I), \qquad \forall \zeta' \in \mathbb{R}^l. \tag{11}$$

Define

$$\alpha(\zeta') = \min_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \overline{z}(\zeta';\ x_I), \qquad \forall \zeta' \in \mathbb{R}^l.$$

So the inequality in Property 3.11 simply says $z(\zeta') \geq \alpha(\zeta')$. Now suppose that Property 3.11 does *not* hold. Then for every $\epsilon > 0$ there exists $x_I \in \mathcal{S}_{\min}$ with $\overline{z}(\zeta';\ x_I) < \alpha(\zeta')$ for some $\zeta' \in B(\zeta;\ \epsilon)$. This is because $z(\zeta') = \overline{z}(\zeta';\ x_I)$ for some $x_I \in \mathcal{S}_{\min}$, by (11). Since there are only a finite number of $x_I \in \mathcal{S}_{\min}$, and each bounding function $\overline{z}(\cdot; x_I)$ is polyhedral, actually there must exist one $\hat{x}_I \in \mathcal{S}_{\min}$ so that for all $\epsilon > 0$, some $\zeta' \in B(\zeta;\ \epsilon)$ has $\overline{z}(\zeta';\ \hat{x}_I) < \alpha(\zeta')$. Now $\overline{z}(\cdot;\ \hat{x}_I)$ is continuous on its finite domain, which is a closed set, so it must be that $\overline{z}(\zeta;\ \hat{x}_I) \leq \alpha(\zeta)$. But $\alpha(\zeta) = z(\zeta) \leq \overline{z}(\zeta;\ \hat{x}_I)$. So it must be that $\overline{z}(\zeta;\ \hat{x}_I) = z(\zeta)$, and $\hat{x}_I \in \mathcal{S}_I^*(\zeta)$. But then $\hat{x}_I \in \mathcal{S}_I^*(\zeta) \cap \mathcal{S}_{\min}$, so in fact $\overline{z}(\zeta';\ \hat{x}_I) \geq \alpha(\zeta')$ for all $\zeta' \in \mathbb{R}^l$, and we obtain a contradiction. The result follows. ■

In what follows, we show that the intersection of the subdifferentials of these active and necessary bounding functions at a point yields the *local* subdifferential of the value function at that point, in the sense defined below.

Definition 3.13 (Local Subdifferential)**.** *Let $f : \mathbb{R}^l \to \mathbb{R}$ and $\zeta \in \mathbb{R}^l$. Then $q \in \mathbb{R}^l$ is a* local subgradient *of f at ζ if there exists $\epsilon > 0$ such that*

$$f(\zeta') \geq f(\zeta) + q^\top(\zeta' - \zeta), \quad \forall \zeta' \in B(\zeta;\ \epsilon). \tag{12}$$

The local subdifferential *of f at ζ, denoted $\partial_L f(\zeta)$, is defined as the set of all local subgradients of f at ζ.*

Our definition of a local subgradient is closely related to that of the proximal subgradient, given in, e.g., [Rockafellar and Wets, 2009] (Definition 8.45). The main difference is that we drop the quadratic term $\sigma\|\zeta' - \zeta\|^2$ in the definition of the proximal subgradient, since $\sigma = 0$ is always valid in our setting. In particular, for functions that are piecewise affine, our local subdifferential and the proximal subdifferential are identical. Furthermore, for convex functions, our local subdifferential and the usual subdifferential are identical.

Proposition 3.14. *The local subdifferential of z at $\zeta \in \mathcal{C}$ is given by*

$$\partial_L z(\zeta) = \bigcap_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \Big(\mu(\zeta - C_I^{1:l} x_I;\ b - A_I x_I) + \nu(\zeta - C_I^{1:l} x_I;\ b - A_I x_I)\Big),$$

where $\mathcal{S}_{\min}$ is a minimal description of the RVF.

Proof. Let $\mathcal{S}_{\min}$ be a minimal description of the RVF, and consider $\zeta \in \mathcal{C}$. Let $g \in \partial_L z(\zeta)$. Then there exists $\epsilon > 0$ such that

$$z(\zeta') \geq z(\zeta) + g^\top(\zeta' - \zeta), \quad \forall \zeta' \in B(\zeta;\ \epsilon).$$

Let $x_I \in S_I^*(\zeta)$ be chosen arbitrarily. Now for any $\zeta' \in B(\zeta;\ \epsilon)$,

$$\overline{z}(\zeta';\ x_I) \geq z(\zeta') \geq z(\zeta) + g^\top(\zeta' - \zeta) = \overline{z}(\zeta;\ x_I) + g^\top(\zeta' - \zeta),$$

where the first inequality follows from the definition of z, the second since g is a local subgradient of z at ζ, and the final equality follows since $x_I \in S_I^*(\zeta)$. Thus g is also a local subgradient of $\overline{z}(\cdot;\ x_I)$ at ζ. But $\overline{z}(\cdot;\ x_I)$ is a restricted LP value function, and so is a convex function over a convex finite domain. Hence

$$g \in \partial\overline{z}(\zeta;\ x_I) = \partial z_{\mathrm{LP}}(\zeta - C_I^{1:l} x_I;\ b - A_I x_I).$$

Since x_I was chosen arbitrarily from $S_I^*(\zeta)$, we have proved that

$$\partial_L z(\zeta) \subseteq \bigcap_{x_I \in S_I^*(\zeta)} \partial z_{\mathrm{LP}}(\zeta - C_I^{1:l} x_I;\ b - A_I x_I) \subseteq \bigcap_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \partial z_{\mathrm{LP}}(\zeta - C_I^{1:l} x_I;\ b - A_I x_I).$$

To prove containment in the opposite direction, let

$$g \in \bigcap_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \partial z_{\mathrm{LP}}(\zeta - C_I^{1:l} x_I;\ b - A_I x_I) = \bigcap_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \partial\overline{z}(\zeta;\ x_I).$$

By Property 3.11, which z satisfies by Lemma 3.12, there must exist $\epsilon > 0$ so that (10) holds. Choose $\zeta' \in B(\zeta;\ \epsilon)$ arbitrarily. Then, by (10),

$$z(\zeta') \geq \min_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \overline{z}(\zeta';\ x_I),$$

and so there exists some $x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}$ with

$$z(\zeta') \geq \overline{z}(\zeta';\ x_I) \geq \overline{z}(\zeta;\ x_I) + g^\top(\zeta' - \zeta) = z(\zeta) + g^\top(\zeta' - \zeta),$$

where the second inequality follows since $g \in \partial \overline{z}(\zeta;\ x_I)$ and the final equality follows since $x_I \in S_I^*(\zeta)$. Thus $g \in \partial_L z(\zeta)$ and it must be that

$$\partial_L z(\zeta) \supseteq \bigcap_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \partial z_{\mathrm{LP}}(\zeta - C_I^{1:l} x_I;\ b - A_I x_I).$$

Hence

$$\partial_L z(\zeta) = \bigcap_{x_I \in S_I^*(\zeta) \cap \mathcal{S}_{\min}} \partial z_{\mathrm{LP}}(\zeta - C_I^{1:l} x_I;\ b - A_I x_I).$$

The result follows by Proposition 3.5. ■

4 The RVF and the EF

We are now ready to formalize the relationship between the RVF and the EF with the following theorem. Although a rigorous formulation of this theorem involves some technicalities that obscure intuition, we can informally interpret the theorem as saying that the boundary of the epigraph of the RVF and the EF effectively encode the same information. The difference is only that the boundary of the epigraph of the RVF may include some "flat" parts (regions over which the RVF has a directional derivative of zero) that are not technically part of the EF because they consist of weak NDPs, which are *not* nondominated, or because there is no corresponding feasible point, meaning they are not objective space images of points in X_{MO}.

Theorem 4.1 formalizes this relationship.

Theorem 4.1. *The EF associated with* (MO-MILP) *is a (possibly strict) subset of the boundary of the epigraph of the RVF. In particular, the following statements hold.*

1. *If $C_I x_I + C_C x_C$ belongs to the EF for some $(x_I, x_C) \in X_{\mathrm{MO}}$, then $(C_I^{1:l} x_I + C_C^{1:l} x_C, c_I^0 x_I + c_C^0 x_C)$ is a point on the boundary of the epigraph of z.*

2. *If $(\zeta, z(\zeta))$ is a point on the boundary of the epigraph of z, then there exists an efficient solution $(x_I, x_C) \in X_{\mathrm{MO}}$ such that $c_I^0 x_I + c_C^0 x_C = z(\zeta)$ and $C_I^{1:l} x_I + C_C^{1:l} x_C \leq \zeta$. Further, $C_I^{1:l} x_I + C_C^{1:l} x_C = \zeta$ if and only if $\nabla_d z(\zeta) > 0$ for all $d \in \mathbb{R}^l_- \setminus \{\mathbf{0}\}$.*

Proof. 1. To prove statement 1, let $(\hat{x}_I, \hat{x}_C) \in X_{\mathrm{MO}}$ be a given efficient solution and let $\hat{\zeta} = C_I^{1:l} \hat{x}_I + C_C^{1:l} \hat{x}_C$. Now $(\hat{x}_I, \hat{x}_C) \in X_{\mathrm{MO}}$ and $\hat{\zeta} = C_I^{1:l} \hat{x}_I + C_C^{1:l} \hat{x}_C$ implies, from definitions, that $(\hat{x}_I, \hat{x}_C) \in \mathcal{S}(\hat{\zeta})$. We want to show that $(\hat{\zeta}, c_I^0 \hat{x}_I + c_C^0 \hat{x}_C)$ is a point on the boundary of the epigraph of the RVF z. Since $z(\hat{\zeta}) = \min_{(x_I, x_C) \in \mathcal{S}(\hat{\zeta})} c_I^0 x_I + c_C^0 x_C$, by definition, and since $(\hat{x}_I, \hat{x}_C) \in \mathcal{S}(\hat{\zeta})$, it must be that $z(\hat{\zeta}) \leq c_I^0 \hat{x}_I + c_C^0 \hat{x}_C$. Assume, for the sake of contradiction, that $c_I^0 \hat{x}_I + c_C^0 \hat{x}_C \neq z(\hat{\zeta})$. Then it must be that $c_I^0 \hat{x}_I + c_C^0 \hat{x}_C > z(\hat{\zeta})$. Now, by the definition of $z(\hat{\zeta})$, there must exist $(x_I, x_C) \in \mathcal{S}(\hat{\zeta})$ with $z(\hat{\zeta}) = c_I^0 x_I + c_C^0 x_C$. Furthermore, $(x_I, x_C) \in \mathcal{S}(\hat{\zeta})$ implies $C_I^{1:l} x_I + C_C^{1:l} x_C \leq \hat{\zeta} = C_I^{1:l} \hat{x}_I + C_C^{1:l} \hat{x}_C$, while $c_I^0 x_I + c_C^0 x_C = z(\hat{\zeta}) < c_I^0 \hat{x}_I + c_C^0 \hat{x}_C$, so $C_I x_I + C_C x_C \lneq C_I \hat{x}_I + C_C \hat{x}_C$. This contradicts the hypothesis that $(\hat{x}_I, \hat{x}_C)$ is an efficient solution, and the result follows.

2. To prove the first part of 2, let $\zeta \in \mathcal{C}$ be given, so that $(\zeta, z(\zeta))$ is a point on the boundary of the epigraph of z. Then there exists $(\tilde{x}_I, \tilde{x}_C) \in \mathcal{S}(\zeta)$ such that $c_I^0 \tilde{x}_I + c_C^0 \tilde{x}_C = z(\zeta)$. There are two cases. If $(\tilde{x}_I, \tilde{x}_C)$ is an efficient solution, the result follows trivially. Otherwise, there must exist an efficient solution $(x_I, x_C) \in X_{\text{MO}}$ that dominates $(\tilde{x}_I, \tilde{x}_C)$ at least weakly, i.e., such that $C_I x_I + C_C x_C \leq C_I \tilde{x}_I + C_C \tilde{x}_C$. Then $C_I^{1:\ell} x_I + C_C^{1:\ell} x_C \leq C_I^{1:\ell} \tilde{x}_I + C_C^{1:\ell} \tilde{x}_C \leq \zeta$, which means $(x_I, x_C) \in \mathcal{S}(\zeta)$ and we also have $c_I^0 x_I + c_C^0 x_C \leq c_I^0 \tilde{x}_I + c_C^0 \tilde{x}_C = z(\zeta)$. But by the optimality of $z(\zeta)$, we must also have that $c_I^0 x_I + c_C^0 x_C \geq z(\zeta)$, so we conclude that $c_I^0 x_I + c_C^0 x_C = z(\zeta)$. Since (x_I, x_C) is an efficient solution, the result follows.

   For the second part, there are two directions.

   - $\Leftarrow$ Suppose that $\nabla_d z(\zeta) > 0$ for all $d \in \mathbb{R}_-^l \setminus \{\mathbf{0}\}$ and assume for the sake of contradiction that $\zeta \gneqq C_I^{1:l} x_I + C_C^{1:l} x_C$. Let $\tilde{\zeta} = C_I^{1:l} x_I + C_C^{1:l} x_C$. Now $\mathcal{S}(\tilde{\zeta}) \subseteq \mathcal{S}(\zeta') \subseteq \mathcal{S}(\zeta)$ for all ζ' with $\tilde{\zeta} \leq \zeta' \leq \zeta$. Thus, since (x_I, x_C) minimizes the objective over $\mathcal{S}(\zeta)$ and $(x_I, x_C) \in \mathcal{S}(\tilde{\zeta})$, it must be that $z(\zeta') = c_I^0 x_I + c_C^0 x_C$ for all $\zeta' \in [\tilde{\zeta}, \zeta]$. So $z(\zeta') = z(\zeta)$ for all $\zeta' \in [\tilde{\zeta}, \zeta]$. Recalling that $\zeta \gneqq \tilde{\zeta}$, it must thus be that $\hat{d} = \tilde{\zeta} - \zeta \in \mathbb{R}_-^l \setminus \{\mathbf{0}\}$ and $\nabla_{\hat{d}} z(\zeta) = 0$, which is a contradiction to the initial hypothesis. Therefore, $\zeta = C_I^{1:l} x_I + C_C^{1:l} x_C$.
   - $\Rightarrow$ We prove the contrapositive. Therefore, suppose there exists $d \in \mathbb{R}_-^l \setminus \{\mathbf{0}\}$ such that $\nabla_d z(\zeta) = 0$. Then, by Proposition 3.9, there must exist $\tilde{\zeta} \lneqq \zeta$ with $z(\tilde{\zeta}) = z(\zeta)$. By the proof of the first part of 2, above, there must exist $(\tilde{x}_I, \tilde{x}_C)$ an efficient solution with $(\tilde{x}_I, \tilde{x}_C) \in \mathcal{S}(\tilde{\zeta})$ and $c_I^0 \tilde{x}_I + c_C^0 \tilde{x}_C = z(\tilde{\zeta})$. Now the former condition implies $C_I^{1:\ell} \tilde{x}_I + C_C^{1:\ell} \tilde{x}_C \leq \tilde{\zeta} \lneqq \zeta$ while the latter implies that $c_I^0 \tilde{x}_I + c_C^0 \tilde{x}_C = z(\zeta)$, since $z(\tilde{\zeta}) = z(\zeta)$. The result follows.

