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Explicit Convex Hull Description of Bivariate Quadratic Sets With Indicator Variables

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Explicit convex hull description of bivariate quadratic sets with indicator variables

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Abstract

We consider the nonconvex set $\mathcal{S}_n = \{(x, X, z) : X = xx^T,\ x(1-z) = 0,\ x \geq 0,\ z \in \{0,1\}^n\}$, which is closely related to the feasible region of several difficult nonconvex optimization problems such as the best subset selection and constrained portfolio optimization. Utilizing ideas from convex analysis and disjunctive programming, we obtain an explicit description for the closure of the convex hull of $\mathcal{S}_2$ in the space of original variables. In order to generate valid inequalities corresponding to supporting hyperplanes of the convex hull of $\mathcal{S}_2$, we present a simple separation algorithm that can be incorporated in branch-and-cut based solvers to enhance the quality of existing relaxations.

Key words. *Quadratic optimization; Indicator variables; Convex hull; Disjunctive programming; Perspective function; Projection.*

AMS subject classifications. 90C11, 90C20, 90C25, 90C26.

1 Introduction

Consider a mixed-integer quadratic optimization problem with indicator variables of the form:

$$\begin{aligned} \min \quad & x^T Q x + c^T x && \text{(MIQPI)} \\ \text{s.t.} \quad & x_i(1-z_i) = 0, \quad \forall i \in [n] \\ & Ax \leq \alpha,\ Bz \leq \beta \\ & x \in \mathbb{R}^n,\ z \in \{0,1\}^n, \end{aligned}$$

where Q, A, B are matrices of appropriate dimensions and Q is symmetric. The binary variables z are often referred to as *indicator variables*. Problem (MIQPI) subsumes several classes of difficult nonconvex optimization problems such as best subset selection [6], constrained portfolio optimization [7], and quadratic facility location [14], among others. It is important to note that we are not assuming that Q is positive semi-definite. In this paper, we are interested in the quality of convex relaxations for Problem (MIQPI).

To build convex relaxations for nonconvex problems, a common approach is to first linearize the objective function and move all the nonlinearity and nonconvexity to the feasible set. To this end, we introduce auxiliary variables $X_{ij} := x_i x_j$ for all $i, j \in [n]$ to obtain a reformulation of Problem (MIQPI) in a lifted space:

$$\min \quad \langle Q, X \rangle + c^T x \qquad \text{(MIQPI')}$$

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$$\begin{aligned} \text{s.t.} \quad & X = xx^T \\ & x_i(1-z_i) = 0, \quad \forall i \in [n] \\ & Ax \leq \alpha,\ Bz \leq \beta \\ & x \in \mathbb{R}^n,\ z \in \{0,1\}^n,\ X \in \mathbb{S}^n_+, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard matrix inner product, and $\mathbb{S}^n_+$ denotes the set of $n \times n$ positive semi-definite matrices. Throughout this paper, we assume that $Ax \leq \alpha$ implies $x \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ denotes the set of nonnegative real n-vectors. It then follows that, to construct strong convex relaxations for Problem (MIQPI'), regardless of the structure of Q, it is essential to effectively convexify the following nonconvex set:

$$\mathcal{S}_n = \Big\{(x, X, z) : \ X = xx^T,\ x(1-z) = 0,\ x \in \mathbb{R}^n_+,\ z \in \{0,1\}^n\Big\}. \tag{1}$$

Throughout this paper, given a set $\mathcal{C}$, we denote its convex hull by $\operatorname{conv}(\mathcal{C})$, and the closure of its convex hull by $\overline{\operatorname{conv}}(\mathcal{C})$. The simplest case of $\mathcal{S}_n$; i.e., the case with $n = 1$ has been studied in [1, 14]; leveraging on disjunctive programming techniques [19, 5, 10], the authors obtain an explicit characterization of the closure of the convex hull of $\mathcal{S}_1$ in the space of the original variables:

Proposition 1 ([1, 14])**.** *The closure of the convex hull of $\mathcal{S}_1$ is given by*

$$\overline{\operatorname{conv}}(\mathcal{S}_1) = \Big\{(x, X, z) : \ X_{11}z_1 \geq x_1^2,\ x_1 \geq 0,\ X_{11} \geq 0,\ z_1 \in [0,1]\Big\}. \tag{2}$$

Using Proposition 1, one obtains the following convex relaxation of Problem (MIQPI'):

$$\begin{aligned} \min \quad & \langle Q, X\rangle + c^T x \qquad \text{(Persp)} \\ \text{s.t.} \quad & X \succeq xx^T \\ & X_{ii} z_i \geq x_i^2, \quad \forall i \in [n] \\ & Ax \leq \alpha,\ Bz \leq \beta \\ & x \in \mathbb{R}^n,\ z \in [0,1]^n,\ X \in \mathbb{S}^n_+. \end{aligned}$$

Problem (Persp) is often referred to as the *perspective relaxation* and has shown to be very effective for problems with $Q \in \mathbb{S}^n_+$ such that Q is diagonal or is strongly diagonally dominant [12, 14, 11].

To further improve the quality of convex relaxations for Problem (MIQPI'), it is natural to study the facial structure of the convex hull of $\mathcal{S}_2$ defined by:

$$\mathcal{S}_2 := \Big\{(x, X, z) \in \mathbb{R}^2_+ \times \mathbb{S}^2_+ \times \{0,1\}^2 : X_{11} = x_1^2,\ X_{12} = x_1x_2,\ X_{22} = x_2^2,\ x_1(1-z_1) = 0, \\ x_2(1-z_2) = 0\Big\}. \tag{3}$$

To this date, obtaining an explicit characterization for the closure of the convex hull of $\mathcal{S}_2$ in the original space remains an open question. Nonetheless, several studies have considered variants of this problem and have made considerable progress [4, 15, 2]. In [4, 15], the authors study the convex hull of the epigraph of bivariate convex quadratic functions. Namely, in [4], the authors consider the nonconvex set $\mathcal{Z}_2^- := \{(x, t, z) : t \geq d_{11}x_1^2 - 2x_1x_2 + d_2x_2^2,\ x_i(1-z_i) = 0, x_i \geq 0, z_i \in \{0,1\}, i \in \{1,2\}\}$ with $d_1, d_2 > 0$, $d_1d_2 \geq 1$, and derive $\overline{\operatorname{conv}}(\mathcal{Z}_2^-)$ in the original space. Subsequently, in [15], they consider $\mathcal{Z}_2^+ := \{(x, t, z) : t \geq d_{11}x_1^2 + 2x_1x_2 + d_2x_2^2,\ x_i(1 - z_i) = 0, x_i \geq 0, z_i \in \{0,1\}, i \in \{1,2\}\}$, and derive $\overline{\operatorname{conv}}(\mathcal{Z}_2^+)$ in original space. Furthermore, utilizing an extended formulation for a relaxation of the convex hull of $\mathcal{S}_2$, they show that the resulting convex relaxation outperforms Problem (Persp) in terms of relaxation quality but is more expensive to solve due to the added variables associated with the extended formulation. In [2], the authors consider the set $\mathcal{S}_2$ with the additional constraint $x \leq 1$, and obtain an extended formulation for the convex hull of this set that is SDP representable. More results regarding the convexification of Problem (MIQPI) can be found in [11, 13, 21, 20].

In this paper, we obtain an explicit characterization for the closure of the convex hull of $\mathcal{S}_2$ in the space of original variables. Using disjunctive programming techniques, we first obtain a convex extended formulation for $\overline{\text{conv}}(\mathcal{S}_2)$. We then project out the auxiliary variables to express $\overline{\text{conv}}(\mathcal{S}_2)$ in the original space; namely, by building on ideas from convex analysis, we show that this projection can be performed via the analytical solution of a convex optimization problem. While the convex hull $\mathcal{S}_2$ has a complex structure, we show that the separation problem over this convex set can be solved via a simple algorithm. This algorithm can readily be incorporated in branch-and-cut based global solvers [18] to improve the quality of existing relaxations for nonconvex problems containing quadratic sets with indicator variables.

The rest of this paper is organized as follows. In Section 2 we provide the statement of our main result. Sections 3-6 contain the proof of our convex hull characterization. Finally, in Section 7 we present our separation algorithm.

2 Main result

The purpose of this paper is to characterize the closure of the convex hull of $\mathcal{S}_2$ in the space of original variables. To present this characterization, we first introduce some notation. Given a set $\mathcal{C}$, we denote its relative interior by $\text{ri}(\mathcal{C})$, and its closure by $\text{cl}(\mathcal{C})$. Consider the function $f(u,v) := \frac{u^2}{v}$, $u \in \mathbb{R}$ and $v > 0$. We define the closure of $f(u,v)$, denoted by $\hat{f}(u,v)$ as follows:

$$\hat{f}(u,v) = \begin{cases} \frac{u^2}{v}, & \text{if } v > 0 \\ 0, & \text{if } u = v = 0 \\ +\infty & \text{if } u \neq 0,\ v = 0. \end{cases}$$

Moreover, consider the function $f(u,v,w) := \frac{uv}{w}$, $u, v \geq 0$ and $w > 0$. We define the closure of $f(u,v)$, denoted by $\hat{f}(u,v)$ over $\mathbb{R}^3_+$ as follows:

$$\hat{f}(u,v) = \begin{cases} \frac{uv}{w}, & \text{if } w > 0 \\ 0, & \text{if } u = v = w = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

For notational simplicity, in the following when we write a function of the form $\frac{u^2}{v}$ or $\frac{uv}{w}$, we imply its closure as defined above.

The following theorem states our main result:

Theorem 1. *Consider the set $\mathcal{S}_2$ defined by (3). Then there exists a convex set $\tilde{\mathcal{S}}$ such that $\text{ri}(\text{conv}(\mathcal{S}_2)) \subseteq \tilde{\mathcal{S}} \subseteq \overline{\text{conv}}(\mathcal{S}_2)$ and the closure of the convex hull of $\mathcal{S}_2$ is given by:*

$$\overline{\text{conv}}(\mathcal{S}_2) = \bigcup_{i=1}^{8} \text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_i),$$

where the sets $\mathcal{R}_i$, $i \in \{1, \ldots, 8\}$ satisfy $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{8} \mathcal{R}_i \supseteq \overline{\text{conv}}(\mathcal{S}_2)$, and

I. if $i \in \{1, 2, 6\}$, then

$$\text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_i) = \left\{(x, X, z) : X_{11} \geq \frac{x_1^2}{z_1},\ X_{22} \geq \frac{x_2^2}{z_2},\ x_1, x_2 \geq 0,\ X_{12} \geq 0,\ z_1, z_2 \in [0,1]\right\} \cap \text{cl}(\mathcal{R}_i),$$

where

$$\mathcal{R}_1 := \Big\{(x,X,z) : x_1x_2(z_1+z_2-1) \le X_{12}z_1z_2 \le x_1x_2\min\{z_1,z_2\}\Big\},$$

$$\mathcal{R}_2 := \Big\{(x,X,z) : z_1 \le z_2,\ X_{12}z_2 > x_1x_2,\ X_{12}z_1 \le x_1x_2, \\ x_1^2(z_2-z_1)(X_{22}z_2-x_2^2) \ge z_1(X_{12}z_2-x_1x_2)^2\Big\},$$

$$\mathcal{R}_6 := \Big\{(x,X,z) : X_{12}z_1z_2 < x_1x_2(z_1+z_2-1), \\ (1-z_1)(z_1+z_2-1)x_1^2(X_{22}z_2-x_2^2) \ge \Big(X_{12}z_1z_2 - x_1x_2(z_1+z_2-1)\Big)^2\Big\}.$$

II. If $i \in \{3,4\}$, then

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_i) = \Big\{(x,X,z) : X_{22} \ge \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_2}\Big)\Big(X_{22} - \frac{x_2^2}{z_2}\Big) \ge \Big(X_{12} - \frac{x_1x_2}{z_2}\Big)^2,\ x_1,x_2 \ge 0, \\ X_{12} \ge 0,\ z_1,z_2 \in [0,1] \le 1\Big\} \cap \mathrm{cl}(\mathcal{R}_i),$$

where

$$\mathcal{R}_3 := \Big\{(x,X,z) :\ z_1 < z_2,\ X_{12}x_2 > X_{22}x_1,\ z_1(X_{12}z_2 - x_1x_2)^2 > x_1^2(z_2-z_1)(X_{22}z_2-x_2^2)\Big\},$$

$$\mathcal{R}_4 := \{(x,X,z) :\ z_2 \le z_1,\ X_{12}x_2 > X_{22}x_1\}.$$

III. If $i = 5$, then

$$\mathrm{cl}(\tilde{S} \cap \mathcal{R}_5) = \Big\{(x,X,z) : X_{22} \ge \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_1}\Big)\Big(X_{22} - \frac{x_2^2}{z_1}\Big) \ge \Big(X_{12} - \frac{x_1x_2}{z_1}\Big)^2,\ x_1,x_2 \ge 0, \\ X_{12} \ge 0,\ z_1,z_2 \in [0,1]\Big\} \cap \mathrm{cl}(\mathcal{R}_5),$$

where

$$\mathcal{R}_5 = \{(x,X,z) :\ X_{12}z_1 > x_1x_2,\ X_{22}x_1 \ge X_{12}x_2\},$$

IV. If $i = 7$, then

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_7) = \Big\{(x,X,z) : X_{11} \ge x_1^2,\ X_{22} \ge \frac{x_2^2}{z_2},\ (X_{11}-x_1^2)(X_{22}-x_2^2) \ge (X_{12}-x_1x_2)^2, \\ x_1,x_2 \ge 0,\ X_{12} \ge 0,\ z_1,z_2 \in [0,1]\Big\} \cap \mathrm{cl}(\mathcal{R}_7),$$

where

$$\mathcal{R}_7 := \Big\{(x,X,z) :\ X_{12}z_1z_2 < x_1x_2(z_1+z_2-1),\ x_1^2(x_2^2 - X_{22}(1-z_1))(X_{22}z_2-x_2^2) \\ > 2x_1x_2X_{12}z_1(X_{22}z_2-x_2^2) - X_{12}^2(X_{22}(z_1+z_2-1) + x_2^2(1-2z_1-z_2(1-z_1)))\Big\}.$$

V. If $i = 8$, then

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_8) = \Big\{(x,X,z) :\ X_{11} \ge \frac{x_1^2}{z_1},\ X_{22} \ge \frac{x_2^2}{z_2},\ x_1,x_2 \ge 0,\ X_{12} \ge 0,\ z_1,z_2 \in [0,1], \\ z_1(1-z_2)\Big(X_{11} - \frac{x_1^2}{z_1}\Big)x_2^2 \ge (z_1+z_2-1)\Big(X_{12}\frac{z_1z_2}{W} - x_1x_2\Big)^2\Big\} \cap \mathrm{cl}(\mathcal{R}_8),$$

where

$$\mathcal{R}_8 := \Big\{(x,X,z) :\ X_{12}z_1z_2 < x_1x_2(z_1+z_2-1)\Big\} \setminus (\mathcal{R}_6 \cup \mathcal{R}_7),$$

and

$$W := (z_1+z_2-1) - \frac{1}{x_2}\sqrt{(X_{22}z_2-x_2^2)(1-z_1)(z_1+z_2-1)}.$$

The proof of Theorem 1 is given in Sections 3-6.

Clearly, the convex hull of $\mathcal{S}_2$ has a complicated structure; however, it can be easily incorporated in branch-and-cut based global solvers [18] for generating strong cutting planes that enhance the quality of existing relaxations. While implementing a separation algorithm and testing its impact on global solvers is beyond the scope of this paper, in Section 7, we outline a simple separation algorithm over the closure of the convex hull of $\mathcal{S}_2$.

We conclude this section by presenting a simple yet strong class of valid inequalities for $\mathcal{S}_2$, which are a direct consequence of Theorem 1.

Corollary 1. *Consider the set $\mathcal{S}_2$ defined by* (1)*. We have the following cases:*

(i) over the region defined by

$$\Big\{(x, X, z): \ X_{12}z_1 > x_1x_2,\ X_{22}x_1 \geq X_{12}x_2,\ x_1, x_2 \geq 0,\ z_1, z_2 \in [0, 1]\Big\}, \tag{4}$$

any supporting hyperplane to the convex set:

$$\begin{pmatrix} z_1 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix} \succeq 0 \tag{5}$$

is a valid inequality for $\overline{\text{conv}}(\mathcal{S}_2)$.

(ii) over the region defined by

$$\Big\{(x, X, z): \ X_{12}z_2 > x_1x_2,\ X_{11}x_2 \geq X_{12}x_1,\ x_1, x_2 \geq 0,\ z_1, z_2 \in [0, 1]\Big\}, \tag{6}$$

any supporting hyperplane to the convex set:

$$\begin{pmatrix} z_2 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix} \succeq 0 \tag{7}$$

is a valid inequality for $\overline{\text{conv}}(\mathcal{S}_2)$.

The proof of Corollary 1 is given in Section 7.

In [3], the authors propose the following *Rank-1 inequality*:

$$\begin{pmatrix} z_1 + z_2 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix} \succeq 0, \tag{8}$$

and demonstrate that the addition of such constraints to Problem (Persp), significantly improves the quality of convex relaxations for Problem (MIQPI). Corollary 1 implies that at any (x, X, z) belonging to either set (4) or (6), the supporting hyperplane of sets (5) or (7) implies the supporting hyperplane of set (8). The incorporation of the proposed relaxations in a branch-and-cut based global solver and performing a computational study is a subject of future research.

3 Disjunctive Formulation

The purpose of this and the next two sections is to prove Theorem 1. Due to the length of the proof, we will redefine the notations used in the statement of Theorem 1 along the way. We start by obtaining an extended formulation for the convex hull of $\mathcal{S}_2$ using a standard disjunctive programming technique [19, 5, 10, 16, 17]. Subsequently, by projecting out the auxiliary variables we obtain an explicit description of the convex hull in the space of original variables. Moreover, to obtain the closure of the convex hull, we make use of the following lemma:

Lemma 1. *Let $k \geq 2$ and consider the sets $\mathcal{C}, \mathcal{P}_1, \ldots, \mathcal{P}_k$ in $\mathbb{R}^n$ such that $\mathcal{C} = \bigcup_{i=1}^k \mathcal{P}_i$. Then*

$$\overline{\mathrm{conv}}(\mathcal{C}) = \overline{\mathrm{conv}}\Big(\bigcup_{i=1}^{k} \overline{\mathrm{conv}}(\mathcal{P}_i)\Big).$$

Proof. Since $\mathrm{conv}(\mathcal{C}) \subseteq \mathrm{conv}(\overline{\mathrm{conv}}(\mathcal{P}_1) \cup \ldots \cup \overline{\mathrm{conv}}(\mathcal{P}_k))$, we have $\overline{\mathrm{conv}}(\mathcal{C}) \subseteq \overline{\mathrm{conv}}(\overline{\mathrm{conv}}(\mathcal{P}_1) \cup \ldots \cup \overline{\mathrm{conv}}(\mathcal{P}_k))$. Hence it suffices to prove the reverse inclusion.

Since for any set $\mathcal{C}$ we have $\mathrm{conv}(\overline{\mathcal{C}}) \subseteq \overline{\mathrm{conv}}(\mathcal{C})$, it follows that $\mathrm{conv}(\overline{\mathrm{conv}}(\mathcal{P}_1) \cup \ldots \cup \overline{\mathrm{conv}}(\mathcal{P}_k)) \subseteq \overline{\mathrm{conv}}(\mathcal{C})$. Moreover, since $\overline{\mathrm{conv}}(\mathcal{C})$ is closed, we get $\overline{\mathrm{conv}}(\overline{\mathrm{conv}}(\mathcal{P}_1) \cup \ldots \cup \overline{\mathrm{conv}}(\mathcal{P}_k)) \subseteq \overline{\mathrm{conv}}(\mathcal{C})$, and this completes the proof. □

3.1 Defining the disjunctions

Let us rewrite the set $\mathcal{S}_2$ defined by (3) as

$$\mathcal{S}_2 = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4,$$

where we define

$$\begin{aligned}
\mathcal{P}_1 &:= \{(x, X, z) : z_1 = z_2 = 0,\ x_1 = x_2 = X_{11} = X_{12} = X_{22} = 0\},\\
\mathcal{P}_2 &:= \{(x, X, z) : z_1 = 1, z_2 = 0,\ X_{11} = x_1^2,\ x_2 = X_{12} = X_{22} = 0,\ x_1 \geq 0\},\\
\mathcal{P}_3 &:= \{(x, X, z) : z_1 = 0, z_2 = 1,\ x_1 = X_{11} = X_{12} = 0,\ X_{22} = x_2^2,\ x_2 \geq 0\},\\
\mathcal{P}_4 &:= \{(x, X, z) : z_1 = z_2 = 1,\ X_{11} = x_1^2,\ X_{12} = x_1 x_2,\ X_{22} = x_2^2,\ x_1, x_2 \geq 0\}.
\end{aligned}$$