■

The formal statement of Theorem 4.1 may obscure the intuition, so we further explain the result here and illustrate it with our previous example. Part 1 is a straightforward statement that every point on the EF is also a point on the boundary of the epigraph of the RVF. Part 2 is more technical because of the aforementioned fact that there are points on the boundary of the epigraph of the RVF that are not part of the EF. A point on the boundary of the epigraph that is not contained in the EF nevertheless corresponds to a weak NDP and can thus be associated with the one or more NDP that weakly dominate it, as indicated in the statement of the theorem.

To further illustrate, consider $(\zeta, z_{\text{LP}}(\zeta))$ on the boundary of the epigraph of z. If the function value strictly increases whenever any component of the argument ζ decreases (i.e., the directional derivative is positive in all negative directions), then this point must correspond to an NDP since this is the same as saying that improving (decreasing, since we are minimizing) the value of any one element must result in a worsening (an increase) of some other element. This is exactly the property that characterizes NDPs. For example, in Example 1, the directional derivative at $\zeta = -11$, which is in the middle of the stability region for the point $(1, 1)$ (see Figure 3) is positive in the direction $d = -1$ and thus corresponds to an NDP.

On the other hand, if there is a direction d in which the directional derivative is zero at ζ, then the point $(\zeta, z_{\text{LP}}(\zeta)$ can only be a weak NDP, since moving in the direction d from ζ corresponds to strictly improving the value of one or more of the multiple objectives that correspond to the parametric constraints, while the objective of (MILP) remains unchanged. This is the case where the point $(\zeta, z_{\mathrm{LP}}(\zeta))$ is in one of the "flat" parts of the RVF. For example, in Example 1, points in the interior of the stability region associated with $(0,1)$ (e.g., $\zeta = -9$) correspond to weakly nondominated points since they are dominated by the point $(-10, 5)$.

The conditions involving directional derivatives also have another interpretation that is possibly more intuitive. Recall from the previous section that when $\nabla_d z(\hat{\zeta})$ is finite, we have that $\nabla_d z(\hat{\zeta}) = d^\top u$ for some optimal solution to (D-RLP). This allows us to re-interpret the above conditions involving directional derivatives in terms of solutions to (D-RLP). In particular, the condition $\nabla_d z(\hat{\zeta}) > 0$ for all $d \in \mathbb{R}^l_- \setminus \{\mathbf{0}\}$ is equivalent to $u < \mathbf{0}$ for all alternative optimal solutions $(u, v) \in \mathcal{P}_D$ for (D-RLP) associated with $\hat{\zeta}$, while a zero directional derivative implies that the dual variable associated with one of the constraints is zero. This makes sense, as a zero dual value implies that the constraint can be tightened without changing the optimal solution, and this is exactly the condition that would indicate a given solution is not nondominated in the multiobjective context.

Note that when all directional derivatives in directions $d \in \mathbb{R}^l_- \setminus \{\mathbf{0}\}$ are strictly positive (the function is strictly decreasing everywhere), then the boundary of the epigraph of the RVF and the EF exactly coincide. This is the case when there are no "flat" parts. This case is illustrated in Example 6 below.

Example 6. *We consider the following instance of* (RVF)*:*

$$
\begin{aligned}
z(\zeta) = \min\ & x_1 + \frac{1}{4}x_2 + \frac{1}{2}y_1 - \frac{3}{4}y_2 \\
& \frac{4}{5}x_1 + \frac{1}{2}x_2 + \frac{1}{3}y_1 + 0y_2 \leq \zeta \\
& \frac{3}{5}x_1 + \frac{1}{3}x_2 + \frac{1}{4}y_1 - \frac{1}{5}y_2 = 4 \\
& x_i \in \mathbb{Z}_+, \quad \forall i \in \{1, 2\} \\
& y_j \in \mathbb{R}_+, \quad \forall j \in \{1, 2\}.
\end{aligned}
$$

Figure 5 illustrates that the EF of the multiobjective optimization problem associated with the RVF in Example 6 exactly coincides with the boundary of the epigraph of the associated RVF.

It is worth noting that it is possible to avoid the difficulty of the additional "flat pieces" of the RVF that aren't part of the EF by changing from "$\leq$" to equality for the constraints associated with the objectives of the multiobjective version of the problem. However, in that case, a different difficulty is introduced—there may then be parts of the RVF that are *increasing* (strictly positive directional derivative in the direction $d \in \mathbb{R}^l_+$), and we then have that the boundary of the epigraph for *those* parts of the RVF is not part of the frontier. This approach does not make the statement of the theorem any cleaner. In addition, we establish the equivalence of Theorem 4.1 in Appendix C, noting that the only deviation from the original statement is replacing $\leq$ with $=$ for the parametric constraints.

Before closing this section, we use results of Theorem 4.1 to connect the representation of the RVF, from Theorem 3.8, and the EF of the (MO-MILP). Specifically, we will see that not only does a minimal subset $\mathcal{S}_{\min}$ give us a representation of the value function but also a representation of

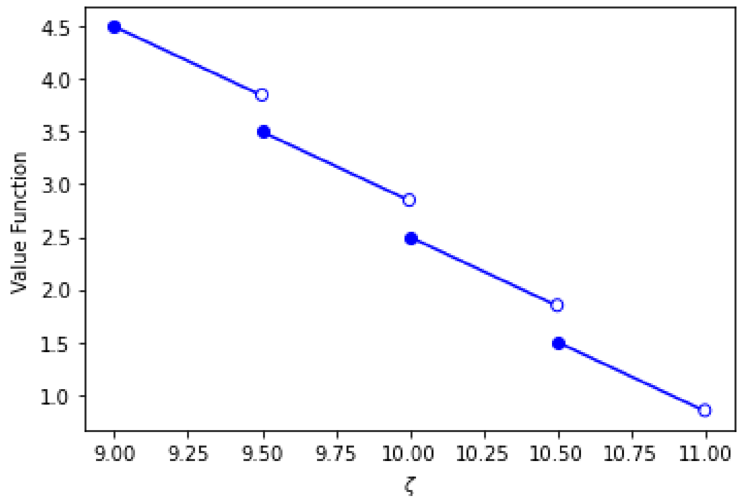


Figure 5: EF and RVF for the MILP in Example 6

the EF. In doing so, we provide some insight into the role of the c^0x objective, which is distinguished in the restricted value function problem but is indistinguishable from other objectives in the (MO-MILP).

First, we define the notation $\mathcal{F} = \{Cx \ : \ x \in X_{\text{MO}} \text{ is an efficient solution}\}$ to be the EF, and suppose that the (MO-MILP) has been solved in some way, and we wish to determine the RVF. If the solution to the (MO-MILP) provides $\mathcal{F}$, then the first part of Theorem 4.1 says

$$z(\zeta) = \min_{\gamma \in \mathcal{F}} \left\{ \gamma_0 \ : \ \gamma^{1:l} \le \zeta \right\}, \qquad \forall \zeta \in \mathcal{C}.$$

Alternatively, the (MO-MILP) may be solved by providing some set of integer parts of efficient solutions, denoted as $\mathcal{S}_I^E$, which is a subset of $\mathcal{S}_I$, i.e., $\mathcal{S}_I^E \subseteq \mathcal{S}_I$ so that for all $\gamma \in \mathcal{F}$ there must exist an efficient solution $(x_I, x_C) \in X_{\text{MO}}$ with $x_I \in \mathcal{S}_I^E$ and $\gamma = C_I x_I + C_C x_C$. Then, by Theorem 3.8, we have $\mathcal{S}_{\min} \subseteq \mathcal{S}_I^E$.

Now we take the opposite perspective: suppose that we can determine a RVF (by finding $\mathcal{S}_{\min}$ as per Theorem 3.8) and wish to determine a solution to a given (MO-MILP) that has p objectives, encoded as rows $1, \ldots, p$ in the matrix $D \in \mathbb{R}^{p \times n}$. It is well known in multiobjective optimization (and straightforward to prove) that adding an objective of the form $\lambda^\top D$ to the (MO-MILP) does not change the set of efficient solutions, provided $\lambda \ge 0$, and nor does removing any duplicate objective. We may thus consider any restricted value problem with (i) $c^0 = D^{k:k}$, the kth objective, and $C^{1:l}$ consisting of D with the kth row deleted, so $l = p-1$, for any $k = 1, \ldots, p$, or (ii) $c^0 = \lambda^\top D$ and $C^{1:l} = D$, so $l = p$, for any $\lambda \ge 0$. In any case of (i) or (ii), the set of efficient solutions of $\operatorname{vinf}\{Dx \ : \ x \in X_{\text{MO}}\}$ is precisely the set of efficient solutions of $\operatorname{vinf}\{Cx \ : \ x \in X_{\text{MO}}\}$. We may thus determine the EF of the former by finding $\mathcal{S}_{\min}$ for the RVF associated with the latter. We are assured, by the first part of Theorem 4.1, that if $\gamma \in \mathcal{F}$ where (x'_I, x'_C) is an efficient solution with $\gamma = C_I x'_I + C_C x'_C$, then there exists $\zeta \in \mathcal{C}$ with $\zeta = \gamma^{1:l}$ and $z(\zeta) = \gamma_0$. Now Theorem 3.8 ensures that for such ζ, there must exist $(x_I, x_C) \in X_{\text{MO}}$ with $x_I \in \mathcal{S}_{\min}$ so that $\gamma = C_I x_I + C_C x_C$. Thus integer parts in $\mathcal{S}_{\min}$ are sufficient to describe the EF, $\mathcal{F}$.

5 Finite Algorithm for Construction

In this section, we present an algorithm for constructing a discrete representation of both the RVF and the EF, as introduced in the previous section. We start with a discussion of the general case before discussing the pure integer case, which has particular properties.

5.1 General (Mixed Integer) Case

The purpose of the proposed algorithm is to construct a subset of $\mathcal{S}_I$, the elements of which are sufficient to fully describe the RVF. The proposed algorithm is a generalized cutting-plane method that iteratively improves an upper approximation of the RVF until it converges to the true function. The "cuts" in this context refer to the convex bounding functions described in (RLPVF). At iteration k, The upper approximation is the function $\bar{z}^k$, given by

$$\bar{z}^k(\zeta) = \min\left\{ \min_{x_I \in S^k} \bar{z}(\zeta;\ x_I),\ U \right\}, \quad \forall \zeta \in \mathcal{C}, \tag{13}$$

where $\mathcal{S}^k$ is the set of points in $\mathcal{S}_I$ identified so far and U is an upper bound on the value function, i.e. $U \geq z(\zeta)$ for all $\zeta \in \mathcal{C}$. The selection of the U will be addressed in Section 5.1.3.

The algorithm proceeds by identifying in iteration k the value $\zeta^* \in \mathcal{C}$ that maximizes the difference between the approximation $\bar{z}^k(\zeta^*)$ and the true value function $z(\zeta^*)$. This value is given by

$$\zeta^* = C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1},$$

where

$$(x_I^{k+1}, x_C^{k+1}) \in \operatorname*{arg\,max}_{(x_I, x_C) \in X_{\mathrm{MO}}} \left(\bar{z}^k(C_I^{1:l} x_I + C_C^{1:l} x_C) - (c_I^0 x_I + c_C^0 x_C) \right). \tag{14}$$

Here $\bar{z}^k(C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}) = \bar{z}^k(\zeta^*)$ is the value of the upper approximation at ζ^* and—as we shall argue later—the true value function $z(\zeta^*) = c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}$. Their difference is recorded in the algorithm as θ^{k+1}, so

$$\begin{aligned} \theta^{k+1} &= \bar{z}^k(C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}) - (c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}) \\ &= \max_{(x_I, x_C) \in X_{\mathrm{MO}}} \left(\bar{z}^k(C_I^{1:l} x_I + C_C^{1:l} x_C) - (c_I^0 x_I + c_C^0 x_C) \right). \end{aligned}$$

Now (x_I^{k+1}, x_C^{k+1}) found via (14) may be only *weakly* efficient, whereas only (strongly) efficient solutions are required to describe the RVF. Thus, before adding the integer part of (x_I^{k+1}, x_C^{k+1}) to $\mathcal{S}^k$, we first "convert it" to a (strong) efficient solution via a process common in multiobjective optimization: we replace (x_I^{k+1}, x_C^{k+1}) by an element of

$$\operatorname*{arg\,min}_{(x_I, x_C) \in X_{\mathrm{MO}}} \left\{ \mathbf{1}^\top (C_I x_I + C_C x_C) \ :\ C_I x_I + C_C x_C \leq C_I x_I^{k+1} + C_C x_C^{k+1} \right\}. \tag{15}$$

We prove later that any optimal solution to (15) must also solve (14); in the case of multiple optimal solutions for (14), solving (15) ensures an efficient solution for the multiobjective problem is selected. Provided that the maximum difference between the upper approximation to the value function and the true value function is strictly positive, indicated by $\theta^{k+1} > 0$, this yields a new stability region associated with a new member of $\mathcal{S}_I$, which we add to obtain $\mathcal{S}^{k+1}$. The algorithm is designed to iterate until the approximation is exact, detected by finding $\theta^k = 0$. A high-level overview of the algorithm is provided below.

RVF Algorithm : Algorithm for constructing the RVF and the associated EF

Input: X_{MO}, $C \in \mathbb{Q}^{(l+1)\times n}$, U an upper bound on $z(\zeta)$ over $\zeta \in \mathcal{C}$.
Output: $\mathcal{S}^k$ such that $z(\zeta) = \bar{z}^k(\zeta) = \min_{x_I \in \mathcal{S}^k} \bar{z}(\zeta;\ x_I), \quad \forall \zeta \in \mathcal{C}$.

1 Initialize $\bar{z}^0(\zeta) = U$ for all $\zeta \in \mathbb{R}^l$, $k = 0$, $\mathcal{S}^0 = \emptyset$, $\theta^0 = +\infty$.
while $\theta^k > 0$ **do**
2 Determine $(x_I^{k+1}, x_C^{k+1}) \in \operatorname*{arg\,max}_{(x_I,x_C)\in X_{\text{MO}}} \left(\bar{z}^k(C_I^{1:l}x_I + C_C^{1:l}x_C) - (c_I^0 x_I + c_C^0 x_C)\right)$.
Convert (x_I^{k+1}, x_C^{k+1}) to an efficient solution (for example, by using optimization problem (15)).
Set $\mathcal{S}^{k+1} \leftarrow \mathcal{S}^k \cup \{x_I^{k+1}\}$.
Set $\theta^{k+1} = \bar{z}^k(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}) - (c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1})$.
$\bar{z}^{k+1}(\zeta) = \min\left\{\bar{z}^k(\zeta),\ \bar{z}(\zeta;\ x_I^{k+1})\right\}$ for all $\zeta \in \mathcal{C}$.
$k \leftarrow k + 1$.
3 **end**

A few important details regarding the RVF Algorithm are worth noting. The RVF Algorithm generates a sequence of points in the multiobjective space X_{MO}, but only stores the integer parts of these points in the set $\mathcal{S}^k$, as this is sufficient to describe the RVF. The upper approximation to the RVF at each iteration, denoted by $\bar{z}^k$, is initialized and then updated inductively at each iteration k according to (13). It is important to note that the upper approximation $\bar{z}^k$ is not explicitly constructed: the expression (13) is a *description* of the upper approximating function; it is the set $\mathcal{S}^k$ that is constructed by the algorithm. Thus, in the first step of each iteration, we need a method for optimization of the upper approximating function. We defer the explanation of this to Section 5.1.2 below. Here we first illustrate the steps of the algorithm on Example 1 and then establish its correctness.