To construct the closure of the convex hull of $\mathcal{S}_2$, we employ a sequential approach, where we first convexify each $\mathcal{P}_i$, $i \in \{1, \ldots, 4\}$ and then by Lemma 1, let

$$\overline{\mathrm{conv}}(\mathcal{S}_2) = \overline{\mathrm{conv}}\Big(\overline{\mathrm{conv}}(\mathcal{P}_1) \cup \overline{\mathrm{conv}}(\mathcal{P}_2) \cup \overline{\mathrm{conv}}(\mathcal{P}_3) \cup \overline{\mathrm{conv}}(\mathcal{P}_4)\Big). \tag{9}$$

It then follows that:

$$\begin{aligned}
\overline{\mathrm{conv}}(\mathcal{P}_1) &= \{(x, X, z) : z_1 = z_2 = 0,\ x_1 = x_2 = X_{11} = X_{12} = X_{22} = 0\}\\
\overline{\mathrm{conv}}(\mathcal{P}_2) &= \{(x, X, z) : z_1 = 1, z_2 = 0,\ X_{11} \geq x_1^2,\ x_2 = X_{12} = X_{22} = 0,\ x_1 \geq 0\},\\
\overline{\mathrm{conv}}(\mathcal{P}_3) &= \{(x, X, z) : z_1 = 0, z_2 = 1,\ x_1 = X_{11} = X_{12} = 0,\ X_{22} \geq x_2^2,\ x_2 \geq 0\},\\
\overline{\mathrm{conv}}(\mathcal{P}_4) &= \left\{(x, X, z) : z_1 = z_2 = 1,\ \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix} \succeq 0, x_1 \geq 0, x_2 \geq 0, X_{12} \geq 0 \right\},
\end{aligned} \tag{10}$$

where to construct $\overline{\mathrm{conv}}(\mathcal{P}_4)$, we made use of a well-known result stating that for $n \leq 4$, the cone of doubly nonnegative matrices coincides with the cone of completely positive matrices (see for example [9]). Before proceeding further, we obtain an equivalent description for $\overline{\mathrm{conv}}(\mathcal{P}_4)$. To this end we make use of the following lemma:

Lemma 2. *The constraint $\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix} \succeq 0$ is equivalent to the following inequalities:*

$$X_{11} \geq x_1^2, \quad X_{22} \geq x_2^2, \quad (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1 x_2)^2.$$

Proof. By characterization of positive semidefinite matrices in terms of the sign of their principal minors, the constraint $\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix} \succeq 0$ can be equivalently written as:

$$X_{11} \geq x_1^2, \quad X_{22} \geq x_2^2 \tag{11}$$

$$X_{11}X_{22} \geq X_{12}^2 \tag{12}$$
$$(X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1x_2)^2 \tag{13}$$

It then suffices to show that inequalities (11) and (13) imply inequality (12). Two cases arise:

- $X_{12} \leq x_1x_2$:

$$X_{11}X_{22} - X_{12}^2 \geq x_1^2X_{22} - 2x_1x_2X_{12} + x_2^2X_{11} \geq 2x_1x_2(x_1x_2 - X_{12}) \geq 0,$$

  where the first inequality follows from (13) and the second inequality follows from (11).

- $X_{12} \geq x_1x_2$: since by (11) we have $X_{11} \geq 0$ and $X_{22} \geq 0$, it follows that (13) can be equivalently written as:

$$\left(\sqrt{X_{11}X_{22}} - X_{12}\right)\left(\sqrt{X_{11}X_{22}} + X_{12} - 2x_1x_2\right) \geq \left(x_1\sqrt{X_{22}} - x_2\sqrt{X_{11}}\right)^2$$

  Moreover, by (11) we have $\sqrt{X_{11}X_{22}} + X_{12} - 2x_1x_2 \geq X_{12} - x_1x_2 \geq 0$, where the second inequality follows from assumption. Hence, we must have $\sqrt{X_{11}X_{22}} - X_{12} \geq 0$; i.e., inequality (12) is satisfied.

□

Therefore by Lemma 2, the closure of the convex hull of $\mathcal{P}_4$ can be equivalently written as:

$$\begin{aligned}\overline{\text{conv}}(\mathcal{P}_4) = \Big\{&(x, X, z) : z_1 = z_2 = 1,\ X_{11} \geq x_1^2,\ X_{22} \geq x_2^2,\\ &(X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1x_2)^2, x_1 \geq 0,\ x_2 \geq 0,\ X_{12} \geq 0\Big\}.\end{aligned} \tag{14}$$

3.2 A convex extended formulation for the convex hull

The convex hull of $\mathcal{S}_2$ can be obtained by taking the convex hull of the union of four convex sets as defined by (9) and (10). Using the standard disjunctive programming technique [19, 5], the closure of the convex hull of $\mathcal{S}_2$ is given by:

$$\overline{\text{conv}}(\mathcal{S}_2) = \text{cl}\Big\{(x, X, z) :\ \exists(x, X, z, \tilde{x}, \tilde{X}, \lambda) \in \Sigma\Big\},$$

where

$$\begin{aligned}\Sigma := \Big\{&(x, X, z, \tilde{x}, \tilde{X}, \lambda) : x = \sum_{i=1}^4 \tilde{x}^i,\ X = \sum_{i=1}^4 \tilde{X}^i,\ z = \sum_{i=1}^4 \lambda_i z^i,\ \sum_{i=1}^4 \lambda_i = 1, \lambda_i \geq 0,\ i \in \{1, \ldots, 4\},\\ &\left(\frac{\tilde{x}^i}{\lambda_i}, \frac{\tilde{X}^i}{\lambda_i}, z^i\right) \in \overline{\text{conv}}(\mathcal{P}_i),\ \forall\, i : \lambda_i > 0,\ \tilde{x}^i = \tilde{X}^i = 0,\ \forall\, i : \lambda_i = 0\Big\}.\end{aligned} \tag{15}$$

It is important to note that Σ is a convex set; indeed, the convexity of Σ follows from the fact that the inverse image of a convex set under the perspective function is convex (see for example Section 2.3.3 of [8]). By (10), (14), and (15), we have:

$$\begin{aligned}\Sigma = \Big\{&(x, X, z, \tilde{x}, \tilde{X}, \lambda) :\\ &\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,\ \lambda_i \geq 0, i \in \{1, \ldots, 4\},\ z_1 = \lambda_2 + \lambda_4,\ z_2 = \lambda_3 + \lambda_4,\\ &x_1 = \tilde{x}_1^2 + \tilde{x}_1^4,\ X_{11} = \tilde{X}_{11}^2 + \tilde{X}_{11}^4,\ x_2 = \tilde{x}_2^3 + \tilde{x}_2^4,\ X_{22} = \tilde{X}_{22}^3 + \tilde{X}_{22}^4,\ X_{12} = \tilde{X}_{12}^4\\ &\begin{cases} \lambda_2\tilde{X}_{11}^2 \geq (\tilde{x}_1^2)^2,\ \tilde{x}_1^2 \geq 0, & \text{if } \lambda_2 \neq 0\\ \tilde{x}_1^2 = \tilde{X}_{11}^2 = 0, & \text{otherwise}\end{cases}\end{aligned}$$

$$
\begin{cases} \lambda_3 \tilde{X}^3_{22} \geq (\tilde{x}^3_2)^2,\ \tilde{x}^3_2 \geq 0, & \text{if } \lambda_3 \neq 0 \\ \tilde{x}^3_2 = \tilde{X}^3_{22} = 0, & \text{otherwise} \end{cases}
$$

$$
\begin{cases} \lambda_4 \tilde{X}^4_{11} \geq (\tilde{x}^4_1)^2,\ \lambda_4 \tilde{X}^4_{22} \geq (\tilde{x}^4_2)^2,\ (\lambda_4 \tilde{X}^4_{11} - (\tilde{x}^4_1)^2)(\lambda_4 \tilde{X}^4_{22} - (\tilde{x}^4_2)^2) \geq (\lambda_4 \tilde{X}^4_{12} - \tilde{x}^4_1 \tilde{x}^4_2)^2, & \\ \tilde{x}^4_1 \geq 0,\ \tilde{x}^4_2 \geq 0,\ \tilde{X}^4_{12} \geq 0, & \text{if } \lambda_4 \neq 0 \\ \tilde{x}^4_1 = \tilde{x}^4_2 = \tilde{X}^4_{11} = \tilde{X}^4_{12} = \tilde{X}^4_{22} = 0, & \text{otherwise} \end{cases}
$$

$$
\Big\}. \tag{16}
$$

Before proceeding with the projection step, we make use of the following result, which allows us to simplify the description of Σ defined by (16) by discarding some parts of its boundary.

Theorem 2 (Theorem 6.9 in [19])**.** *Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be nonempty convex sets in $\mathbb{R}^n$, and let $\mathcal{C}_0 = \mathrm{conv}(C_1 \cup \ldots \cup C_m)$. Then*

$$
\mathrm{ri}(\mathcal{C}_0) = \bigcup \Big\{ \lambda_1 \mathrm{ri}(\mathcal{C}_1) + \ldots \lambda_m \mathrm{ri}(\mathcal{C}_m) : \ \lambda_i > 0, \forall i \in [m],\ \sum_{i=1}^m \lambda_i = 1 \Big\}.
$$

For any set convex set $\mathcal{D}$ such that $\mathrm{ri}(\mathcal{C}) \subseteq \mathcal{D} \subseteq \mathrm{cl}(\mathcal{C})$, we have $\mathrm{cl}(\mathcal{D}) = \mathrm{cl}(\mathcal{C})$ (see for example, Theorem 6.3 in [19]). Together with Theorem 2, this implies that to construct $\overline{\mathrm{conv}}(\mathcal{S}_2)$,

(i) in (10), we can replace all constraints of the form $x_1 \geq 0$, $x_2 \geq 0$, and $X_{12} \geq 0$, by $x_1 > 0$, $x_2 > 0$, and $X_{12} > 0$, respectively,

(ii) in (16), we can replace $\lambda_4 \geq 0$ by $\lambda_4 > 0$.

Hence we obtain the following description of the closure of the convex hull of $\mathcal{S}_2$:

$$
\overline{\mathrm{conv}}(\mathcal{S}_2) = \mathrm{cl}\Big\{ (x, X, z) : \ \exists (x, X, z, \tilde{x}, \tilde{X}, \lambda) \in \Sigma' \Big\},
$$

where

$$
\begin{aligned}
\Sigma' := \Big\{ & (x, X, z, \tilde{x}, \tilde{X}, \lambda) : x_1, x_2, X_{12} > 0, \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,\ \lambda_i \geq 0, i \in \{1,2,3\},\ \lambda_4 > 0,\ z_1 = \lambda_2 + \lambda_4,\ z_2 = \lambda_3 + \lambda_4, \\
& x_1 = \tilde{x}^2_1 + \tilde{x}^4_1,\ X_{11} = \tilde{X}^2_{11} + \tilde{X}^4_{11},\ x_2 = \tilde{x}^3_2 + \tilde{x}^4_2,\ X_{22} = \tilde{X}^3_{22} + \tilde{X}^4_{22},\ X_{12} = \tilde{X}^4_{12} \\
& \begin{cases} \lambda_2 \tilde{X}^2_{11} \geq (\tilde{x}^2_1)^2,\ \tilde{x}^2_1 \geq 0, & \text{if } \lambda_2 \neq 0 \\ \tilde{x}^2_1 = \tilde{X}^2_{11} = 0, & \text{otherwise} \end{cases} \\
& \begin{cases} \lambda_3 \tilde{X}^3_{22} \geq (\tilde{x}^3_2)^2,\ \tilde{x}^3_2 \geq 0, & \text{if } \lambda_3 \neq 0 \\ \tilde{x}^3_2 = \tilde{X}^3_{22} = 0, & \text{otherwise} \end{cases} \\
& \lambda_4 \tilde{X}^4_{11} \geq (\tilde{x}^4_1)^2,\ \lambda_4 \tilde{X}^4_{22} \geq (\tilde{x}^4_2)^2,\ (\lambda_4 \tilde{X}^4_{11} - (\tilde{x}^4_1)^2)(\lambda_4 \tilde{X}^4_{22} - (\tilde{x}^4_2)^2) \geq (\lambda_4 \tilde{X}^4_{12} - \tilde{x}^4_1 \tilde{x}^4_2)^2, \\
& \tilde{x}^4_1 \geq 0,\ \tilde{x}^4_2 \geq 0,\ \tilde{X}^4_{12} \geq 0. \\
\Big\}. &
\end{aligned} \tag{17}
$$

In the remainder of the proof, our objective is to project out variables $(\tilde{x}, \tilde{X}, \lambda)$ from the description of Σ' and construct $\overline{\mathrm{conv}}(\mathcal{S}_2)$. We perform the projection in a number of steps:

(i) Project out $\lambda_1, \lambda_2, \lambda_3, \tilde{X}^4_{12}$:

$$
\begin{aligned}
\overline{\mathrm{conv}}(\mathcal{S}_2) = \mathrm{cl}\Big\{ & (x, X, z) : \ \exists (x, X, z, \tilde{x}, \tilde{X}, \lambda_4) : \ x_1, x_2, X_{12} > 0, \\
& z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\},\ \lambda_4 > 0,
\end{aligned}
$$

$$x_1 = \tilde{x}_1^2 + \tilde{x}_1^4,\ X_{11} = \tilde{X}_{11}^2 + \tilde{X}_{11}^4,\ x_2 = \tilde{x}_2^3 + \tilde{x}_2^4,\ X_{22} = \tilde{X}_{22}^3 + \tilde{X}_{22}^4,$$

$$\begin{cases} (z_1 - \lambda_4)\tilde{X}_{11}^2 \geq (\tilde{x}_1^2)^2,\ \tilde{x}_1^2 \geq 0, & \text{if } \lambda_4 \neq z_1 \\ \tilde{x}_1^2 = \tilde{X}_{11}^2 = 0, & \text{otherwise} \end{cases}$$

$$\begin{cases} (z_2 - \lambda_4)\tilde{X}_{22}^3 \geq (\tilde{x}_2^3)^2,\ \tilde{x}_2^3 \geq 0, & \text{if } \lambda_4 \neq z_2 \\ \tilde{x}_2^3 = \tilde{X}_{22}^3 = 0, & \text{otherwise} \end{cases}$$

$$\lambda_4\tilde{X}_{11}^4 \geq (\tilde{x}_1^4)^2,\ \lambda_4\tilde{X}_{22}^4 \geq (\tilde{x}_2^4)^2,\ (\lambda_4\tilde{X}_{11}^4 - (\tilde{x}_1^4)^2)(\lambda_4\tilde{X}_{22}^4 - (\tilde{x}_2^4)^2) \geq (\lambda_4 X_{12} - \tilde{x}_1^4\tilde{x}_2^4)^2,$$

$$\tilde{x}_1^4 \geq 0,\ \tilde{x}_2^4 \geq 0$$

$$\Big\}.$$

(ii) Project out $\tilde{x}_1^2, \tilde{X}_{11}^2, \tilde{x}_2^3, \tilde{X}_{22}^3$:

$$\overline{\text{conv}}(\mathcal{S}_2) = \text{cl}\Big\{(x, X, z) :\ \exists(x, X, z, \tilde{x}, \tilde{X}, \lambda_4) :\ x_1, x_2, X_{12} > 0,$$

$$z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\},\ \lambda_4 > 0,$$

$$\begin{cases} (z_1 - \lambda_4)(X_{11} - \tilde{X}_{11}^4) \geq (x_1 - \tilde{x}_1^4)^2,\ \tilde{x}_1^4 \leq x_1, & \text{if } \lambda_4 \neq z_1 \\ \tilde{x}_1^4 = x_1,\ \tilde{X}_{11}^4 = X_{11}, & \text{otherwise} \end{cases}$$

$$\begin{cases} (z_2 - \lambda_4)(X_{22} - \tilde{X}_{22}^4) \geq (x_2 - \tilde{x}_2^4)^2,\ \tilde{x}_2^4 \leq x_2, & \text{if } \lambda_4 \neq z_2 \\ \tilde{x}_2^4 = x_2,\ \tilde{X}_{22}^4 = X_{22}, & \text{otherwise} \end{cases}$$

$$\lambda_4\tilde{X}_{11}^4 \geq (\tilde{x}_1^4)^2,\ \lambda_4\tilde{X}_{22}^4 \geq (\tilde{x}_2^4)^2,\ (\lambda_4\tilde{X}_{11}^4 - (\tilde{x}_1^4)^2)(\lambda_4\tilde{X}_{22}^4 - (\tilde{x}_2^4)^2) \geq (\lambda_4 X_{12} - \tilde{x}_1^4\tilde{x}_2^4)^2,$$

$$\tilde{x}_1^4 \geq 0,\ \tilde{x}_2^4 \geq 0$$

$$\Big\}.$$

(iii) Project out $\tilde{X}_{11}^4, \tilde{X}_{22}^4$: note that by $\lambda_4 > 0$ and $\lambda_4 \leq \min\{z_1, z_2\}$, we have $z_1, z_2 > 0$.

$$\overline{\text{conv}}(\mathcal{S}_2) = \text{cl}\Big\{(x, X, z) :\ \exists(x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) :\ x_1, x_2, X_{12} > 0$$

$$z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\},\ \lambda_4 > 0,$$

$$\begin{cases} X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} \geq 0,\ X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0, & \\ \left(X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4}\right)\left(X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{(z_2 - \lambda_4)}\right) \geq \left(X_{12} - \frac{\tilde{x}_1^4\tilde{x}_2^4}{\lambda_4}\right)^2, & \\ 0 \leq \tilde{x}_1^4 \leq x_1,\ 0 \leq \tilde{x}_2^4 \leq x_2, & \text{if } \lambda_4 \neq z_1, z_2 \\ X_{11} - \frac{x_1^2}{z_1} \geq 0,\ X_{22} - \frac{(\tilde{x}_2^4)^2}{z_1} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - z_1} \geq 0, & \\ \left(X_{11} - \frac{x_1^2}{z_1}\right)\left(X_{22} - \frac{(\tilde{x}_2^4)^2}{z_1} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - z_1}\right) \geq \left(X_{12} - \frac{x_1\tilde{x}_2^4}{z_1}\right)^2, & \\ 0 \leq \tilde{x}_2^4 \leq x_2, & \text{if } \lambda_4 = z_1,\ z_1 < z_2 \\ X_{11} - \frac{(\tilde{x}_1^4)^2}{z_2} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} \geq 0,\ X_{22} - \frac{x_2^2}{z_2} \geq 0, & \\ \left(X_{11} - \frac{(\tilde{x}_1^4)^2}{z_2} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - z_2}\right)\left(X_{22} - \frac{x_2^2}{z_2}\right) \geq \left(X_{12} - \frac{\tilde{x}_1^4 x_2}{z_2}\right)^2, & \\ 0 \leq \tilde{x}_1^4 \leq x_1, & \text{if } \lambda_4 = z_2,\ z_2 < z_1 \\ X_{11} - \frac{x_1^2}{\lambda_4} \geq 0,\ X_{22} - \frac{x_2^2}{\lambda_4} \geq 0, & \\ \left(X_{11} - \frac{x_1^2}{\lambda_4}\right)\left(X_{22} - \frac{x_2^2}{\lambda_4}\right) \geq \left(X_{12} - \frac{x_1 x_2}{\lambda_4}\right)^2, & \text{if } \lambda_4 = z_1 = z_2 \end{cases}$$

$$\Big\}.$$

Consider the function $f(u, v) := \frac{u^2}{v}$, $u \in \mathbb{R}$ and $v > 0$. We define the closure of $f(u, v)$, denoted by $\hat{f}(u,v)$ as follows:

$$\hat{f}(u,v) = \begin{cases} \frac{u^2}{v}, & \text{if } v > 0 \\ 0, & \text{if } u = v = 0 \\ +\infty & \text{if } u \neq 0,\ v = 0. \end{cases} \tag{18}$$

For notational simplicity, in the following whenever we write a function of the form $\frac{u^2}{v}$, we imply its closure as defined by (18). We use a similar convention when composing this function by an affine mapping. It then follows that the convex hull of $\mathcal{S}_2$ can be equivalently written as:

$$\overline{\text{conv}}(\mathcal{S}_2) = \text{cl}\Big\{(x, X, z):\ \exists (x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) \in \tilde{\Sigma}\Big\}, \tag{19}$$

where we define

$$\begin{aligned} \tilde{\Sigma} :=\Big\{&(x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) : x_1, x_2, X_{12} > 0,\ z_1 + z_2 - 1 \le \lambda_4 \le \min\{z_1, z_2\},\ \lambda_4 > 0 \\ &X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} \ge 0,\ X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \ge 0, \\ &\Big(X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4}\Big)\Big(X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{(z_2 - \lambda_4)}\Big) \ge \Big(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}\Big)^2, \\ &0 \le \tilde{x}_1^4 \le x_1,\ 0 \le \tilde{x}_2^4 \le x_2\Big\}. \end{aligned} \tag{20}$$

The set $\tilde{\Sigma}$ is a convex; to see this, note that $\tilde{\Sigma}$ is the projection of Σ' onto the space $(x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4)$. The set Σ' is convex as it is obtained by removing some boundary points from the convex set Σ. Since convexity is preserved under projection, it follows that $\tilde{\Sigma}$ is a convex set.