Example 7. ***Illustrative Example for the RVF Algorithm:*** *The steps of the RVF Algorithm as applied to Example 1 are depicted graphically in Figure 6 below. In the first iteration, the RVF Algorithm identifies the point* $(\zeta, z(\zeta)) = (4\frac{4}{9}, 0)$*, with stability region corresponding to* $(x_1, x_2) = (0, 0)$*, and updates the upper approximation to the blue convex function. In the subsequent iteration, the algorithm searches for the point with the largest difference between the current upper approximation and the RVF (the red piecewise linear function). The point of largest difference actually occurs at the far left, outside of the region shown in the figure, at* $\zeta = -55.5$ *(refer to Appendix A to view the full extent of the LP EFs). This point lies within the stability region corresponding to* $(x_1, x_2) = (1, 0)$*. The RVF Algorithm then updates the upper approximation again and, in the subsequent iteration, finds the point with the most difference between the upper approximation and the RVF, which occurs at* $\zeta = -12\frac{1}{6}$*. The RVF Algorithm continues in this manner, next finding* $\zeta = -10$*, until, at the next iteration, there is no such point. Thus the RVF Algorithm terminates in iteration 4 with* $\theta = 0$ *when the upper approximation and the VF are the same.*

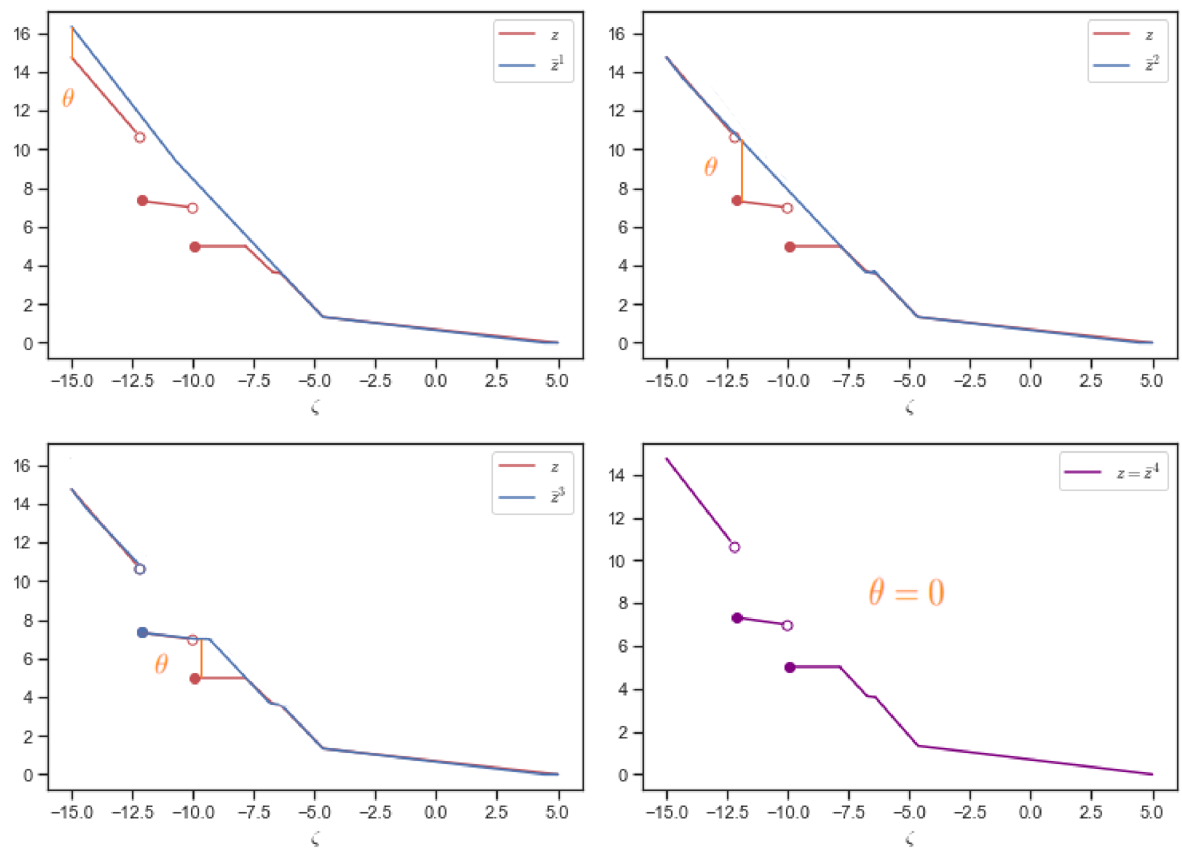


Figure 6: RVF and upper approximation in iterations of the RVF Algorithm with Example 1

5.1.1 Correctness of the RVF Algorithm

The proof of the correctness of the RVF Algorithm relies on the fact that $\bar{z}^k$ is a nonincreasing function for the same reason that the RVF is nonincreasing, namely that it is the minimum of a finite set of nonincreasing bounding functions by (13). The proof of correctness also uses the following simple observation.

Lemma 5.1. *For all* $(\hat{x}_I, \hat{x}_C) \in X_{\text{MO}}$*, we have that*

$$\bar{z}(C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C;\ \hat{x}_I) \leq c_I^0\hat{x}_I + c_C^0\hat{x}_C.$$

Proof. The bounding function (CR) associated with $\hat{x}_I$ is equivalently expressed as

$$\bar{z}(\zeta;\ \hat{x}_I) = \min\left\{c_I^0 x_I + c_C^0 x_C\ :\ (x_I, x_C) \in X_{\text{MO}},\ x_I = \hat{x}_I,\ C_I^{1:l}x_I + C_C^{1:l}x_C \leq \zeta\right\}. \tag{16}$$

The result follows as $(\hat{x}_I, \hat{x}_C)$ is feasible for the problem above for $\zeta = C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C$. ■

The following lemma helps to establish that the integer parts added to $\mathcal{S}^k$ by the algorithm eventually suffice to describe the whole value function.

Lemma 5.2. *At iteration k of the algorithm,* $\theta^{k+1} = \max_{\zeta\in\mathcal{C}} \bar{z}^k(\zeta) - z(\zeta)$.

Proof. Let $\zeta^* = \arg\max_{\zeta\in\mathcal{C}}(\bar{z}^k(\zeta) - z(\zeta))$ and denote $\zeta^{k+1} = C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}$. Since $\zeta^* \in \mathcal{C}$ there must exist $(\hat{x}_I, \hat{x}_C) \in X_{\text{MO}}$ with $\hat{\zeta} := C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C \leq \zeta^*$ and $z(\zeta^*) = c_I^0\hat{x}_I + c_C^0\hat{x}_C = z(\hat{\zeta})$. Further, since $\bar{z}^k$ is nonincreasing, $\bar{z}^k(\hat{\zeta}) \geq \bar{z}^k(\zeta^*)$, and hence

$$\begin{aligned}
\bar{z}^k(\hat{\zeta}) - z(\hat{\zeta}) &\geq \bar{z}^k(\zeta^*) - z(\hat{\zeta}) \\
&= \bar{z}^k(\zeta^*) - z(\zeta^*) \\
&\geq \bar{z}^k(\zeta^{k+1}) - z(\zeta^{k+1})
\end{aligned}$$

The last of these inequalities follows from the fact that $\zeta^{k+1} \in \mathcal{C}$. Since

$$(x_I^{k+1}, x_C^{k+1}) \in \underset{(x_I,x_C)\in X_{\text{MO}}}{\arg\max} \left(\bar{z}^k(C_I^{1:l}x_I + C_C^{1:l}x_C) - (c_I^0 x_I + c_C^0 x_C)\right),$$

it follows that

$$\bar{z}^k(\zeta^{k+1}) - z(\zeta^{k+1}) \geq \bar{z}^k(\hat{\zeta}) - z(\hat{\zeta}) \Rightarrow \bar{z}^k(\zeta^{k+1}) - z(\zeta^{k+1}) = \bar{z}^k(\hat{\zeta}) - z(\hat{\zeta}).$$

Finally, by the definition of θ^{k+1}, we have

$$\theta^{k+1} = \bar{z}^k(\zeta^{k+1}) - z(\zeta^{k+1}) = \bar{z}^k(\zeta^*) - z(\zeta^*)$$

and the result follows. ■

The crucial property for establishing termination of the algorithm is that as long as $\bar{z}^k \neq x$, then in iteration k, we are guaranteed to produce a point not already contained in $\mathcal{S}^k$. This is established in the lemma below, which states that unless the optimal value of the optimization problem in (14) is zero, the integer part of the solution obtained is not contained in $\mathcal{S}^k$.

Lemma 5.3. *Any optimal solution* (x_I^{k+1}, x_C^{k+1}) *to the optimization problem in* (14) *having optimal value* $\theta^{k+1} > 0$ *has the property that* $x_I^{k+1} \notin \mathcal{S}^k$*.*

Proof. We show the contrapositive of the lemma as follows. Let (x_I^{k+1}, x_C^{k+1}) solve the optimization problem in (14) having optimal value θ^{k+1} and suppose that $x_I^{k+1} \in \mathcal{S}^k$. Then

$$\bar{z}^k(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}) \leq \bar{z}(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1};\ x_I^{k+1}) \leq c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1},$$

where the first inequality follows from (13), since $x_I^{k+1} \in \mathcal{S}^k$, and the second inequality follows from Lemma 5.1, since $(x_I^{k+1}, x_C^{k+1}) \in X_{\text{MO}}$. It thus follows, by the definition of θ^{k+1} and x^{k+1}, that

$$\theta^{k+1} = \bar{z}^k(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}) - (c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}) \leq 0,$$

as required to obtain the contrapositive. ∎

Now, with Theorem 5.4, we show the correctness of the RVF Algorithm by proving that it terminates finitely and returns the correct value function.

Theorem 5.4. *(Correctness of the RVF Algorithm) At termination, we have that* $z(\zeta) = \bar{z}^k(\zeta)$ *for all* $\zeta \in \mathcal{C}$*, so that* $\mathcal{S}^k$ *describes both the value function and the EF. Furthermore, the RVF Algorithm terminates in finitely many iterations under the assumption that* X_{MO} *is bounded.*

Proof. When X_{MO} is bounded, $\mathcal{S}_I$ is finite. Therefore, by Lemma 5.3, which states that each iteration of the algorithm produces a new element of $\mathcal{S}_I$, the number of iterations must be finite. To show that $z = \bar{z}^k$ at termination, assume not for the sake of contradiction. Then there must exist $\zeta \in \mathbb{R}^l$ such that $z(\zeta) < \bar{z}^k(\zeta)$. But, by Lemma 5.2, this is a contradiction to the assumption that $\theta^k = 0$ at termination. This completes the proof. ∎

Proposition G.3 in Appendix G shows that the algorithm has one seemingly advantageous property that we briefly describe here. As the description is being constructed, each newly added integer part is guaranteed to be associated with a nonempty stability region. In fact, it is not difficult to see that the added region must be *non-redundant* at the time it is added. In other words, we must have that $\bar{z}^k \neq \bar{z}^{k+1}$ at iteration k. Equivalently, there exists $\zeta \in \mathcal{C}$ such that $\bar{z}^{k+1}(\zeta) < \bar{z}^k(\zeta)$, namely ζ^{k+1} from the proof of Lemma 5.2. Unfortunately, this property does not translate into a guarantee that the stability region associated with x^k is non-redundant in the end, so the algorithm is not guaranteed to produce a minimal description. It would be possible to postprocess the description to make it minimal if this was important for a particular application.

5.1.2 Solving the Subproblem

The RVF Algorithm was presented in the previous section at a high level of abstraction to simplify the exposition. In the next two sections, we provide further details on how the algorithm can actually be implemented in practice. In this section, we start by clarifying how the subproblem that needs to be solved in each iteration can be solved in practice by formulating it as a standard mathematical optimization problem.

The optimization problem in (14), solved in Step 2 of the algorithm, can be formulated as a mixed integer nonlinear optimization problem (MINLP) as follows. First, we model it using an auxiliary variable θ to move $\bar{z}^k$ into the constraints:

$$\begin{array}{ll} \theta^{k+1} = \max & \theta \\ \text{subject to} & \theta \le \bar{z}^k(C_I^{1:l} x_I + C_C^{1:l} x_C) - (c_I^0 x_I + c_C^0 x_C) \\ & (x_I, x_C) \in X_{\mathrm{MO}} \\ & \theta \in \mathbb{R}. \end{array} \tag{17}$$

Next, we use (13) and (CR) to expand (17) to the $k+1$ constraints

$$\theta + c_I^0 x_I + c_C^0 x_C \le U, \tag{18}$$

and

$$\theta + c_I^0 x_I + c_C^0 x_C \le c_I^0 x_I^i + z_{\mathrm{LP}}(C_I^{1:l} x_I + C_C^{1:l} x_C - C_I^{1:l} x_I^i;\ b - A_I x_I^i), \quad i = 1, \ldots, k. \tag{19}$$

(Recall $\mathcal{S}^k = \{x_I^1, \ldots, x_I^k\}$.) We can model the term involving z_{LP} for each i by using its LP dual problem, which via (D-RLP) is given by

$$\max_{(u^i, v^i) \in \mathcal{P}_D} (C_I^{1:l} x_I + C_C^{1:l} x_C - C_I^{1:l} x_I^i)^\top u^i + (b - A_I x_I^i)^\top v^i. \tag{20}$$

If $(x_I, x_C) \in X_{\mathrm{MO}}$ satisfies (19) for some i, then by strong LP duality, there must exist $(u^i, v^i) \in \mathcal{P}_D$ (optimal for the dual of the LP after fixing the values of the integer variables to x_I^i) such that

$$\begin{array}{rcl} \theta + c_I^0 x_I + c_C^0 x_C & \le & c_I^0 x_I^i + z_{\mathrm{LP}}(C_I^{1:l} x_I + C_C^{1:l} x_C - C_I^{1:l} x_I^i;\ b - A_I x_I^i) \\ & = & c_I^0 x_I^i + (C_I^{1:l} x_I + C_C^{1:l} x_C - C_I^{1:l} x_I^i)^\top u^i + (b - A_I x_I^i)^\top v^i. \end{array}$$

Conversely, if there exists some $(u^i, v^i) \in \mathcal{P}_D$ such that $(x_I, x_C) \in X_{\mathrm{MO}}$ satisfies

$$\theta + c_I^0 x_I + c_C^0 x_C \le c_I^0 x_I^i + (C_I^{1:l} x_I + C_C^{1:l} x_C - C_I^{1:l} x_I^i)^\top u^i + (b - A_I x_I^i)^\top v^i,$$

then by weak duality, it must be that (x_I, x_C) satisfies (19) for this i. This shows that the following explicit MINLP is a valid formulation of the optimization problem in (14):

$$\theta^{k+1} = \max \quad \theta \tag{21}$$

$$\text{subject to} \quad \theta + c_I^0 x_I + c_C^0 x_C \le c_I^0 x_I^i + (C_I^{1:l} x_I + C_C^{1:l} x_C - C_I^{1:l} x_I^i)^\top u^i + (b - A_I x_I^i)^\top v^i, \quad i = 1, \ldots, k \tag{22}$$

$$\theta + c_I^0 x_I + c_C^0 x_C \le U \tag{23}$$

$$(u^i, v^i) \in \mathcal{P}_D, \quad i = 1, \ldots, k \tag{24}$$

$$(x_I, x_C) \in X_{\mathrm{MO}} \tag{25}$$

$$\theta \in \mathbb{R}. \tag{26}$$

Note that the constraint (22) makes the problem nonlinear: it has bilinear terms $(C_I^{1:l} x_I)^\top u^i$ and $(C_C^{1:l} x_C)^\top u^i$ linking the u^i and (x_I, x_C) variables for each i. However, the problem (21)–(26), which we refer to as the *MINLP subproblem*, can be solved either with an off-the-shelf nonconvex quadratic solver or possibly with a customized algorithm.