3.3 A piece-by-piece characterization of the convex hull: the simple piece

Henceforth, we denote by $\tilde{\mathcal{S}}$ the projection of $\tilde{\Sigma}$ defined by (20) onto the space (x, X, z). The convexity of $\tilde{\mathcal{S}}$ follows from the convexity of $\tilde{\Sigma}$. Moreover, by Theorem 2 we have $\text{ri}(\text{conv}\mathcal{S}_2) \subseteq \tilde{\mathcal{S}} \subseteq \overline{\text{conv}}(\mathcal{S}_2)$, implying

$$\overline{\text{conv}}(\mathcal{S}_2) = \text{cl}(\tilde{\mathcal{S}}).$$

As we detail next, given a partition $\mathcal{R}_k,\ k \in K$ such that $\bigcup_k \mathcal{R}_k \supset \overline{\text{conv}}(\mathcal{S}_2)$, we characterize $\tilde{\mathcal{S}} \cap \mathcal{R}_k$ for all $k \in K$. Subsequently, to obtain the closure of the convex hull of $\mathcal{S}_2$, we use the following lemma:

Lemma 3. *Consider a convex set $\mathcal{C}$ and k sets $\mathcal{R}_1, \dots, \mathcal{R}_k$ such that $\cup_{i=1}^k \mathcal{R}_i \supseteq \text{cl}(\mathcal{C})$. Then*

$$\text{cl}(\mathcal{C}) = \bigcup_{i=1}^k \text{cl}(\mathcal{C} \cap \mathcal{R}_i).$$

Proof. We have

$$\text{cl}(\mathcal{C}) = \text{cl}\Big(\bigcup_{i=1}^k (\mathcal{C} \cap \mathcal{R}_i)\Big) = \bigcup_{i=1}^k \text{cl}(\mathcal{C} \cap \mathcal{R}_i),$$

where the last equality holds because the inclusion $\bigcup_{i=1}^k \text{cl}(\mathcal{C} \cap \mathcal{R}_i) \subset \text{cl}(\bigcup_{i=1}^k (\mathcal{C} \cap \mathcal{R}_i))$ is trivial, while if $x \in \text{cl}(\bigcup_{i=1}^k (\mathcal{C} \cap \mathcal{R}_i))$, then there exists a sequence $\{x_n\} \subset \bigcup_{i=1}^k (\mathcal{C} \cap \mathcal{R}_i)$ such that $x_n \to x$. Hence, up to passing to a subsequence, there exists $j \in [k]$ such that $\{x_n\} \subset \mathcal{C} \cap \mathcal{R}_j$, implying $x \in \text{cl}(\mathcal{C} \cap \mathcal{R}_i) \subset \bigcup_{i=1}^k \text{cl}(\mathcal{C} \cap \mathcal{R}_i)$. □

Define the set

$$\mathcal{C} := \Big\{(x, X, z) : X_{11} \geq \frac{x_1^2}{z_1},\ X_{22} \geq \frac{x_2^2}{z_2},\ X_{12} \geq 0,\ x_1, x_2 \geq 0,\ z_1, z_2 \in [0, 1]\Big\}.$$

The set $\mathcal{C}$ is closed and convex; moreover from Proposition 1 it follows that $\mathcal{C} \supseteq \overline{\text{conv}}(\mathcal{S}_2)$. Define

$$\mathcal{R}_1 := \Big\{(x, X, z) :\ x_1 x_2(z_1 + z_2 - 1) \leq X_{12} z_1 z_2 \leq x_1 x_2 \min\{z_1, z_2\}\Big\}. \tag{21}$$

We claim that

$$\mathcal{C} \cap \mathcal{R}_1 = \text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_1). \tag{22}$$

Since $\text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_1) \subseteq \overline{\text{conv}}(\mathcal{S}_2) \cap \mathcal{R}_1$, to prove the claim, it suffices to show that $\mathcal{C} \cap \mathcal{R}_1 \subseteq \text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_1)$. To this aim, for any $x_1, x_2, X_{12}, z_1, z_2 > 0$, consider the point

$$\tilde{x}_1^4 = \lambda_4 \frac{x_1}{z_1}, \quad \tilde{x}_2^4 = \lambda_4 \frac{x_2}{z_2}, \quad \lambda_4 = X_{12} \frac{z_1 z_2}{x_1 x_2}, \tag{23}$$

It then follows that over the region $\mathcal{R}_1$ defined by (21) the point defined by (23) belongs to $\tilde{\Sigma}$. Substituting this point in the inequalities defining (20) yields:

$$\text{cl}\Big(\Big\{(x, X, z) :\ X_{11} - \frac{x_1^2}{z_1} \geq 0,\ X_{22} - \frac{x_2^2}{z_2} \geq 0,\ X_{12} > 0,\ x_1 > 0,\ x_2 > 0,\ z_1, z_2 \in (0, 1]\Big\} \cap \mathcal{R}_1\Big)$$
$$\subseteq \text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_1).$$

Hence to prove (22), it remains to show that:

$$\mathcal{C} \cap \mathcal{R}_1 = \text{cl}\Big(\Big\{(x, X, z) :\ X_{11} - \frac{x_1^2}{z_1} \geq 0,\ X_{22} - \frac{x_2^2}{z_2} \geq 0,\ X_{12} > 0,\ x_1 > 0,\ x_2 > 0,\ z_1, z_2 \in (0, 1]\Big\} \cap \mathcal{R}_1\Big). \tag{24}$$

For every $(x, X, z) \in \mathcal{C} \cap \mathcal{R}_1$, the sequence (x^n, X^n, z^n) defined as

$$x_1^n = \max\Big\{x_1, \frac{1}{n}\Big\},\ x_2^n = \max\Big\{x_2, \frac{1}{n}\Big\},\ X_{12}^n = \max\Big\{X_{12}^n, \frac{1}{n}\Big\},\ X_{11}^n = \max\Big\{X_{11}, \frac{1}{n}\Big\},$$
$$X_{22}^n = \max\Big\{X_{22}, \frac{1}{n}\Big\},\ z_1^n = \max\Big\{z_1, \frac{1}{n}\Big\},\ z_2^n = \max\Big\{z_2, \frac{1}{n}\Big\},$$

converges to (x, X, z) as $n \to \infty$ and clearly $\{(x^n, X^n, z^n)\} \subset \{X_{11} - \frac{x_1^2}{z_1} \geq 0,\ X_{22} - \frac{x_2^2}{z_2} \geq 0,\ X_{12} > 0,\ x_1 > 0,\ x_2 > 0,\ z_1, z_2 \in (0, 1]\} \cap \mathcal{R}_1$. This proves (24) which in turn proves (22).

This concludes the proof of Part (I) of Theorem 1 for $i = 1$.

By (21) and (22) to characterize the closure of the convex hull of $\mathcal{S}_2$ using Lemma 3, it suffices to characterize $\tilde{\mathcal{S}}$ over the following two regions:

- Region (1):
$$\Big\{(x, X, z) :\ X_{12} \max\{z_1, z_2\} > x_1 x_2\Big\} \tag{25}$$

- Region (2):
$$\Big\{(x, X, z) :\ X_{12} z_1 z_2 < x_1 x_2(z_1 + z_2 - 1)\Big\} \tag{26}$$

4 Projection by convex optimization

In this section, we show that the projection of the set of points $(x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4)$ satisfying System (20) onto the space of (x, X, z) can be obtained by solving a convex optimization problem. Define the functions

$$g_1 := X_{11} - \frac{(\tilde{x}_1^4)^2}{\lambda_4} - \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4}, \quad g_2 := X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4}, \quad h := X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}. \tag{27}$$

We claim that the set of points satisfying System (20), i.e.,

$$\begin{aligned} & g_1 . g_2 \geq h^2, \quad g_1 \geq 0, \quad g_2 \geq 0, \\ & z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\}, \quad \lambda_4 > 0, \\ & 0 \leq \tilde{x}_1^4 \leq x_1, \quad 0 \leq \tilde{x}_2^4 \leq x_2, \quad x_1, x_2, X_{12} > 0, \end{aligned} \tag{I}$$

coincides with the set of points satisfying:

$$\begin{aligned} & g_1 \geq \text{cl}\Big(\frac{h^2}{g_2}\Big), \quad g_2 \geq 0, \\ & z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\}, \quad \lambda_4 > 0, \\ & 0 \leq \tilde{x}_1^4 \leq x_1, \quad 0 \leq \tilde{x}_2^4 \leq x_2, \quad x_1, x_2, X_{12} > 0, \end{aligned} \tag{II}$$

where we define

$$\text{cl}\Big(\frac{h^2}{g_2}\Big) := \begin{cases} \frac{h^2}{g_2}, & \text{if } g_2 > 0 \\ 0, & \text{if } h = g_2 = 0 \\ +\infty, & \text{if } \ h \neq 0, \ g_2 = 0. \end{cases} \tag{28}$$

To see the equivalence of Systems (I) and (II), notice that if $g_2 > 0$, then the two clearly coincide. Hence, let $g_2 = 0$. Two cases arise:

(i) $h = 0$: in this case System (I) simplifies to $g_1 \geq 0$ and $g_2 = 0$, and by (28), System (II) simplifies to $g_1 \geq 0$ and $g_2 = 0$ as well.

(ii) $h \neq 0$: in this case, the first inequality in System (I) simplifies to $0 \geq h^2$, which together with $h \neq 0$ implies that the system is infeasible. By (28), the first inequality in System (II) simplifies to $g_1 \geq +\infty$. It can be checked that g_1 is a concave function and its maximum value equals $X_{11} - \frac{(x_1)^2}{z_1}$. Hence in this case, System (II) is infeasible as well.

Henceforth, we work with System (II); as before, for notational simplicity, whenever we write $\frac{h^2}{g_2}$, we imply $\text{cl}(\frac{h^2}{g_2})$ as defined by (28). It then follows that to characterize the convex hull of $\mathcal{S}_2$, it suffices to project out $(\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4)$ from the set of points $(x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4)$ satisfying:

$$\begin{aligned} & X_{11} \geq \frac{(\tilde{x}_1^4)^2}{\lambda_4} + \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} + \frac{\Big(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}\Big)^2}{X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{(z_2 - \lambda_4)}}, \\ & X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0, \\ & z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\}, \quad \lambda_4 > 0 \\ & 0 \leq \tilde{x}_1^4 \leq x_1, \quad 0 \leq \tilde{x}_2^4 \leq x_2, \quad x_1, x_2, X_{12} > 0, \end{aligned} \tag{29}$$

over regions defined by (25) and (26). To this end we make use of the following two lemmas.

Lemma 4. *Consider two functions $f, g : (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ and two sets $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^m$. Define*

$$Q := \{x \in C \,:\, \exists y \in D \text{ s.t. } f(x, y) \geq 0,\ g(x, y) \geq 0\}.$$

Consider the following optimization problem:

$$\begin{aligned} \max_{y \in D} \quad & f(x, y) \\ \text{s.t.} \quad & g(x, y) \geq 0. \end{aligned} \tag{30}$$

Assume for every $x \in C$, a maximizer of Problem (30) denoted by y_x exists. Then

$$Q = Q' := \{x \in C \,:\, f(x, y_x) \geq 0,\ g(x, y_x) \geq 0\}.$$

Proof. For every $x \in Q$ there exists $y \in D$ such that $f(x, y) \geq 0, g(x, y) \geq 0$. Hence, by definition of y_x, it holds $f(x, y_x) \geq f(x, y) \geq 0$ and $g(x, y_x) \geq 0$. We deduce that $x \in Q'$. Analogously, for every $x \in Q'$ it holds $f(x, y_x) \geq 0$, and $g(x, y_x) \geq 0$. Hence $x \in Q$. □

Lemma 5. *Consider a convex set $D \subset \mathbb{R}^n$, and define a set $Q \subset D$ as*

$$Q = [l_1, u_1] \times \cdots \times [l_n, u_n].$$

Consider a convex function $f : D \to \mathbb{R} \cup \{\infty\}$ that attains its minimum value at a point $z \in D \setminus Q$. Assume there exists $i < j \in [n]$ such that $u_k \geq l_k > z_k$ for every $k = 1, \ldots, i$, $z_k > u_k \geq l_k$ for every $k = i+1, \ldots, j$ and $u_k \geq z_k \geq l_k$ for every $k = j+1, \ldots, n$. Then

$$\min_Q f = \min_B f, \tag{31}$$

where

$$B := \Big(\bigcup_{k=1}^{i} [l_1, u_1] \times \cdots \times \{l_k\} \times \cdots \times [l_n, u_n]\Big) \cup \Big(\bigcup_{k=i+1}^{j} [l_1, u_1] \times \cdots \times \{u_k\} \times \cdots \times [l_n, u_n]\Big) \subset \partial Q.$$

Proof. Since D is convex, for every $x \in Q$ the line segment $\{(1-s)z + sx : s \in [0, 1]\} \subset D$. Moreover, for every $x \in Q$ there exists $s_x \in [0, 1]$ such that $(1 - s_x)z + s_x x \in B$. Using that f is convex and z is a minimizer, we conclude that

$$f((1 - s_x)z + s_x x) \leq (1 - s_x)f(z) + s_x f(x) \leq (1 - s_x)f(x) + s_x f(x) = f(x).$$

As $B \subset Q$, equality (31) holds. □

Therefore, by Lemma 4, to characterize the convex hull of $\mathcal{S}_2$ in the original space, it suffices to solve the following optimization problem for all (x, X, z) with $x_1, x_2, X_{12} > 0$, and satisfying inequalities (25) or (26):

$$\begin{aligned} \min_{\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4} \quad & \frac{(\tilde{x}_1^4)^2}{\lambda_4} + \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} + \frac{\Big(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}\Big)^2}{X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4}}, && \text{(P)} \\ \text{s.t.} \quad & X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0, && (32) \\ & z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\}, \quad \lambda_4 > 0 && (33) \\ & 0 \leq \tilde{x}_1^4 \leq x_1, \quad 0 \leq \tilde{x}_2^4 \leq x_2. && (34) \end{aligned}$$

Problem (P) is a convex optimization problem. To see this, first note that the set of points defined by System (29) forms a convex set as it coincides with the set of points satisfying System (20); i.e., the convex set $\tilde{\Sigma}$; let us denote by Σ_x the convex set defined as the restriction of $\tilde{\Sigma}$ to a fixed (x, X, z); then Σ_x can be written as

$$\Sigma_x = \{(\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) : X_{11} \geq f(\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) : (\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) \in \mathcal{Q}\},$$

where we define

$$f := \frac{(\tilde{x}_1^4)^2}{\lambda_4} + \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} + \frac{\left(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}\right)^2}{X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4}} \tag{35}$$

and

$$\mathcal{Q} := \Big\{(\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) : X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0,$$
$$z_1 + z_2 - 1 \leq \lambda_4 \leq \min\{z_1, z_2\},\ \lambda_4 > 0,\ 0 \leq \tilde{x}_1^4 \leq x_1,\ 0 \leq \tilde{x}_2^4 \leq x_2\Big\}.$$

It is simple to check that $\mathcal{Q}$ is a convex set. It then follows that the convex set Σ_x can be described as the epigraph of the function f over the convex set $\mathcal{Q}$. Therefore, f is a convex function over $\mathcal{Q}$; f is precisely the objective function of Problem (P) while $\mathcal{Q}$ is its feasible set. Hence, Problem (P) is a convex optimization problem. Now, consider the following relaxation of Problem (P):

$$\min_{\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4} \quad \frac{(\tilde{x}_1^4)^2}{\lambda_4} + \frac{(x_1 - \tilde{x}_1^4)^2}{z_1 - \lambda_4} + \frac{\left(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{\lambda_4}\right)^2}{X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4}}, \tag{PR}$$

$$\text{s.t.} \quad X_{22} - \frac{(\tilde{x}_2^4)^2}{\lambda_4} - \frac{(x_2 - \tilde{x}_2^4)^2}{z_2 - \lambda_4} \geq 0, \tag{36}$$

$$0 < \lambda_4 \leq \min\{z_1, z_2\}. \tag{37}$$

Problem (PR) is a convex optimization problem as well. The proof follows from a similar line of arguments to those for Problem (P) by redefining $\tilde{\Sigma}$ as follows $\tilde{\Sigma} = \{(x, X, z, \tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) : (x, X, z, \tilde{x}, \tilde{X}, \lambda) \in \Sigma\}$, where $\Sigma = \text{conv}(\mathcal{P}'_2 \cup \mathcal{P}'_3 \cup \mathcal{P}'_4)$, and

$$\mathcal{P}'_2 := \{(x, X, z) : z_1 = 1, z_2 = 0,\ X_{11} = x_1^2,\ x_2 = X_{12} = X_{22} = 0,\ x_1 \in \mathbb{R}\},$$
$$\mathcal{P}'_3 := \{(x, X, z) : z_1 = 0, z_2 = 1,\ x_1 = X_{11} = X_{12} = 0,\ X_{22} = x_2^2,\ x_2 \in \mathbb{R}\},$$
$$\mathcal{P}'_4 := \{(x, X, z) : z_1 = z_2 = 1,\ X_{11} = x_1^2,\ X_{12} = x_1 x_2,\ X_{22} = x_2^2,\ x_1, x_2 \in \mathbb{R}\}.$$

However, we should remark that constraints (36) and (37) are essential for convexity of Problems (P) and (PR).

In the following, we solve Problem (P) analytically for all (x, X, z) satisfying inequalities (25) or (26). First let us consider Problem (PR); for any positive x_1, X_{12}, z_1, it can be checked that $f \geq \frac{(x_1)^2}{z_1}$ and this lower bound is attained when:

$$\tilde{x}_1^4 = \lambda_4 \frac{x_1}{z_1}, \quad \tilde{x}_2^4 = \frac{X_{12} z_1}{x_1}, \tag{38}$$

where in addition λ_4 must satisfy constraints (36) and (37). To optimally determine λ_4, we consider the following auxiliary optimization problem:

$$\max \quad X_{22} - \Big(\frac{X_{12} z_1}{x_1}\Big)^2 \frac{1}{\lambda_4} - \Big(x_2 - \frac{X_{12} z_1}{x_1}\Big)^2 \frac{1}{z_2 - \lambda_4} \tag{39}$$

$$\text{s.t.} \quad 0 < \lambda_4 \leq \min\{z_1, z_2\}.$$

If the optimal value of Problem (39) is nonnegative, then the maximizer λ_4 together with $\tilde{x}_1^4, \tilde{x}_1^4$ defined by (38) are an optimal solution of Problem (PR). The objective function of Problem (39) is concave over $(0, z_2)$. Setting the derivative of the objective function to zero, it follows that the maximizer of the objective function over $(0, z_2)$ is attained at:

$$\bar{\lambda}_4 = \begin{cases} \frac{X_{12}z_1z_2}{x_1x_2}, & \text{if } X_{12}z_1 < x_1x_2 \\ \frac{X_{12}z_1z_2}{2X_{12}z_1 - x_1x_2}, & \text{if } X_{12}z_1 > x_1x_2. \end{cases} \tag{40}$$

In the special case where $X_{12}z_1 = x_1x_2$, Problem (39) simplifies to the problem of maximizing $X_{22} - \frac{x_2^2}{\lambda_4}$ over $0 < \lambda_4 \leq \min\{z_1, z_2\}$. The optimal value of this problem is attained at $\lambda_4^* = \min\{z_1, z_2\}$.

Now let us consider Problem (P) and examine cases under which an optimal solution of this problem is given by (38) and (40). To proceed further, we should consider the two regions defined by (25) and (26) separately.