5.1.3 Initialization and Termination

The most important aspect of algorithm initialization is determining an initial upper bound U. This can be done using any appropriate method. One possible approach is to solve the LP relaxation of the MILP $\max\{c_I^0 x_I + c_C^0 x_C \ : \ (x_I, x_C) \in X_{\text{MO}}\}$. This would provide a valid initial upper bound for the optimization problem.

At the termination of the algorithm, it is possible that the final set $\mathcal{S}^k$ is not minimal. This can occur in the presence of continuous variables when there are multiple efficient solutions that yield the same NDP. This scenario is not possible in the pure integer case, as shown in Lemma 5.5. For example, consider an instance with $l = 2$ in which there are three integer parts, denoted x_I^1, x_I^2, and x_I^3, that have nonempty stability regions. Suppose the part of the epigraph agreeing with the bounding function associated with each of these is given by (1) the line segment joining $(\zeta, z(\zeta)) = (2, 2)$ to $(4, 0)$, (2) the line segment joining $(1, 3)$ to $(3, 1)$ and (3) the line segment joining $(0, 4)$ to $(2, 2)$, respectively. Then the RVF Algorithm will generate each of these solutions in order, adding x_I^{k+1} to $\mathcal{S}^k$ for each $k = 0, 1, 2$, so that at the completion of the algorithm, $\mathcal{S}^k = \{x_I^1, x_I^2, x_I^3\}$. However, the minimal set $\mathcal{S}_{\min}$ in this instance is $\{x_I^1, x_I^3\}$. Therefore, it may be necessary to perform a postprocessing step to remove elements from the final $\mathcal{S}^k$ until a minimal set is obtained.

It is important to note that in the final iteration, when θ^{k+1} takes value 0, there is no need to convert the solution (x_I^{k+1}, x_C^{k+1}) into an NDP and add it to $\mathcal{S}^k$ *unless* its objective value $c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}$ happens to coincide with the initial upper bound, U.

Finally, observe that if the RVF Algorithm is terminated early, while $\theta^k > 0$, then the value of the final optimization problem solved in Step 2 provides a natural performance guarantee on the quality of the current upper approximation to the value function: it measures the maximum error between the true value function, z, and the upper approximation, $\bar{z}^k$, over any point in its domain.

5.2 Pure Integer Case

The pure integer case, in which the problem has no continuous variables, is useful for illustrating connections to existing algorithms and providing insight into algorithm mechanisms. Here, we focus on the latter, leaving the former to Section 5.3.

In the pure integer case, the algorithm can be simplified substantially, as there is no continuous part. In this case, the primal LP parameterized by ζ and formed by fixing the integer variables to $\hat{x}_I$, for which $\mathcal{P}_D$ forms the dual LP feasible set, has no variables but can be thought of as the problem

$$z_{\text{LP}}(\zeta;\ \hat{x}_I) = \min \left\{ 0 \ : \ 0 \leq \zeta - C_I^{1:l} \hat{x}_I \right\}.$$

From strong duality, we have that

$$\begin{aligned} z_{\text{LP}}(\zeta;\ \hat{x}_I) &:= \max \left\{ (\zeta - C_I^{1:l} \hat{x}_I)^\top u \ : \ u \in \mathbb{R}^l_- \right\} \\ &= \begin{cases} 0, & \text{if } \zeta \geq C_I^{1:l} \hat{x}_I, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{27}$$

We may now rewrite problem (21)–(26) for the pure integer case explicitly as follows.

$$
\begin{aligned}
\theta^{k+1} = \max \quad & \theta && && (28)\\
\text{subject to} \quad & \theta + c_I^0 x_I \le c_I^0 x_I^i + (\zeta - C_I^{1:l} x_I^i)^\top u^i, && i = 1, \dots, k && (29)\\
& \zeta = C_I^{1:l} x_I && && (30)\\
& \theta + c_I^0 x_I \le U && && (31)\\
& u^i \in \mathbb{R}^l_-, && i = 1, \dots, k && (32)\\
& A_I x_I = b && && (33)\\
& x_I \in \mathbb{Z}^r_+ && && (34)\\
& \theta \in \mathbb{R}. && && (35)
\end{aligned}
$$

After making use of (27) to eliminate the u^i variables and (30) to eliminate ζ, we have

$$
\begin{aligned}
\theta^{k+1} = \max \quad & \theta && (36)\\
\text{subject to} \quad & \theta + c_I^0 x_I \le c_I^0 x_I^i, \quad i = 1, \dots, k, \text{ s.t. } C_I^{1:l} x_I \ge C_I^{1:l} x_I^i && (37)\\
& (31),\ (33),\ (34) \text{ and } (35). && (38)
\end{aligned}
$$

Note that the above formulation includes conditional constraints (37), which cannot be directly solved by a typical MILP or MINLP solver. We include it to facilitate proofs of algorithm properties in what follows.

It can be observed that the RVF Algorithm generates all NDPs of the EF in nondecreasing order of their $c_I^0 x_I$ value. We formalize this in Theorem 5.7. We give a linearization of the problem to be solved in each iteration of the pure integer case, (36)–(38), in Appendix D.

The following result states that unless the optimal value of (28)–(35) is zero, the objective space image of its solution is new, meaning that it cannot already be in the objective space image of points in $\mathcal{S}^k$. Here we define the set of objective space images of points in $\mathcal{S}^k$ by

$$F(\mathcal{S}^k) = \left\{ C_I x_I^i \ : \ i = 1, \dots, k \right\}.$$

Lemma 5.5. *Any optimal solution (x_I, θ) to (28)–(35) (equivalently (36)–(38)) with $\theta > 0$ has the property that $C_I x_I \notin F(\mathcal{S}^k)$.*

Proof. If $C_I x_I \in F(\mathcal{S}^k)$ then there exists $i \in \{1, \dots, k\}$ such that $C_I x_I = C_I x_I^i$. This means that $C_I^{1:l} x_I = C_I^{1:l} x_I^i$, so constraint (37) applies for this i, and $c_I^0 x_I = c_I^0 x_I^i$. But constraint (37) says $\theta + c_I^0 x_I \le c_I^0 x_I^i$ so with $c_I^0 x_I^i = c_I^0 x_I$ it must be that $\theta \le 0$. The results follow by contradiction. ■

We now consider the set of all NDPs of the (pure integer) multiobjective problem as

$$\operatorname{vinf} \left\{ C_I x_I : A_I x_I = b,\ x_I \in \mathbb{Z}^n_+ \right\}.$$

Let $\mathcal{S}^E$ denote the set of all efficient solutions for the problem. Recall that in the RVF Algorithm, after solving the problem (28)–(35) to yield solution $\hat{x}_I$, an auxiliary problem is solved to "convert it" to a strongly efficient solution, x_I^k, which, by Lemma G.2, also solves the problem (28)–(35), to add to $\mathcal{S}^k$ to create $\mathcal{S}^{k+1}$. Thus we may safely assume $\mathcal{S}^k \subseteq \mathcal{S}^E$ at any iteration k.

Proposition 5.6. *Suppose in the RVF Algorithm at iteration k, the elements of $\mathcal{S}^k$ are efficient solutions with the property that*

$$\min\left\{c_I^0 y \ : \ y \in \mathcal{S}^E,\ C_I y \notin F(\mathcal{S}^k\Big)\right\} \geq \max\left\{c_I^0 y \ : \ y \in \mathcal{S}^k\right\}, \tag{39}$$

in other words, the points in $\mathcal{S}^k$ induce NDPs, which have the least value of the first objective. Then optimal solution (x_I, θ) to (28)–(35) having $x_I \in \mathcal{S}^E$ and $\theta > 0$ satisfies $c_I^0 x_I = \min\{c_I^0 y \ : \ y \in \mathcal{S}^E,\ C_I y \notin F(\mathcal{S}^k)\}$, so giving an NDP with next-smallest first objective value.

Proof. Optimal solution (x_I, θ) to (28)–(35) having $x_I \in \mathcal{S}^E$ and $\theta > 0$ implies that $C_I x_I \notin F(\mathcal{S}^k)$ by Lemma 5.5. Now suppose that for some $i \in \{1, \ldots, k\}$, we have that $C_I^{1:l} x_I \geq C_I^{1:l} x_I^i$. Then since $x_I \in \mathcal{S}^E$, it must be that $c_I^0 x_I < c_I^0 x_I^i$, since otherwise $C_I x_I^i$ would dominate $C_I x_I$, contradicting $x_I \in \mathcal{S}^E$. But $c_I^0 x_I < c_I^0 x_I^i$ contradicts equation (39). Thus it must be that $C_I^{1:l} x_I \ngeq C_I^{1:l} x_I^i$ for all $i = 1, \ldots, k$. Hence (x_I, θ) must solve

$$\begin{array}{ll} \max & \theta' \\ \text{subject to} & \theta' + c_I^0 x_I' \leq U \\ & x_I' \in \mathcal{S}^E \\ & C_I x_I' \notin F(\mathcal{S}^k) \\ & \theta' \in \mathbb{R}. \end{array}$$

Now if any $x_I' \in \mathcal{S}^E$ with $C_I x_I' \notin F(\mathcal{S}^k)$ has $c_I^0 x_I' < c_I^0 x_I$, then (x_I', θ') is feasible for this problem, where

$$\theta' := U - c_I^0 x_I' > U - c_I^0 x_I \geq \theta,$$

which contradicts optimality of (x_I, θ). Thus it must be that

$$c_I^0 x_I = \min\left\{c_I^0 y \ : \ y \in \mathcal{S}^E,\ C_I y \notin F(\mathcal{S}^k)\right\},$$

as required. ■

Theorem 5.7. *In the pure integer case, the RVF Algorithm generates NDPs in the nondecreasing order of the first objective.*

Proof. We proceed by induction on k. In the case $k = 0$, the set $\mathcal{S}^k = \emptyset$, so any solution of (28)–(35) for $k = 0$ also solves $\min\{c_I^0 x_I \ : \ A_I x_I = b,\ x_I \in \mathbb{Z}_+^n\} = \min\{c_I^0 x_I \ : \ x_I \in \mathcal{S}^E\}$. The next iterate $x_I^1 \in \mathcal{S}^E$ is a solution of the latter and $\mathcal{S}^1 = \{x_I^1\}$. Thus the base step for the induction, with $k = 1$, is proved. The inductive step is proved by Proposition 5.6. ■

5.3 Comparison to Other Algorithms

While many algorithms exist for solving multiobjective MILPs, and some for finding the VF of an MILP (see Section 2), the RVF Algorithm has some distinctive features. First, it is very general, applying across both pure and mixed integer cases and any number of objective functions. This is in contrast to multiobjective algorithms, which typically apply in either the pure or mixed cases, but not both, and which often apply only to a specific number of objectives. Second, the RVF

Algorithm fills a gap in the multiobjective MILP field in its applicability to problems that include continuous variables and any number of objectives. This area of multiobjective MILP is still in its infancy, and the RVF Algorithm constitutes a significant advance in it. Thirdly, the RVF Algorithm solves two optimization problems per stability region: one MINLP, which identifies a new stability region, and one "conversion" MILP, which converts a weakly nondominated solution to an NDP and is usually easily solved in practice. The size of the MINLP, in terms of its variables and constraints, grows linearly in the number of stability regions found: $l + m$ variables and $n - r + 1$ extra constraints (one including l bilinear terms and the rest linear) are added per new stability region. This is comparable to existing multiobjective algorithms in the pure integer case and offers advantages in the mixed case, even for small numbers of objectives. We provide more details for each of these cases below in comparison with well-known multiobjective algorithms.

5.3.1 Pure Integer Case

In the pure integer case, the stability regions are precisely the NDPs, and any minimal set of efficient solutions, denoted as $\mathcal{S}_{\min}$, consists of one efficient solution per NDP. Therefore, for an instance with N NDPs, the RVF Algorithm will solve N MINLPs and N conversion integer linear programmings (ILPs), with the size of the MINLPs increasing by l variables and only one constraint per NDP found, where the constraint added includes l bilinear terms. It is worth noting that none of these subproblems (MINLP or ILP) will be infeasible problems. In multiobjective algorithms, it is usually considered desirable to avoid infeasible ILPs, since these are often observed to be harder to solve in practice.

The RVF Algorithm performs comparably to earlier algorithms for multiobjective (pure integer) ILP with any number of objectives, $p = l + 1$. For example, in theoretical analysis of their algorithm performance, following on from Lemma 4.4 of [Özlen and Azizoğlu, 2009] suggest their algorithm solves of the order of N choose $p - 1$ ILPs. Similarly, Theorem 4 of [Kirlik and Sayın, 2014] suggests that their algorithm solves at most N^{p-1} "two-stage" ILPS, meaning an ILP minimizing one of the objectives over the ILP feasible set followed by a conversion ILP. The method of [Sylva and Crema, 2004] solves one ILP per NDP, but the size of the ILP increases by p binary variables, and $p + 1$ constraints per NDP found, which model the disjunctive requirement that ILP feasible points are not dominated by NDPs already found. The big-M values required in the model present challenges in addition to its size. Improvements were made in [Lokman and Köksalan, 2013], who suggest a variant limiting the ILP size to have at most $N(p - 1)$ variables and Np constraints (reducing the objective dimension by one), and who propose using the disjunctions to define a search tree. Although some practical improvements are suggested, the number of ILPs solved in the search procedure is given in [Lokman and Köksalan, 2013] to be of the order of N^{p-1} in the worst case.

The RVF Algorithm can be related to the variant of the method proposed by [Sylva and Crema, 2004] and developed by [Lokman and Köksalan, 2013] in the pure integer case by modeling its MINLP subproblem as an MILP. In [Lokman and Köksalan, 2013] variant, NDPs are found in non-decreasing order of one objective, more specifically the $c^0 x$ objective, and the size of the subproblem grows in a similar manner to the RVF Algorithm, as we showed in Theorem 5.7.

More recent algorithms for the pure integer case, such as the Disjunctive Constraints Method (DCM) proposed by [Boland et al., 2017] (given as Algorithm 1 in the paper), have shown good performance in practice, but their theoretical analysis in the paper provides a worst-case number of ILPs to be solved that is exponential in N. Furthermore, the ILPs increase in size through the need to model more disjunctions, and ILPs solved that do not yield a new NDP are infeasible. Hence the worst-case analysis suggests that the number of infeasible ILPs solved could be exponential in N. The authors of [Boland et al., 2017] observe in computational experiments that, in practice, a small number of ILPs is solved per NDP and conjecture that the number is at most $pN + 1$, with $(p - 1)N + 1$ of these being infeasible ILP. In fact, a closer examination of DCM, as presented in Appendix E, reveals that irrespective of p, at most $2N + 1$ ILPs need to be solved, with at most $N + 1$ infeasible ILPs, and that the number of disjunctions modeled in any ILP is N, in the worst case. (In practice, this number is found to be several orders of magnitude smaller than N.)

The goal of this discussion is to demonstrate that the performance of the RVF Algorithm remains within contention of existing multiobjective algorithms for the pure integer case and any number of objectives, in theory. In practice, the state-of-the-art in MINLP solution versus ILP will play a part. While the MINLP solution is currently less well-developed, it is an active area of research. In addition to advances in MINLP, it is expected that future research on the RVF Algorithm will focus on extending the concept to improve its practical performance, as has been the case with earlier versions of multiobjective algorithms.

5.3.2 General Case

For the setting in which we have continuous variables, only the case of two objectives has existing algorithms with theoretical performance analysis. These are two variants of the Boxed Line Method (BLM), which were proposed by [Perini et al., 2020]. It is important to note that the performance analysis of the BLM variants depends on the number of line segments in the EF, not on the number of stability regions. Let L denote the number of line segments in the frontier. Then the basic iterative variant of BLM solves at most $3L + 2$ two-stage MILPs and at most $\frac{1}{2}L^2 + \frac{5}{2}L - 2$ scalarized (single-objective) MILPs. The recursive variant of BLM has better theoretical performance guarantees, with the number of two-stage MILPs remaining the same but the number of scalarized MILPs being at most $2L - 1$. This results in a total of $5L + 1$ MILPs plus $3L + 2$ conversion MILPs as a worst-case bound. It is worth noting that the size of the MILPs does not change as the algorithm progresses. In comparison, the RVF Algorithm solves increasingly larger MINLPs, which may be more challenging to solve, but may also be required to solve far fewer of them. If N denotes the number of stability regions found by the RVF Algorithm, so $N = \text{card}(\mathcal{S}_{\min})$, then $N \leq L$ and the number of MINLPs it solves is bounded above by $N \leq L < 5L + 1$. However, each integer vector in $\mathcal{S}_{\min}$ (each stability region) could contribute *many* line segments, so the RVF Algorithm may need to solve far fewer subproblems in practice. Overall, the RVF Algorithm offers a promising approach for solving multiobjective MILPs with any number of objectives.

To illustrate the difference in performance between the RVF Algorithm and the BLM variants, consider the example presented in Figures 7 and 8, with numerical details provided in Appendix F. The example has two objectives, and its EF has 24 line segments but only 4 stability regions. This means that the RVF Algorithm would be expected to solve only 4 MINLPs, with the first being solvable as an MILP, while the best variant of the BLM method would require 121—which is over *thirty times* more—MILPs to be solved.

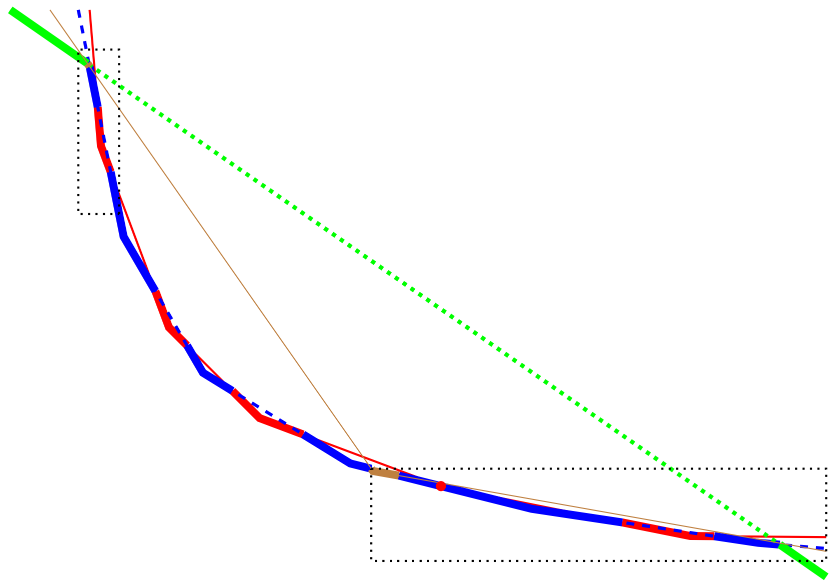
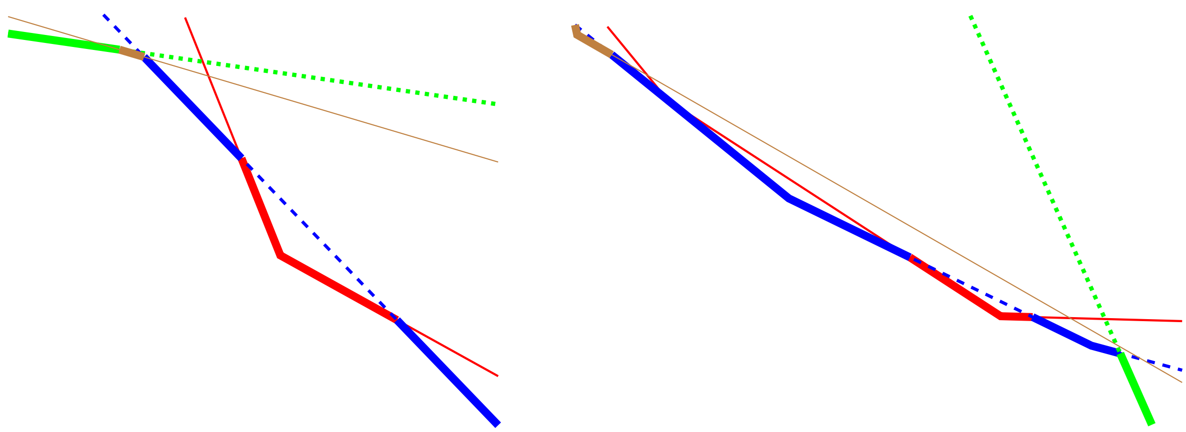


Figure 7: An example with two objectives, four elements of $\mathcal{S}_{\min}$, all with disconnected stability regions, having a total of 24 linear pieces in the EF, shown with thicker lines. The LP EFs for the four elements of $\mathcal{S}_{\min}$ are shown as red solid, blue dashed, sepia solid, and green dotted lines, respectively. Numerical details are given in Appendix F. Close-up of the areas in the top-left and bottom-right parts are given in Figure 8



(a) Top-left

(b) Bottom right

Figure 8: Close-ups of the regions in the top-left and bottom-right ends of shown in Figure 7 (Note that these have been asymmetrically scaled to allow better visibility of LP frontier intersections)

In a recent study, Rasmi et al. [2019] presented an algorithm for identifying all the integer vectors required to construct the frontier for a (MO-MILP) with an arbitrary number of objectives. They also provided a method for identifying the facets of multiobjective linear programming (MOLP) EF, which contribute at least one point to the overall (MO-MILP) frontier. For all pairs of extreme points of such facets that are adjacent in the facet, the authors demonstrated how to determine the parts of the line segment connecting the two points that are nondominated, if any. If only a portion of a facet of a MOLP EF appears in the (MO-MILP) frontier, Rasmi et al. [2019] do not attempt to describe the nondominated part explicitly. Their method for generating all integer vectors required for the (MO-MILP) frontier utilizes "no-good" constraints to eliminate integer vectors that have already been identified and disjunctive constraints, similar to those used by [Sylva and Crema, 2004], to eliminate parts of objective space dominated by individual NDPs found. Thus the size of the MILPs they solve grows in two ways as the algorithm progresses. No theoretical performance analysis is provided. It is noted that the algorithm may solve MILPs those yield integer solutions that do not contribute to the frontier, and many MILPs solved may find the same integer solutions repeatedly. Both of these issues are avoided in the RVF algorithm. Computational results on instances with 2, 3, and 4 objectives are given by [Rasmi et al., 2019]. These show that even for only 2 objectives, the algorithm was required to solve more than 25 MILPs per integer solution, and for one class of instances with 3 objectives, on average more than 135 MILPs were solved per integer solution. No information is provided regarding the number of disjunctive constraints added to the MILPs in computation or how the size of these MILPs grows as the algorithm progresses. In another study, Pettersson and Ozlen [2019] developed an algorithm for identifying all supported and non-supported NDPs with an arbitrary number of objectives. The algorithm first identifies a super-set (which contains the EF) using Benson's method, then modifies the set of polytopes so that no two polytopes have a nonempty intersection, and finally, refines the super-set to exclude portions that are not part of the EF. The details of this algorithm are not provided, as it was presented in a short proceedings article, and no performance analysis is given.

5.4 Publicly Available Implementation

We provide a Python package, implemented using the RVF Algorithm, for enumerating all integer parts required to construct the NDPs for instances of multiobjective integer and mixed integer programs with an arbitrary number of objective functions. The MINLP subproblems encountered at each iteration are solved using the Couenne solver [Belotti, 2009]. This package is available at `https://github.com/SamiraFallah/RestrictedValueFunction`.

6 Conclusions and Future Research Directions

In this paper, we discussed the relationship between the restricted value function, RVF, and the EF of a multiobjective integer linear program (MO-MILP). We demonstrated that the EF lies on a subset of the boundary of the epigraph of the RVF. In so doing, we highlight an important relationship that connects two parts of the literature that had been considered distinct. We also demonstrated that the RVF is the minimum of polyhedral functions associated with RLPVF and discussed the structure of the RVF, including its continuity. Finally, we showed that under the assumption that the X_{MO} is bounded, there exists a discrete representation of both RVF and the EF. In doing so, we introduced the notion of a stability region, which is a part of the EF associated with one element of the discrete representation, and discussed its properties.

In this context, we proposed the RVF algorithm for identifying all nondominated points in the EF for a multiobjective optimization problem. Our proposed algorithm offers an alternative to existing algorithms for constructing the frontier that provides a convenient and precise performance guarantee if terminated early. We also provide a Python package, available at `https://github.com/SamiraFallah/RestrictedValueFunction`, that demonstrates the practical effectiveness of the proposed algorithm.

The RVF algorithm is unique in the MO-MILP literature in its degree of generality – it can compute the discrete representation of the EF for arbitrarily many objectives and in the presence of continuous variables – and in its computational performance, requiring only one nonlinear mixed integer problem (MINLP) to be solved per stability region in the frontier. These characteristics provide distinct advantages compared to existing approaches that also handle continuous variables.

The basic algorithm presented here can be improved in several ways, and its computational efficiency in practice can be further enhanced. Ongoing research on developing such enhancements is currently underway. One potential direction for improvement is to develop approaches to warm start the MINLP solver at each iteration since, at each iteration, the MINLP has changed incrementally: only one cut and a set of associated variables have been added. Another potential direction for improvement is to develop a customized branch-and-bound method that incorporates specialized branching, bounding, and search strategies to exploit the structure of the MINLP. Additionally, the algorithm presented here could be integrated into algorithms for related problems that require parametric analysis or value function construction.

We hope that our proposed algorithm will inspire further research into the study of multiobjective programs from a new perspective. Investigating the relationship between the value function and the EF both theoretically and computationally could have significant value for the numerous applications in which the value function (or, equivalently, the EF) plays a role.

References

Shabbir Ahmed, Mohit Tawarmalani, and Nikolaos V Sahinidis. A finite branch-and-bound algorithm for two-stage stochastic integer programs. *Mathematical Programming*, 100(2):355–377, 2004.

Bernd Bank, Jürgen Guddat, Diethard Klatte, Bernd Kummer, and Klaus Tammer. *Non-linear parametric optimization*. Springer, 1982.

Pietro Belotti. Couenne: a user's manual, 2009.

Pietro Belotti, Banu Soylu, and Margaret M Wiecek. A branch-and-bound algorithm for biobjective mixed-integer programs. *Optimization Online*, 2013.

Harold P Benson. Existence of efficient solutions for vector maximization problems. *Journal of Optimization Theory and Applications*, 26(4):569–580, 1978.

Harold P Benson. An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem. *Journal of Global Optimization*, 13(1):1–24, 1998.

Charles Blair. A closed-form representation of mixed-integer program value functions. *Mathematical Programming*, 71(2):127–136, 1995.

Charles E Blair and Robert G Jeroslow. The value function of a mixed integer program: I. *Discrete Mathematics*, 19(2):121–138, 1977.

Charles E Blair and Robert G Jeroslow. The value function of an integer program. *Mathematical Programming*, 23(1):237–273, 1982.

Charles E Blair and Robert G Jeroslow. Constructive characterizations of the value-function of a mixed-integer program i. *Discrete Applied Mathematics*, 9(3):217–233, 1984.

Charles Eugene Blair and Robert G Jeroslow. The value function of a mixed integer program: Ii. *Discrete Mathematics*, 25(1):7–19, 1979.

Merve Bodur, Shabbir Ahmed, Natashia Boland, and George L Nemhauser. Decomposition of loosely coupled integer programs: A multiobjective perspective. *Optimization Online*, 2016.

Natashia Boland, Hadi Charkhgard, and Martin Savelsbergh. A new method for optimizing a linear function over the efficient set of a multiobjective integer program. *European journal of operational research*, 260(3):904–919, 2017.

V Joseph Bowman. On the relationship of the tchebycheff norm and the efficient frontier of multiple-criteria objectives. In *Multiple criteria decision making*, pages 76–86. Springer, 1976.

Seth Brown, Wenxin Zhang, Temitayo Ajayi, and Andrew J Schaefer. A gilmore-gomory construction of integer programming value functions. *Operations Research Letters*, 2021.

LG Chalmet, L Lemonidis, and DJ Elzinga. An algorithm for the bi-criterion integer programming problem. *European Journal of Operational Research*, 25(2):292–300, 1986.

Pasqualina Conti and Carlo Traverso. Buchberger algorithm and integer programming. In *International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes*, pages 130–139. Springer, 1991.

László Csirmaz. Using multiobjective optimization to map the entropy region. *Computational Optimization and Applications*, 63(1):45–67, 2016.

Marianna De Santis and Gabriele Eichfelder. A decision space algorithm for multiobjective convex quadratic integer optimization. *Computers & Operations Research*, 134:105396, 2021.

Marianna De Santis, Gabriele Eichfelder, Julia Niebling, and Stefan Rocktäschel. Solving multi-objective mixed integer convex optimization problems. *SIAM Journal on Optimization*, 30(4): 3122–3145, 2020.

Matthias Ehrgott. *Multicriteria optimization*, volume 491. Springer Science & Business Media, 2005.

Matthias Ehrgott. A discussion of scalarization techniques for multiple objective integer programming. *Annals of Operations Research*, 147(1):343–360, 2006.

Matthias Ehrgott and Xavier Gandibleux. A survey and annotated bibliography of multiobjective combinatorial optimization. *OR-spektrum*, 22(4):425–460, 2000.

Matthias Ehrgott and Margaret M. Wiecek. *Mutiobjective Programming*, pages 667–708. Springer New York, New York, NY, 2005. ISBN 978-0-387-23081-8. doi: 10.1007/0-387-23081-5_17. URL https://doi.org/10.1007/0-387-23081-5_17.

Matthias Ehrgott, Xavier Gandibleux, and Anthony Przybylski. Exact methods for multi-objective combinatorial optimisation. In *Multiple criteria decision analysis*, pages 817–850. Springer, 2016.

Gabriele Eichfelder and Leo Warnow. A hybrid patch decomposition approach to compute an enclosure for multi-objective mixed-integer convex optimization problems. *Preprint*, pages 08–8541, 2021a.