(i) Region (1) defined by (25): consider the case $z_2 \geq z_1$. Then we have $X_{12}z_2 > x_1x_2$. Suppose that $X_{12}z_1 \leq x_1x_2$. If $X_{12}z_1 = x_1x_2$, then $\lambda_4^* = z_1$. Now let $X_{12}z_1 < x_1x_2$, implying the objective function of (39) is maximized at $\bar{\lambda}_4 = \frac{X_{12}z_1z_2}{x_1x_2}$. However, in this case we have $\bar{\lambda}_4 > z_1$, implying that the maximizer of Problem (39) is $\lambda_4^* = z_1$. Substituting λ_4^* in inequality (36), it follows that if

$$x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2) \geq z_1(X_{12}z_2 - x_1x_2)^2,$$

then an optimal solution of Problem (PR) is given by

$$\tilde{x}_1^4 = x_1, \quad \tilde{x}_2^4 = \frac{X_{12}z_1}{x_1}, \quad \lambda_4 = z_1. \tag{41}$$

Since by assumption $X_{12}z_1 \leq x_1x_2$, we conclude that (41) also satisfies constraints (34) and hence is an optimal solution of Problem (P). Therefore, by Lemma 4, over the region

$$\mathcal{R}_2 := \Big\{(x, X, z) : z_1 \leq z_2,\ X_{12}z_2 > x_1x_2,\ X_{12}z_1 \leq x_1x_2, \\ x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2) \geq z_1(X_{12}z_2 - x_1x_2)^2\Big\}, \tag{42}$$

the projection of $\tilde{\Sigma}$ defined by (20) onto the space (x, X, z) is given by

$$\tilde{\mathcal{S}} \cap \mathcal{R}_2 = \Big\{(x, X, z) : X_{11} \geq \frac{x_1^2}{z_1},\ x_1 > 0, x_2 > 0,\ X_{12} > 0,\ z_1 > 0,\ z_2 \leq 1\Big\} \cap \mathcal{R}_2.$$

We claim that

$$\text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_2) = \Big\{(x, X, z) : X_{11} \geq \frac{x_1^2}{z_1},\ X_{22} \geq \frac{x_2^2}{z_2},\ x_1 \geq 0, x_2 \geq 0,\ X_{12} \geq 0,\ z_1 \geq 0,\ z_2 \leq 1\Big\} \\ \cap\, \text{cl}(\mathcal{R}_2) =: \chi. \tag{43}$$

where $\text{cl}(\mathcal{R}_2) = \{(x, X, z) :\ z_1 \leq z_2,\ X_{12}z_2 \geq x_1x_2,\ X_{12}z_1 \leq x_1x_2,\ x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2) \geq z_1(X_{12}z_2 - x_1x_2)^2\}$. To show the validity of (43), first note that χ is a closed superset of $\tilde{\mathcal{S}} \cap \mathcal{R}_2$. Moreover, for every $(x, X, z) \in \chi$, the sequence (x^n, X^n, z^n) defined as

$$x_1^n = \max\Big\{x_1, \frac{1}{n}\Big\},\ x_2^n = \max\Big\{x_2, \frac{1}{n}\Big\},\ X_{12}^n = \max\Big\{X_{12}^n, \frac{1}{n}\Big\},\ X_{11}^n = \max\Big\{X_{11}, \frac{1}{n}\Big\}, \\ X_{22}^n = \max\Big\{X_{22}, \frac{1}{n}\Big\},\ z_1^n = \max\Big\{z_1, \frac{1}{n}\Big\},\ z_2^n = \max\Big\{z_2, \frac{2}{n}\Big\},$$

converges to (x, X, z) as $n \to \infty$ and clearly $\{(x^n, X^n, z^n)\} \subset \tilde{\mathcal{S}} \cap \mathcal{R}_2$. Hence $\mathrm{cl}(\tilde{S} \cap \mathcal{R}_2) = \chi$.

This concludes the proof of Part (I) of Theorem 1 for $i = 2$.

As we will show in the next section, over the remaining parts of Region (1), there always exist optimal solutions of Problem (P) at which constraints (34) are active.

(ii) Region (2) defined by (26): we have $X_{12}z_1 < x_1x_2\frac{z_1+z_2-1}{z_2} \leq x_1x_2$. Hence, the objective function of (39) is maximized at $\bar{\lambda}_4 = \frac{X_{12}z_1z_2}{x_1x_2}$ and in this region we have $0 < \bar{\lambda}_4 < \min\{z_1, z_2\}$. Hence the maximum of Problem (39) is attained at $\bar{\lambda}_4$ and is equal to $X_{22} - \frac{x_2^2}{z_2}$. Since for any $(x, X, z) \in \overline{\mathrm{conv}}(\mathcal{S}_2)$ we have $X_{22} - \frac{x_2^2}{z_2} \geq 0$, it follows that $\bar{\lambda}_4$ together with $\tilde{x}_1^4, \tilde{x}_1^4$ defined by (38) form an optimal solution of Problem (PR).

Now let us consider Problem (P); in Region (2), $\tilde{x}_2^4 = \frac{X_{12}z_1}{x_1} < x_2$; however, we have $\bar{\lambda}_4 < z_1+z_2-1$. Hence by Lemma 5, over Region (2), there exists an optimal solution of Problem (P) with $\lambda_4 = z_1 + z_2 - 1$. We analyze this boundary in the next section.

5 Region (1) boundaries

In this section, we prove that over Region (1) defined by (25), if the system of inequalities defined by (42) is not satisfied, to characterize the convex hull of $\mathcal{S}_2$, it suffices to consider feasible points of Problem (P) at which $\tilde{x}_1^4 = x_1$ or $\tilde{x}_2^4 = x_2$. To this end, we make use of the following lemma:

Lemma 6. *Let C denote a set in $\mathbb{R}^n$; consider a function $g : (x, \varepsilon) \in C \times [0, \delta] \mapsto g(x, \varepsilon) \in \mathbb{R}$ such that, for every $x \in C$, $g(x, \cdot)$ is continuous and strictly decreasing. Denoting*

$$\Omega_\varepsilon := \{x \in C \,:\, g(x, \varepsilon) \geq 0\}, \quad P := \bigcup_{\varepsilon>0} \Omega_\varepsilon, \quad \textit{and} \quad Q := \{x \in C \,:\, g(x, 0) > 0\},$$

we have that

$$P = Q.$$

Proof. For every $x \in P$ there exists $\bar{\varepsilon} > 0$ such that $x \in \Omega_{\bar{\varepsilon}}$, i.e. $x \in C$ and $g(x, \bar{\varepsilon}) \geq 0$. Since $g(x, \cdot)$ is strictly decreasing, then $g(x, 0) > g(x, \bar{\varepsilon}) \geq 0$. Hence $x \in Q$. For every $x \in Q$, we have $x \in C$ and $g(x, 0) > 0$. Since $g(x, \cdot)$ is continuous, there exists $\bar{\varepsilon} > 0$ small enough such that $g(x, \bar{\varepsilon}) > 0$. Hence $x \in \Omega_{\bar{\varepsilon}} \subset P$. □

Consider the function $f(u, v, w) := \frac{uv}{w}$, $u, v \geq 0$ and $w > 0$. We define the closure of $f(u, v)$, denoted by $\hat{f}(u, v)$ over $\mathbb{R}^3_+$ as follows:

$$\hat{f}(u, v) = \begin{cases} \frac{uv}{w}, & \text{if } w > 0 \\ 0, & \text{if } u = v = w = 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{44}$$

For notational simplicity, in the remainder of this paper, whenever we write a function of the form $\frac{uv}{w}$, we imply its closure as defined by (44).

5.1 Boundary $\tilde{x}_1^4 = x_1$:

Since Problem (P) is a convex optimization problem, a feasible point $(\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4)$ with $\tilde{x}_1^4 = x_1$ is an optimal solution, if it satisfies the following conditions:

1. $\frac{\partial f}{\partial \tilde{x}_1^4} \leq 0$: this condition is satisfied, if

$$x_1 g_2 \leq \tilde{x}_2^4 h, \tag{45}$$

2. $\frac{\partial f}{\partial \tilde{x}_2^4} = 0$: this condition is satisfied, if

$$\Big(\frac{\tilde{x}_2^4}{\lambda_4} - \frac{x_2 - \tilde{x}_2^4}{z_2 - \lambda_4}\Big) h = \frac{x_1}{\lambda_4} g_2, \tag{46}$$

3. $\frac{\partial f}{\partial \lambda_4} = 0$: this condition is satisfied, if

$$\Big(\frac{x_1}{\lambda_4}\Big)^2 g_2^2 = \frac{2x_1\tilde{x}_2^4}{\lambda_4^2} h g_2 - \Big(\Big(\frac{\tilde{x}_2^4}{\lambda_4}\Big)^2 - \Big(\frac{x_2 - \tilde{x}_2^4}{z_2 - \lambda_4}\Big)^2\Big) h^2, \tag{47}$$

where f is defined by (35) and g_2, h are defined by (27). We should remark that conditions (46) and (47) are defined for $\lambda_4 \in (0, z_2)$. In the following, we assume that $g_2 \neq 0$, which in turn implies $h \neq 0$. First we show that equation (47) is implied by equation (46). Using (46) and $g_2 \neq 0$, equation (47) can be written as:

$$\begin{aligned}
\Big(\frac{x_1}{\lambda_4}\Big)^2 g_2 &= \frac{2x_1\tilde{x}_2^4}{\lambda_4^2} h - \frac{x_1}{\lambda_4}\Big(\frac{\tilde{x}_2^4}{\lambda_4} + \frac{x_2 - \tilde{x}_2^4}{z_2 - \lambda_4}\Big) h \\
&= \frac{x_1}{\lambda_4}\Big(\frac{\tilde{x}_2^4}{\lambda_4} - \frac{x_2 - \tilde{x}_2^4}{z_2 - \lambda_4}\Big) h \\
&= \frac{x_1}{\lambda_4}\frac{x_1}{\lambda_4} g_2.
\end{aligned}$$

Hence, equation (47) is implied by equation (46). By feasibility we must have $g_2 > 0$; using equation (46), inequality $g_2 > 0$ holds if and only if $\frac{\tilde{x}_2^4}{\lambda_4} - \frac{x_2 - \tilde{x}_2^4}{z_2 - \lambda_4} > 0$ and $h = X_{12} - \frac{x_1 \tilde{x}_2^4}{\lambda_4} > 0$; i.e.,

$$\lambda_4 \frac{x_2}{z_2} < \tilde{x}_2^4 < \lambda_4 \frac{X_{12}}{x_1}. \tag{48}$$