Gabriele Eichfelder and Leo Warnow. On implementation details and numerical experiments for the hypad algorithm to solve multi-objective mixed-integer convex optimization problems. *Preprint*, pages 08–8538, 2021b.

Gabriele Eichfelder, Oliver Stein, and Leo Warnow. A solver for multiobjective mixed-integer convex and nonconvex optimization. Technical Report 2022-02-8796, Institute of Mathematics, Technische Universität Ilmenau, 2022.

Julius Farkas. Theorie der einfachen ungleichungen. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1902(124):1–27, 1902.

Nicolas Forget and Sophie N Parragh. Enhancing branch-and-bound for multi-objective 0-1 programming. *arXiv preprint arXiv:2210.05385*, 2022.

Nicolas Forget, Sune Lauth Gadegaard, Kathrin Klamroth, Lars Relund Nielsen, and Anthony Przybylski. Branch-and-bound and objective branching with three or more objectives. *Computers & Operations Research*, 148:106012, 2022a.

Nicolas Forget, Sune Lauth Gadegaard, and Lars Relund Nielsen. Warm-starting lower bound set computations for branch-and-bound algorithms for multi objective integer linear programs. *European Journal of Operational Research*, 2022b.

Juergen Guddat, F Guerra Vasquez, Klaus Tammer, and Klaus Wendler. Multiobjective and stochastic optimization based on parametric optimization. *Mathematical Research*, 26, 1985.

M. Güzelsoy and T.K. Ralphs. Duality for Mixed-Integer Linear Programs. *International Journal of Operations Research*, 4:118–137, 2007. URL http://coral.ie.lehigh.edu/~ted/files/papers/MILPD06.pdf.

MENAL Guzelsoy and T Ralphs. The value function of a mixed-integer linear program with a single constraint. *To be submitted*, 2006.

Yacov Haimes. On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE transactions on systems, man, and cybernetics*, 1(3):296–297, 1971.

Pascal Halffmann, Luca E Schäfer, Kerstin Dächert, Kathrin Klamroth, and Stefan Ruzika. Exact algorithms for multiobjective linear optimization problems with integer variables: A state of the art survey. *Journal of Multi-Criteria Decision Analysis*, 2022.

Andreas H Hamel, Andreas Löhne, and Birgit Rudloff. Benson type algorithms for linear vector optimization and applications. *Journal of Global Optimization*, 59(4):811–836, 2014.

John Hooker. *Logic-based methods for optimization: combining optimization and constraint satisfaction*. John Wiley & Sons, 2011.

John N Hooker. Logic-based methods for optimization. In *International Workshop on Principles and Practice of Constraint Programming*, pages 336–349. Springer, 1994.

Gokhan Kirlik and Serpil Sayın. A new algorithm for generating all nondominated solutions of multiobjective discrete optimization problems. *European Journal of Operational Research*, 232 (3):479–488, 2014.

Nan Kong, Andrew J Schaefer, and Brady Hunsaker. Two-stage integer programs with stochastic right-hand sides: a superadditive dual approach. *Mathematical Programming*, 108(2):275–296, 2006.

A Löhne and B Weißing. Bensolve-vlp solver, version 2.0. 1. *URL http://bensolve. org*, 2015.

Banu Lokman and Murat Köksalan. Finding all nondominated points of multi-objective integer programs. *Journal of Global Optimization*, 57(2):347–365, 2013.

George Nemhauser and Laurence Wolsey. The scope of integer and combinatorial optimization. *Integer and combinatorial optimization*, pages 1–26, 1988.

Melih Özlen and Meral Azizoğlu. Multi-objective integer programming: A general approach for generating all non-dominated solutions. *European Journal of Operational Research*, 199(1):25–35, 2009.

Adriano Pascoletti and Paolo Serafini. Scalarizing vector optimization problems. *Journal of Optimization Theory and Applications*, 42(4):499–524, 1984.

Tyler Perini, Natashia Boland, Diego Pecin, and Martin Savelsbergh. A criterion space method for biobjective mixed integer programming: The boxed line method. *INFORMS Journal on Computing*, 32(1):16–39, 2020.

William Pettersson and Melih Ozlen. Multi-objective mixed integer programming: An objective space algorithm. In *AIP Conference Proceedings*, volume 2070, page 020039. AIP Publishing LLC, 2019.

Anthony Przybylski and Xavier Gandibleux. Multi-objective branch and bound. *European Journal of Operational Research*, 260(3):856–872, 2017.

Ted K Ralphs and Anahita Hassanzadeh. On the value function of a mixed integer linear optimization problem and an algorithm for its construction. *COR@ L Technical Report 14T–004*, 2014.

Ted K Ralphs, Matthew J Saltzman, and Margaret M Wiecek. An improved algorithm for solving biobjective integer programs. *Annals of Operations Research*, 147(1):43–70, 2006.

Seyyed Rasmi, Amir Babak, and Metin Türkay. Gondef: an exact method to generate all non-dominated points of multi-objective mixed-integer linear programs. *Optimization and Engineering*, 20:89–117, 2019.

Seyyed Amir Babak Rasmi and Metin Türkay. Gondef: an exact method to generate all non-dominated points of multi-objective mixed-integer linear programs. *Optimization and Engineering*, 20(1):89–117, 2019.

R Tyrrell Rockafellar. *Convex analysis*, volume 11. Princeton university press, 1997.

R Tyrrell Rockafellar and Roger J-B Wets. *Variational analysis*, volume 317. Springer Science & Business Media, 2009.

Stefan Ruzika and Margaret M Wiecek. Approximation methods in multiobjective programming. *Journal of optimization theory and applications*, 126(3):473–501, 2005.

Martin WP Savelsbergh. Preprocessing and probing techniques for mixed integer programming problems. *ORSA Journal on Computing*, 6(4):445–454, 1994.

Rüdiger Schultz, Leen Stougie, and Maarten H Van Der Vlerk. Solving stochastic programs with integer recourse by enumeration: A framework using gröbner basis. *Mathematical Programming*, 83(1):229–252, 1998.

John Sylva and Alejandro Crema. A method for finding the set of non-dominated vectors for multiple objective integer linear programs. *European Journal of Operational Research*, 158(1): 46–55, 2004.

Andrew C Trapp and Oleg A Prokopyev. A note on constraint aggregation and value functions for two-stage stochastic integer programs. *Discrete Optimization*, 15:37–45, 2015.

Andrew C Trapp, Oleg A Prokopyev, and Andrew J Schaefer. On a level-set characterization of the value function of an integer program and its application to stochastic programming. *Operations Research*, 61(2):498–511, 2013.

Laurence A Wolsey. Integer programming duality: Price functions and sensitivity analysis. *Mathematical Programming*, 20(1):173–195, 1981.

Po-Lung Yu. A class of solutions for group decision problems. *Management science*, 19(8):936–946, 1973.

Lofti Zadeh. Optimality and non-scalar-valued performance criteria. *IEEE transactions on Automatic Control*, 8(1):59–60, 1963.

Milan Zeleny. Compromise programming. *Multiple criteria decision making*, 1973.

Junlong Zhang and Osman Y Özaltın. Bilevel integer programs with stochastic right-hand sides. *INFORMS Journal on Computing*, 2021.

Appendix A

The entire EF for Example 1 is shown below in Figure 9. The bounding functions associated with each integer vector of Example 1 are illustrated in Figure 10. To highlight the detail in the bottom right parts of the frontiers more clearly, an enlarged view is presented in Figure 11.

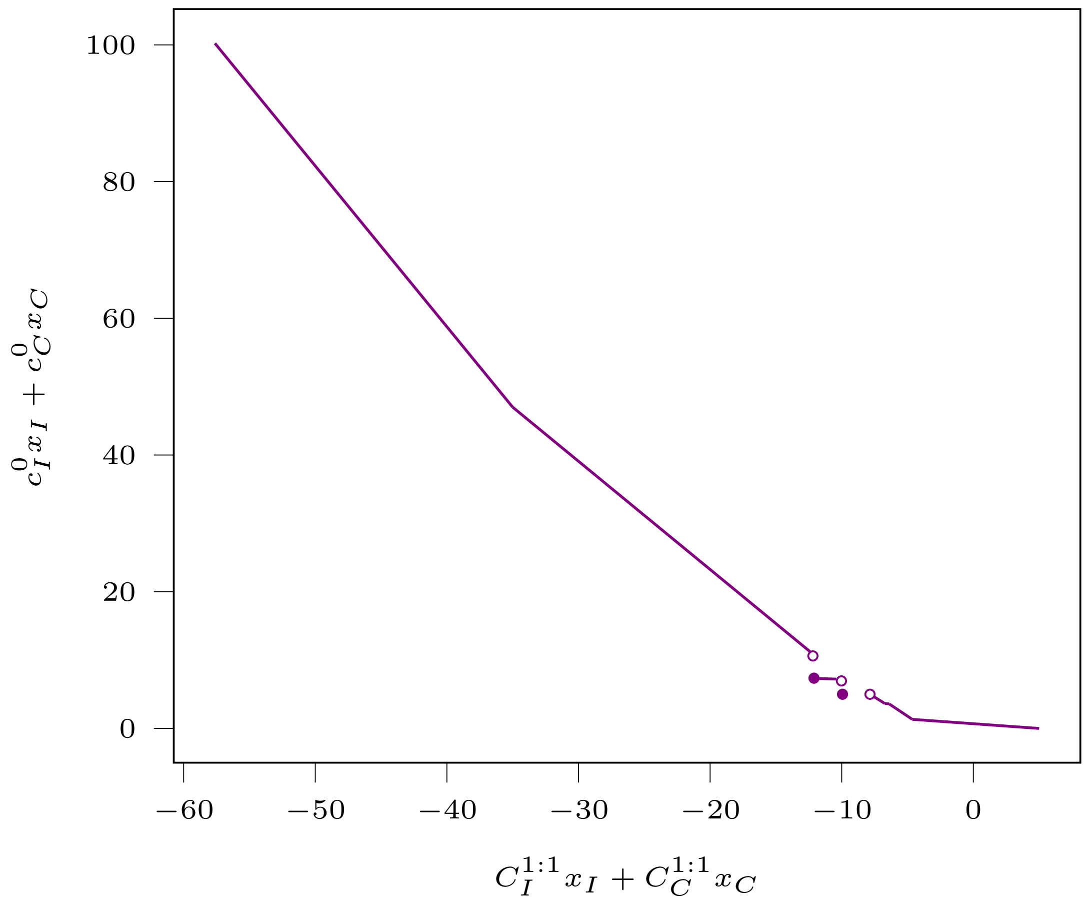


Figure 9: The EF associated with Example 1

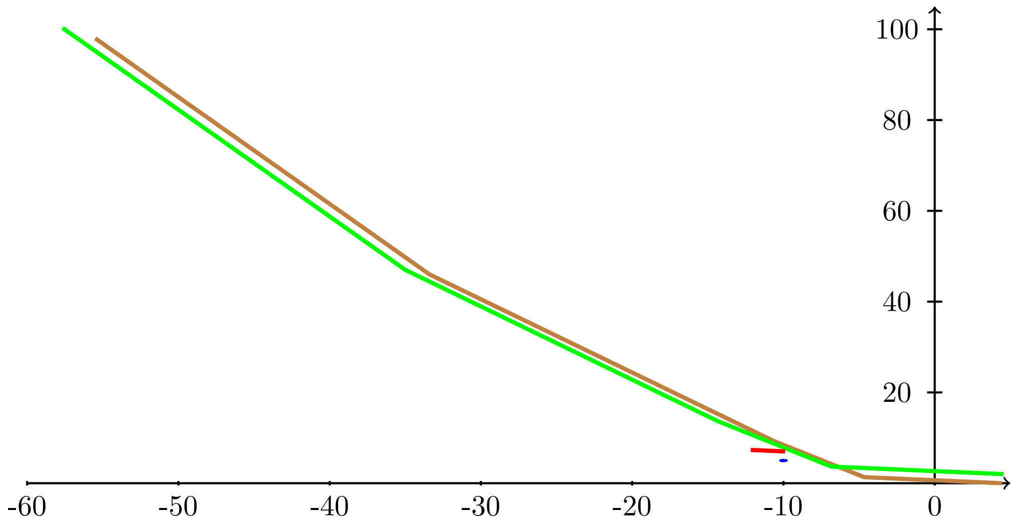


Figure 10: The LP frontiers. Brown is for $(x_1, x_2) = (0, 0)$. Green is for $(x_1, x_2) = (1, 0)$. Red is for $(x_1, x_2) = (1, 1)$. Blue is for $(x_1, x_2) = (0, 1)$. Note the axes are unequally scaled

Appendix B

Here we describe the directional derivatives of a function formed by taking the minimum of a finite set of polyhedral functions. We are particularly interested in functions defined over the extended reals.

The properties we identify here are for any such function; they are not specific to value functions. We think it is very likely that the material we provide here has already been published. However we have been unable to locate a source for it, so include it for the sake of completeness.

First, we give definitions and conventions for directional derivatives of functions defined over the extended reals. Then we briefly review the properties of polyhedral functions, before giving our main result on the directional derivatives of the minimum of a finite set of polyhedral functions.

Consider $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, a function defined over the extended reals that is bounded below, and so cannot have the value $-\infty$. We refer to the set of points $x \in \mathbb{R}^n$ for which $f(x) < +\infty$ as the *finite domain* of f. We are interested in functions with closed finite domain. The directional derivative of f at the point $x \in \mathbb{R}^n$ in direction $d \in \mathbb{R}^n$ is denoted by $\nabla_d f(x)$ and is defined by

$$\nabla_d f(x) = \lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t}.$$

We take $\nabla_d f(x) = 0$ if $f(x) = +\infty$ (meaning x is not in the finite domain of f). If $f(x) < +\infty$ and there exists $\epsilon > 0$ such that $f(x + td) = +\infty$ for all $t \in (0, \epsilon)$, then we say x is on the boundary of the finite domain of f and d points out of it. In this case, $\nabla_d f(x) = +\infty$.

We now turn our attention to the case of a function defined as the minimum of a finite set of polyhedral functions.

Say $z^k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a given polyhedral function for each $k = 1, \dots, K$, which means that

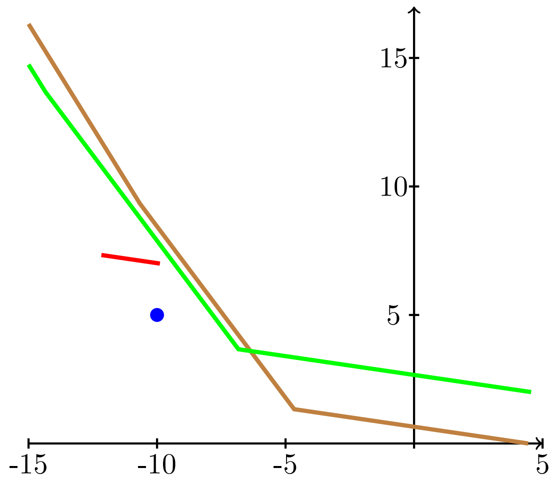


Figure 11: The (truncated at the left) LP frontiers. Brown is for $(x_1, x_2) = (0, 0)$. Green is for $(x_1, x_2) = (1, 0)$. Red is for $(x_1, x_2) = (1, 1)$. Blue is for $(x_1, x_2) = (0, 1)$

the epigraph over its finite domain is a polyhedron. A property of a polyhedral function is that for any point x and direction d, one of the following cases must hold:

1. x is not in the finite domain of z^k, in which case $\nabla_d z^k(x) = 0$,
2. x is on the boundary of the finite domain of z^k and d points out of it, in which case $\nabla_d z^k(x) = +\infty$, or
3. there exists $\epsilon > 0$ such that $x + td$ is in the finite domain of z^k for all $t \in [0, \epsilon]$, and, furthermore, ϵ may be taken to be sufficiently small that all points in the line segment defined as $\{x + td \ : \ 0 \leq t \leq \epsilon\}$ lie on the same facet of the polyhedron created by the epigraph of z^k over its finite domain. In this case, there exists $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $z^k(x + td) = a^\top(x + td) + b$ for all $t \in [0, \epsilon]$. Note that a and b depend on x and d, but not on ϵ, and it follows that $\nabla_d z^k(x) = a^\top d$.

It is also helpful to observe that the finite domain of a polyhedral function must itself be a polyhedron, which is closed.

Let $z : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be defined to be

$$z(x) = \min_{k=1,\dots,K} z^k(x).$$

Since each function z^k bounds z from above, we call z^k a bounding function. Define

$$\kappa^*(x) = \{k \in \{1, \dots, K\} : z^k(x) = z(x)\},$$

to be the index set of the bounding functions that are active at x. Clearly, the finite domain of z is the union of the finite domains of its bounding functions, so is the union of a finite collection of closed sets, and hence is closed.

It is useful to observe that in the case that x is in the finite domain of z and d does *not* point out of it, then it may be that there is a jump, or discontinuity, in z when moving away from x in the direction d. In other words, for such x and d, it may be that

$$\lim_{t\to 0^+} z(x+td) > z(x).$$

In such a case, there must exist $k \in \{1,\ldots,K\}$ and $\epsilon > 0$ so that $z(x+td) = z^k(x+td)$ for all $t \in (0,\epsilon)$, where, importantly, $t = 0$ is *not* included in this interval. So

$$\lim_{t\to 0^+} z(x+td) = \lim_{t\to 0^+} z^k(x+td) = z^k(x) > z(x),$$

and it follows that $\nabla_d z(x) = +\infty$.

Whether or not z has a discontinuity when moving away from x in direction d, the structure of z as a minimum of a finite set of polyhedral functions ensures that when x is in the finite domain of z and d does not point out of it, there must exist $k \in \{1,\ldots,K\}$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ defining a facet of the epigraph of z^k over its finite domain, and $\epsilon > 0$ sufficiently small so that

$$z(x+td) = z^k(x+td) = a^\top(x+td) + b, \qquad \forall t \in (0,\epsilon).$$

We say that the facet of z^k defined by (a,b) *yields* z *when moving away from* x *in direction* d.

We now give the main result.

Proposition B.1. *For any point $x \in \mathbb{R}^n$ and any direction $d \in \mathbb{R}^n$,*

$$\nabla_d z(x) = \min_{k\in\kappa^*(x)} \nabla_d z^k(x).$$

Proof. Let the point $x \in \mathbb{R}^n$ and direction $d \in \mathbb{R}^n$ be given. There are three main cases.
Case 1: x is not in the finite domain of z. In this case, $\nabla_d z(x) = 0$. Furthermore, since $z(x) = +\infty$, it follows from the definition of z that $z^k(x) = +\infty$, and hence $\nabla_d z^k(x) = 0$, for all $k = 1,\ldots,K$. The result follows.
Case 2: x is in the boundary of the finite domain of z and d points out of it. This means that there exists $\epsilon > 0$ so that $z(x+td) = +\infty$ for all $t \in (0,\epsilon)$. By the definition of z, it must be that for all $k = 1,\ldots,K$, $z^k(x+td) = +\infty$ for all $t \in (0,\epsilon)$. Now for all $k \in \kappa^*(x)$, x is in the finite domain of the bounding function z^k, since $z^k(x) = z(x) < +\infty$. Thus $\nabla_d z^k(x) = +\infty$ and the result follows.
Case 3. there exists $\epsilon > 0$ such that $z(x+td) < +\infty$ for all $t \in [0,\epsilon)$. Note that $t = 0$ is included in the definition of this case, since x not in the finite domain of z was covered in Case 1. Assume, without loss of generality, that $k = 1$ is the index of the bounding function having a facet that yields z when moving away from x in direction d. Thus there must exist $\epsilon' > 0$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$z(x+td) = z^1(x+td) = a^\top(x+td) + b, \qquad \forall t \in (0,\epsilon').$$

Let k^* denote the minimizer of $\nabla_d z^k(x)$ over $k \in \kappa^*(x)$. Note that $\nabla_d z^1(x) = a^\top d$, which is finite. So $1 \in \kappa^*(x)$ implies $\nabla_d z^{k^*}(x)$ is finite. Thus if $\nabla_d z^{k^*}(x) = +\infty$ it must be that $1 \notin \kappa^*(x)$. Hence

$$\lim_{t\to 0^+} z(x+td) = \lim_{t\to 0^+} z^1(x+td) = z^1(x) > z(x),$$

and $\nabla_d z(x) = +\infty = \nabla_d z^{k^*}(x)$, as required. Now consider the case that $\nabla_d z^{k^*}(x)$ is finite. By the properties of polyhedral functions discussed above, there must exist $\epsilon'' > 0$, $\hat{a} \in \mathbb{R}^n$ and $\hat{b} \in \mathbb{R}$ such that $z^{k^*}(x + td) = \hat{a}^\top(x + td) + \hat{b}$ for all $t \in [0, \epsilon'')$, so $\nabla_d z^{k^*}(x) = \hat{a}^\top d$. To summarize, we have

$$\min_{k \in \kappa^*(x)} \nabla_d z^k(x) = \hat{a}^\top d \qquad \text{and} \qquad \nabla_d z(x) = a^\top d.$$

Now suppose, for the sake of contradiction, that $\hat{a}^\top d \neq a^\top d$. Define ϵ''' by

$$\epsilon''' = \begin{cases} +\infty, & \text{if } \hat{a}^\top d < a^\top d \\ \frac{z^1(x) - z(x)}{\hat{a}^\top d - a^\top d}, & \text{if } \hat{a}^\top d > a^\top d. \end{cases}$$

Observe that $\epsilon''' > 0$ since $1 \in \kappa^*(x)$ implies $\hat{a}^\top d = \nabla_d z^{k^*}(x) \leq \nabla_d z^1(x) = a^\top d$, so the case $\hat{a}^\top d > a^\top d$ implies that $1 \notin \kappa^*(x)$, and so $z^1(x) > z(x)$. Then for $0 < t < \epsilon'''$ we have that

$$t(\hat{a}^\top d - a^\top d) < z^1(x) - z(x).$$

Now $z(x) = \hat{a}^\top x + \hat{b}$ since $k^* \in \kappa^*(x)$, and by continuity of z^1, we also have $z^1(x) = a^\top x + b$. Substituting these in, we obtain

$$t(\hat{a}^\top d - a^\top d) < a^\top x + b - (\hat{a}^\top x + \hat{b}),$$

which is equivalently written as

$$\hat{a}^\top(x + td) + \hat{b} < a^\top(x + td) + b.$$

Since $t > 0$, $t < \epsilon'$ and $t < \epsilon''$, it must be that

$$z^{k^*}(x + td) = \hat{a}^\top(x + td) + \hat{b} < a^\top(x + td) + b = z(x + td),$$

which contradicts the definition of z. Thus it must be that $\hat{a}^\top d = a^\top d$, as required. ■

Appendix C

As mentioned in Section 4, we can have another representation of Theorem 4.1. Consider the RVF $z' : \mathbb{R}^l \to \mathbb{R} \cup \{\pm\infty\}$ as

$$z'(\zeta) = \inf_{(x_I, x_C) \in \mathcal{S}(\zeta)} c_I^0 x_I + c_C^0 x_C, \tag{RVF$'$}$$

where

$$\mathcal{S}(\zeta) = \left\{ (x_I, x_C) \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} \ : \ C_I^{1:l} x_I + C_C^{1:l} x_C = \zeta, \ A_I x_I + A_C x_C = b \right\}.$$

It is worth noting that the only distinction here is that we have an equality sign for the constraints that serve as our objectives. We formalize the relationship between the RVF and the EF through the following theorem.

Theorem C.1. *The following statements hold for X_{MO} and the (RVF$'$) z'.*

(a) If $(x_I, x_C) \in X_{\text{MO}}$ is an efficient solution (equivalently, $C_I x_I + C_C x_C$ is an NDP), then $(\zeta, c^0 x_I + c^0 x_C)$ is a point on the boundary of the epigraph of z' for $\zeta = C_I^{1:l} x_I + C_C^{1:l} x_C$.

(b) If $(\zeta, z'(\zeta))$ is a point on the boundary of the epigraph of z' and $\nabla_d z'(\zeta) > 0$ for all $d \in \mathbb{R}^l_- \setminus \{\mathbf{0}\}$ for which $\nabla_d z'(\zeta)$ exists, then there exists an efficient solution $(x_I, x_C) \in X_{\text{MO}}$ that yields $z'(\zeta)$ and satisfies $C_I^{1:l} x_I + C_C^{1:l} x_C = \zeta$.

Proof. The proofs for Part (a) and Part (b) are in accordance with the proofs for Part 1 and Part 2 with the exception of the portion designated by $\Leftarrow$ in Theorem 4.1. ■

Here we have a very similar proposition compared to Proposition 3.9 as follows.

Proposition C.2. *The restricted value function (RVF) is a lower semi-continuous function and decreasing over $\mathcal{C}$ where the optimal dual value is negative. Furthermore, it is comprised of a minimum of a finite number of polyhedral functions.*

Proof. The proof for the property of semi-continuity and being a minimum of a finite number of polyhedral functions is the same as that in Proposition 3.9. The decreasing property is straightforward; with negative optimal dual values, the value function is decreasing. ■

Therefore, in the steps of the algorithm, although we have equality for the constraints with the parametric RHS, i.e., the constraints that serve as our objectives, we must have negative dual variables $u \in \mathbb{R}^l_-$, which relates to the constraints with the parametric RHS. In this case, the algorithm generates that part of the value function, which is decreasing and is the same as the related EF. As a result, the value function algorithm remains unchanged in this scenario.

Appendix D

We can linearize the problem to be solved in each iteration of the pure integer case ((36)–(38)) as follows.

$$\theta^k = \max \quad \theta \tag{40}$$
$$\text{subject to} \quad \theta + c_I^0 x_I \le (1-\beta^i) c_I^0 x_I^i + \beta^i U, \qquad i = 0, 1, \ldots, k-1 \tag{41}$$
$$(C_I^{1:l} x_I)_j - (C_I^{1:l} x_I^i)_j + \epsilon_j^i \le \overline{M}^j (1-\alpha_j^i), \qquad i = 0, 1, \ldots, k-1, \; j = 1, \ldots, l \tag{42}$$
$$(C_I^{1:l} x_I^i)_j - (C_I^{1:l} x_I)_j \le \underline{M}^j \alpha_j^i, \qquad i = 0, 1, \ldots, k-1, \; j = 1, \ldots, l \tag{43}$$
$$\beta^i \ge \alpha_j^i, \qquad i = 0, 1, \ldots, k-1, \; j = 1, \ldots, l \tag{44}$$
$$\beta^i \le \sum_{j=1}^{l} \alpha_j^i, \qquad i = 0, 1, \ldots, k-1 \tag{45}$$
$$\alpha^i \in \{0,1\}^l, \qquad i = 0, 1, \ldots, k-1 \tag{46}$$
$$\beta^i \in \{0,1\}, \qquad i = 0, 1, \ldots, k-1 \tag{47}$$
$$(33),\ (34) \text{ and } (35), \tag{48}$$

where for each i and j, $\underline{M}^j$ and ϵ_j^i are set to sufficiently large and small positive values, respectively, so that for any possible choice of x_I having $(C_I^{1:l} x_I)_j < (C_I^{1:l} x_I^i)_j$ it must be that $\epsilon_j^i \le (C_I^{1:l} x_I^i)_j -$

$(C_I^{1:l}x_I)_j \leq \underline{M}^j$, and $\overline{M}^j$ is set to a sufficiently large positive value so that for any possible choice of x_I having $(C_I^{1:l}x_I)_j \geq (C_I^{1:l}x_I^i)_j$ it must be that $(C_I^{1:l}x_I)_j - (C_I^{1:l}x_I^i)_j \leq \overline{M}^j - \epsilon_j^i$. For $j = 1, \ldots, l$, it can be established that $\overline{M}^j = \underline{M}^j = \sum_{i=0}^{r-1} C_I^{1:l}[j][i]$. Note that in the case that $C^{1:l}$ is entirely integer-valued, $\epsilon_j^i = 1$ for all i and j suffices. Furthermore, for each i, α^i is a vector of binary variables such that for all $j = \{1, 2, \ldots, l\}$

$$\alpha_j^i = \begin{cases} 0, & (C_I^{1:l}x_I)_j \geq (C_I^{1:l}x_I^i)_j, \\ 1, & (C_I^{1:l}x_I)_j < (C_I^{1:l}x_I^i)_j, \end{cases} \tag{49}$$

if and only if (42) and (43) are satisfied. Note that we substituted out (37) and (31) in the problem set (36)–(38) with (41), (42), and (43).

The RVF algorithm with the problem set (40)–(48) would be equivalent to the method given in [Sylva and Crema, 2004], which enumerates all NDPs of a pure integer program using a sequence of progressively more constrained MILP that generates a new solution at each iteration. In their study, Sylva and Crema employ a reformulation of disjunctive constraints using a big-M approach in conjunction with artificial binary variables. This modification results in the production of nondominated solutions, as opposed to just weakly nondominated solutions, by transforming the objective function into a weighted sum. However, this process also leads to an increase in computational effort as p binary variables and $p + 1$ constraints are added in each iteration, where $p = l + 1$. The authors only present complete representations for biobjective cases and limit their numerical examination for three objectives to the generation of incomplete representations.

The first approach in [Sylva and Crema, 2004] builds upon the methodology proposed by [Sylva and Crema, 2004] by incorporating the other objectives into the objective function, scaled by a small constant. This modification results in the elimination of one constraint and one binary variable in each iteration. Despite this improvement, the algorithm still experiences a rapid increase in the number of constraints and binary variables.

Appendix E

The Disjunctive Constraints Method (DCM), which is outlined as Algorithm 1 in [Boland et al., 2017], is a technique for solving pure integer (MO-MILP)s with an arbitrary number of objectives. In [Boland et al., 2017], the performance of DCM is analyzed, and it is shown that the worst-case number of ILPs that must be solved is exponential in N, the number of NDPs. We demonstrate that the number of ILPs that need to be solved is actually upper bounded by $2N + 1$, regardless of the number of objectives.

As is noted in [Boland et al., 2017], the workings of the DCM correspond to a binary tree (actually a Fibonacci tree, which is a special case) in which each node of the tree represents an object generated by the algorithm and is searched using an ILP subproblem. If the ILP subproblem has a solution, then it is an NDP that is excluded from the search region for all child nodes created (via Proposition 11 in [Boland et al., 2017]), and at least one child node is created. Otherwise, no child nodes are created; the ILP at this node is infeasible, and the node is a leaf node.

Therefore, the DCM algorithm guarantees that each NDP is found exactly once at an internal node of the binary tree, and the total number of ILPs solved is equal to the number of NDPs plus the number of leaf nodes.

Before starting our formal proof, we recall some basic definitions related to binary trees. A binary tree is a finite tree in which each node has 0, 1, or 2 child nodes. An internal node is a node that has at least one child node, while a leaf node is a node that has no child nodes. A *full* binary tree is a binary tree in which every node has 0 or 2 child nodes.

We use the following lemma.

Lemma E.1. *The number of leaf nodes in a full binary tree is precisely the number of internal nodes plus 1.*

Proof. This lemma can be proved by induction on the number of internal nodes in the tree. The inductive step considers the removal of an internal node of maximum depth. ∎

It is worth noting that a binary tree that is not full may have fewer leaf nodes per internal node. For instance, consider a tree consisting of a "chain" of nodes, each having only one child until the single leaf node is reached. The following proposition demonstrates that such a tree cannot have more leaf nodes per internal node.

Proposition E.2. *The number of leaf nodes in a binary tree is no greater than the number of internal nodes plus 1.*

Proof. Let L denote the number of leaf nodes and I the number of internal nodes in a binary tree. We prove the result by induction on I. Clearly, the case $I = 0$ is the tree consisting of a single node, which is a leaf, so $L = 1 = I + 1$ as required. Now suppose that the result holds for all binary trees with $I - 1$ or fewer internal nodes, and consider a binary tree with I internal nodes and L leaf nodes. If every internal node has 2 children, then the tree is a full binary tree and $L = I + 1$ by Lemma E.1, so $L \leq I + 1$, as required. Thus, suppose there exists an internal node with 1 child. We create a new binary tree by replacing this internal node with its descendent subtree. This operation does not change the number of child nodes of the parent node, as its descendent subtree starts from a single child. Thus, the new binary tree has $I - 1$ internal nodes but still L leaf nodes. By the inductive hypothesis, $L \leq I - 1 + 1 = I$, which implies $L \leq I + 1$, as required. ∎

We can now present the worst-case performance result for the DCM algorithm.

Theorem E.3. *The total number of ILP subproblems solved by DCM is at most $2N + 1$, where N denotes the number of NDPs of the instance solved.*

Proof. From results in [Boland et al., 2017], each ILP subproblem solved by the DCM algorithm is either infeasible or yields a new NDP. Therefore, the total number of ILP subproblems solved by DCM is $N + M$, where M is the number of infeasible ILPs solved by DCM. Furthermore, the ILP subproblems are organized in a binary tree structure, with an internal node corresponding to each feasible ILP and a leaf node corresponding to each infeasible ILP. Thus, by Proposition E.2, we have $M \leq N + 1$. As a result, the total number of ILP subproblems solved by DCM is at most $N + (N + 1) = 2N + 1$. ∎

Appendix F

The (MO-MILP) illustrated in Figures 7 and 8, having four elements of $\mathcal{S}_{\min}$, has the LP EF for each formed as follows. The LP frontier shown in red in the figures, which has the right-most upper left corner point, is created by joining its extreme supported points given by

$$F_1 = \{(3\frac{1}{2}, 27), (4, 21), (7, 13), (11, 9), (19, 6), (30, 3\frac{4}{5}), (36, 3\frac{3}{4})\}.$$

The LP frontier shown in blue in the figures, which has the second right-most upper left corner point, is created by joining its extreme supported points given by

$$F_2 = \{(3, 27), (5, 17), (8\frac{1}{2}, 11), (15, 7), (23, 5), (33, 3\frac{1}{2}), (36, 3\frac{1}{4})\}.$$

The LP frontier shown in brown in the figures, which has the third right-most upper left corner point, is created by joining its extreme supported points given by

$$F_3 = \{(1\frac{3}{4}, 27), (16, 6\frac{2}{3}), (36, 3\frac{1}{8})\}.$$

The LP frontier shown in green in the figures, which has the left-most upper left corner point, is created by joining its extreme supported points given by

$$F_4 = \{(0, 27), (36, 2)\}.$$

The four individual LP frontiers are shown in Figures 12–14, with the fourth LP frontier, shown in green, duplicated in each, to provide a common reference in addition to the axes.

There are various methods for constructing a multiobjective optimization model of the form given in (MO-MILP) that corresponds to these LP EFs. One straightforward approach involves introducing four binary variables, one for each of the LP frontiers, to switch the corresponding LP constraints on or off, as well as two continuous variables to represent the two objectives.

Appendix G

In this appendix, we show the details of the proof that the integer part added to $\mathcal{S}^k$ in iteration k has a nonempty stability region *at the time it is added*. Note, however, that this stability region may end up being redundant by the end of the algorithm. First, we show that the true value function $z(\zeta^*) = c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}$ whenever (x_I^{k+1}, x_C^{k+1}) solves the optimization problem (14) and $\zeta^* = C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}$.

Lemma G.1. *If* (x_I^{k+1}, x_C^{k+1}) *is defined by* (14) *then* $z(\zeta^*) = c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}$ *where* $\zeta^* = C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}$.

Proof. Let (x_I^{k+1}, x_C^{k+1}) be defined by (14) and suppose, for the sake of contradiction, that $z(C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}) \neq c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}$. Then there must exist $(\hat{x}_I, \hat{x}_C)$ a solution of the optimization problem

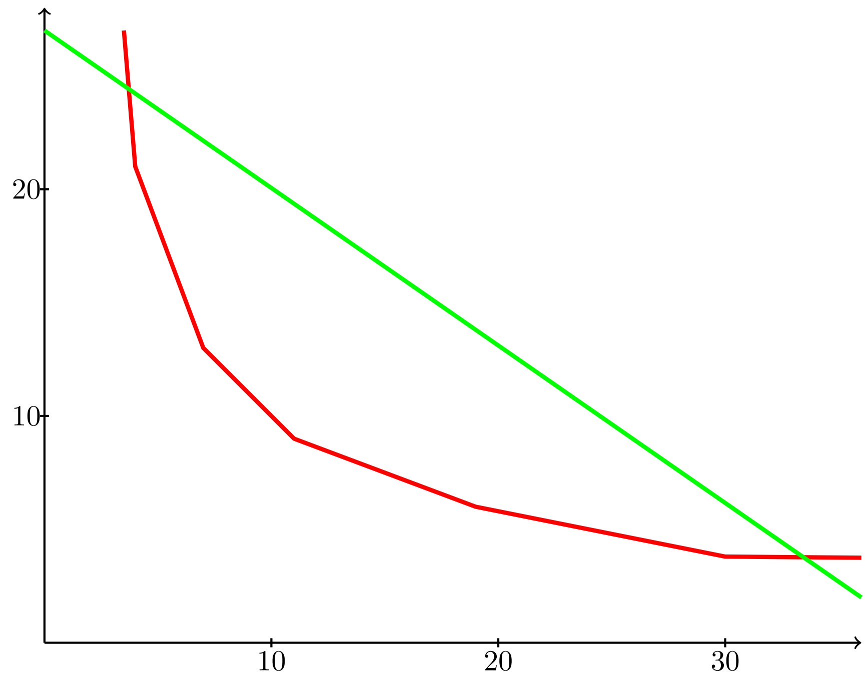


Figure 12: The complete LP frontier for the parts shown in red in Figure 7, together with the complete LP frontier for the part shown in green

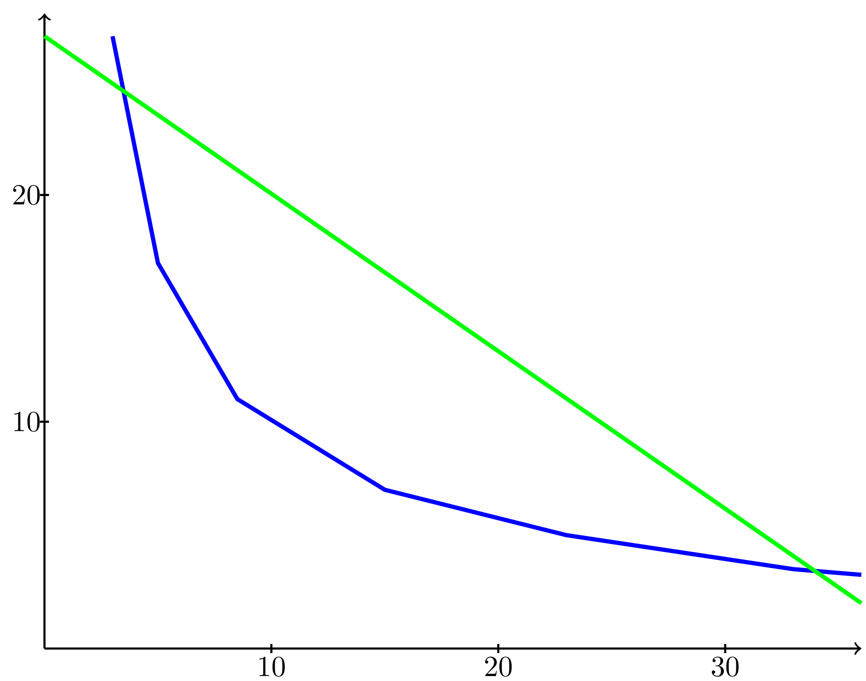


Figure 13: The complete LP frontier for the parts shown in blue in Figure 7, together with the complete LP frontier for the part shown in green

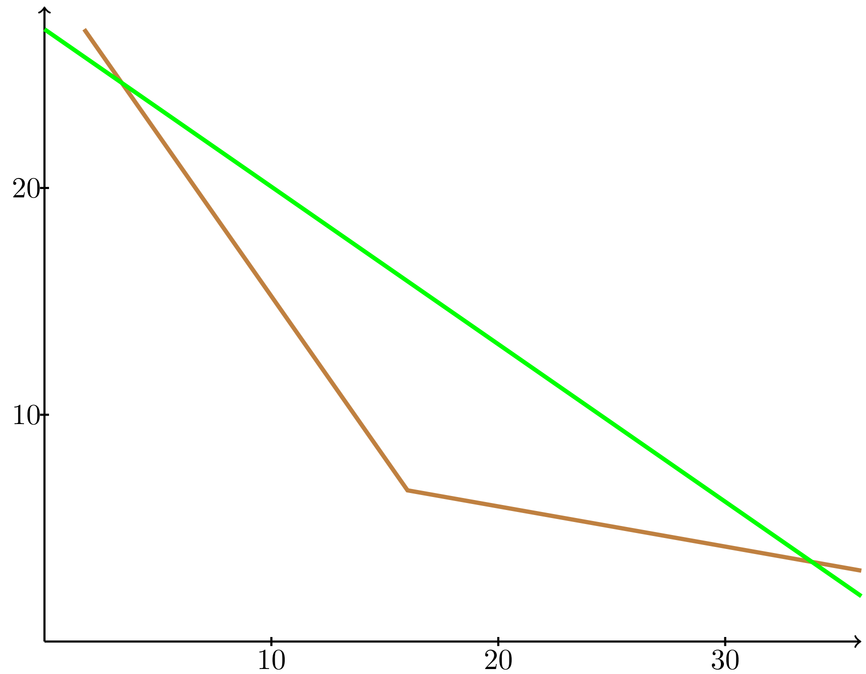


Figure 14: The complete LP frontier for the parts shown in brown in Figure 7, together with the complete LP frontier for the part shown in green

that defines the RVF, $z(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1})$, given by

$$z(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}) = \min\big\{c_I^0 x_I + c_C^0 x_C \ : (x_I, x_C) \in X_{\text{MO}}, \\ C_I^{1:l}x_I + C_C^{1:l}x_C \leq C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}\big\},$$

and it must be that

$$z(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}) = c_I^0\hat{x}_I + c_C^0\hat{x}_C < c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}.$$

Now since $\bar{z}^k$ is nonincreasing, and, by definition, $(\hat{x}_I, \hat{x}_C)$ satisfies

$$C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C \leq C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1},$$

it must be that

$$\bar{z}^k(C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C) \geq \bar{z}^k(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}).$$

Thus

$$\bar{z}^k(C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C) - (c_I^0\hat{x}_I + c_C^0\hat{x}_C) > \bar{z}^k(C_I^{1:l}x_I^{k+1} + C_C^{1:l}x_C^{k+1}) - (c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}),$$

and $(\hat{x}_I, \hat{x}_C)$ must be a feasible solution for the optimization problem in (14) having better objective value than (x_I^{k+1}, x_C^{k+1}), which is a contradiction. ∎

Next, we justify our claim that converting to a (strong) efficient solution via (15) again yields a solution to (14). We omit formal proof that the resulting solution must be efficient for (MO-MILP) since this is a straightforward and well-known result from multiobjective optimization.

Lemma G.2. *If* (x_I^{k+1}, x_C^{k+1}) *is defined by* (14) *and* $(\hat{x}_I, \hat{x}_C)$ *solves* (15) *then* $(\hat{x}_I, \hat{x}_C)$ *is also an optimal solution for the optimization problem in* (14)*.*

Proof. Let (x_I^{k+1}, x_C^{k+1}) be defined by (14) and suppose $(\hat{x}_I, \hat{x}_C)$ solves (15). Then $(\hat{x}_I, \hat{x}_C) \in X_{\text{MO}}$, so is feasible for the optimization problem in (14). Furthermore,

$$C_I\hat{x}_I + C_C\hat{x}_C \leq C_I x_I^{k+1} + C_C x_C^{k+1},$$

so

$$c_I^0\hat{x}_I + c_C^0\hat{x}_C \leq c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1},$$

and

$$C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C \leq C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}.$$

Since $\bar{z}^k$ is nonincreasing it must thus be that

$$\bar{z}^k(C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C) \geq \bar{z}^k(C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}),$$

and so

$$\bar{z}^k(C_I^{1:l}\hat{x}_I + C_C^{1:l}\hat{x}_C) - (c_I^0\hat{x}_I + c_C^0\hat{x}_C) \geq \bar{z}^k(C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}) - (c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1}).$$

Since $(\hat{x}_I, \hat{x}_C)$ is feasible for the optimization problem in (14) and has at least as good an objective value as that of (x_I^{k+1}, x_C^{k+1}), it must also solve the optimization problem in (14). ■

These two lemmas combine to prove that the integer part added to $\mathcal{S}^k$ at each iteration k of the algorithm has a nonempty stability region.

Proposition G.3. *If* x_I^{k+1} *is added to* $\mathcal{S}^k$ *in iteration* k *of the RVF Algorithm then its stability region* $\mathcal{C}(x_I^{k+1})$ *is nonempty.*

Proof. By Lemmas G.1 and G.2, if x_I^{k+1} is added to $\mathcal{S}^k$ in iteration k of the RVF Algorithm, then x_C^{k+1} found in the process has $c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1} = z(\zeta)$ where $\zeta = C_I^{1:l} x_I^{k+1} + C_C^{1:l} x_C^{k+1}$. But by the characterization of the value function as the minimum of a finite set of bounding functions (as per Theorem 3.8), we have that

$$z(\zeta) \leq \bar{z}(\zeta;\ x_I^{k+1}) \leq c_I^0 x_I^{k+1} + c_C^0 x_C^{k+1},$$

where the latter inequality follows by Lemma 5.1, since $(x_I^{k+1}, x_C^{k+1}) \in X_{\text{MO}}$. Thus it must be that $z(\zeta) = \bar{z}(\zeta;\ x_I^{k+1})$, so $\zeta \in \mathcal{C}(x_I^{k+1})$ and the stability region $\mathcal{C}(x_I^{k+1})$ is nonempty, as required. ■