Next, we examine the validity of inequality (45). By equation (46) and inequality $h \geq 0$, inequality (45) can be equivalently written as $\lambda_4(\frac{\tilde{x}_2^4}{\lambda_4} - \frac{x_2 - \tilde{x}_2^4}{z_2 - \lambda_4}) \leq \tilde{x}_2^4$, which using $\lambda_4 > 0$ simplifies to $\frac{x_2 - \tilde{x}_2^4}{z_2 - \lambda_4} \geq 0$, whose validity follows from inequalities (33) and (34). Finally, we consider equation (46); first note that for $\lambda_4 \in (0, z_2)$, this equation is equivalent to $x_1(X_{22}z_2 - x_2^2) + \lambda_4(x_2 X_{12} - x_1 X_{22}) = \tilde{x}_2^4(X_{12}z_2 - x_1 x_2)$; moreover, by inequality (48) we have $X_{12}z_2 - x_1x_2 > 0$; it then follows that:

$$\tilde{x}_2^4 = \frac{x_1(X_{22}z_2 - x_2^2) + \lambda_4(x_2 X_{12} - x_1 X_{22})}{X_{12}z_2 - x_1 x_2}. \tag{49}$$

Hence the point $(x_1, \tilde{x}_2^4, \lambda_4)$ is an optimal solution of Problem (P) if it satisfies (33), (34), (48) and (49). Substituting (49) into (48), it follows that $\lambda_4 \frac{x_2}{z_2} < \tilde{x}_2^4$ simplifies to $\lambda_4 < z_2$ which holds by (33), and and $\tilde{x}_2^4 < \lambda_4 \frac{X_{12}}{x_1}$ is equivalent to

$$(X_{12}z_2 - x_1x_2)^2 > \Big(\frac{z_2}{\lambda_4} - 1\Big) x_1^2 (X_{22}z_2 - x_2^2). \tag{50}$$

Next, we examine constraints (33); using (49) it follows that $\tilde{x}_2^4 \leq x_2$ is equivalent to

$$X_{12}x_2 \geq X_{22}x_1, \tag{51}$$

which together with the fact that $X_{22}z_2 \geq x_2^2$ for all points in $\overline{\text{conv}}(\mathcal{S}_2)$ implies $\tilde{x}_2^4 \geq 0$. Finally, to determine the optimal λ_4, we consider two cases:

- If $z_1 < z_2$, then we let $\lambda_4 = z_1$. Hence we conclude that over the region

$$\mathcal{R}_3 := \Big\{(x, X, z) :\ z_1 < z_2,\ X_{12}x_2 > X_{22}x_1,\ z_1(X_{12}z_2 - x_1x_2)^2 > x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2)\Big\}, \tag{52}$$

an optimal solution of Problem (P) is given by

$$(\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) = \Big(x_1,\ \frac{X_{12}x_2z_1 + x_1(X_{22}(z_2 - z_1) - x_2^2)}{X_{12}z_2 - x_1x_2},\ z_1\Big).$$

Therefore, by Lemma 4, over the region $\mathcal{R}_3$, the projection of $\tilde{\Sigma}$ defined by (20) onto the space (x, X, z) is given by

$$\tilde{\mathcal{S}} \cap \mathcal{R}_3 = \Big\{(x, X, z) : X_{22} \geq \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_2}\Big)\Big(X_{22} - \frac{x_2^2}{z_2}\Big) \geq \Big(X_{12} - \frac{x_1x_2}{z_2}\Big)^2,\ x_1 > 0,\ x_2 > 0,\ X_{12} > 0,\ z_1 > 0,\ z_2 \leq 1\Big\} \cap \mathcal{R}_3.$$

Moreover, we claim that

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_3) = \Big\{(x, X, z) : X_{22} \geq \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_2}\Big)\Big(X_{22} - \frac{x_2^2}{z_2}\Big) \geq \Big(X_{12} - \frac{x_1x_2}{z_2}\Big)^2,\ x_1 \geq 0,\ x_2 \geq 0,\ X_{12} \geq 0,\ z_1 \geq 0,\ z_2 \leq 1\Big\} \cap \mathrm{cl}(\mathcal{R}_3), \tag{53}$$

where $\mathrm{cl}(\mathcal{R}_3) = \{(x, X, z) :\ z_1 \leq z_2,\ X_{12}x_2 \geq X_{22}x_1,\ z_1(X_{12}z_2 - x_1x_2)^2 \geq x_1^2(z_2 - z_1)(X_{22}z_2 - x_2^2)\}$. To prove (53), one can employ a similar line of arguments to those used to prove (43).

This concludes the proof of Part (II) of Theorem 1 for $i = 3$.

- If $z_2 \leq z_1$: as we discussed before, equations (46) and (47) are defined over $\lambda_4 \in (0, z_2)$. Hence, in this case, we let $\lambda_4 = z_2 - \epsilon$ for some $\epsilon > 0$. It then follows that over the region

$$\Omega_\epsilon := \Big\{(x, X, z) : z_2 \leq z_1, X_{12}x_2 > X_{22}x_1,\ (X_{12}z_2 - x_1x_2)^2 > \frac{\epsilon}{z_2 - \epsilon}x_1^2(X_{22}z_2 - x_2^2)\Big\}, \tag{54}$$

an optimal solution of Problem (P) is given by

$$(\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4) = \Big(x_1,\ x_2 - \epsilon(\frac{X_{12}x_2 - X_{22}x_1}{X_{12}z_2 - x_1x_2}),\ z_2 - \epsilon\Big).$$

Therefore, by Lemma 4, over the region defined by (54), the projection of $\tilde{\Sigma}$ defined by (20) onto the space (x, X, z) is given by

$$\tilde{\mathcal{S}} \cap \Omega_\epsilon = \Big\{(x, X, z) : X_{22} \geq \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_2}\Big)\Big(X_{22} - \frac{x_2^2}{z_2}\Big) \geq \Big(X_{12} - \frac{x_1x_2}{z_2}\Big)^2,\ x_1 > 0,\ x_2 > 0,\ X_{12} > 0,\ z_2 > 0,\ z_1 \leq 1\Big\} \cap \Omega_\epsilon.$$

We now utilize Lemma 6 to complete the projection in this region. Letting $g(x, \epsilon) := (X_{12}z_2 - x_1x_2)^2 - \frac{\epsilon}{z_2 - \epsilon}x_1^2(X_{22}z_2 - x_2^2)$, $C = \{(x, X, z) : z_2 \leq z_1,\ X_{12}x_2 > X_{22}x_1\}$, and

$$\mathcal{R}_4 := \{(x, X, z) :\ z_2 \leq z_1,\ X_{12}x_2 > X_{22}x_1\}, \tag{55}$$

by Lemma 6 we deduce that

$$\tilde{\mathcal{S}} \cap \mathcal{R}_4 = \Big\{(x, X, z) : X_{22} \geq \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_2}\Big)\Big(X_{22} - \frac{x_2^2}{z_2}\Big) \geq \Big(X_{12} - \frac{x_1x_2}{z_2}\Big)^2,\ x_1 > 0,\ x_2 > 0,$$

$$X_{12} > 0,\ z_2 > 0,\ z_1 \leq 1\Big\} \cap \mathcal{R}_4. \tag{56}$$

Using a similar line of arguments to those used to prove (43), we obtain:

$$\begin{aligned} \mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_4) = \Big\{(x, X, z) : X_{22} \geq \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_2}\Big)\Big(X_{22} - \frac{x_2^2}{z_2}\Big) \geq \Big(X_{12} - \frac{x_1 x_2}{z_2}\Big)^2, \\ x_1 \geq 0,\ x_2 \geq 0,\ X_{12} \geq 0,\ z_2 \geq 0,\ z_1 \leq 1\Big\} \cap \mathrm{cl}(\mathcal{R}_4), \end{aligned}$$

where $\mathrm{cl}(\mathcal{R}_4) = \{(x, X, z) :\ z_2 \leq z_1,\ X_{12}x_2 \geq X_{22}x_1\}$.

This concludes the proof of Part (II) of Theorem 1 for $i = 4$.

5.2 Boundary $\tilde{x}_2^4 = x_2$

Since Problem (P) is a convex optimization problem, a feasible point $(\tilde{x}_1^4, \tilde{x}_2^4, \lambda_4)$ with $\tilde{x}_2^4 = x_2$ is an optimal solution, if it satisfies the following conditions:

1. $\frac{\partial f}{\partial \tilde{x}_1^4} = 0$: this condition is satisfied, if

$$\Big(\frac{\tilde{x}_1^4}{\lambda_4} - \frac{x_1 - \tilde{x}_1^4}{z_1 - \lambda_4}\Big) g_2 = \frac{x_2}{\lambda_4} h, \tag{57}$$

2. $\frac{\partial f}{\partial \tilde{x}_2^4} \leq 0$: this condition is satisfied, if

$$\Big(X_{12} - \frac{\tilde{x}_1^4 x_2}{\lambda_4}\Big)\Big(X_{12}x_2 - X_{22}\tilde{x}_1^4\Big) \leq 0, \tag{58}$$

3. $\frac{\partial f}{\partial \lambda_4} = 0$: this condition is satisfied, if

$$\Big(\Big(\frac{\tilde{x}_1^4}{\lambda_4}\Big)^2 - \Big(\frac{x_1 - \tilde{x}_1^4}{z_1 - \lambda_4}\Big)^2\Big) g_2^2 = \frac{2\tilde{x}_1^4 x_2}{\lambda_4^2} h g_2 - \Big(\frac{x_2}{\lambda_4}\Big)^2 h^2, \tag{59}$$

where f is defined by (35) and g_2, h are defined by (27). We should remark that conditions (57) and (59) are defined for $\lambda_4 \in (0, z_1)$. In the following, we assume $g_2 \neq 0$, which in turn implies $h \neq 0$. First we show that equation (59) is implied by equation (57). Using (57) and $h \neq 0$, equation (59) can be written as

$$\begin{aligned} \Big(\Big(\frac{\tilde{x}_1^4}{\lambda_4}\Big)^2 - \Big(\frac{x_1 - \tilde{x}_1^4}{z_1 - \lambda_4}\Big)^2\Big) g_2^2 &= \frac{2\tilde{x}_1^4 x_2}{\lambda_4^2} h g_2 - \Big(\frac{x_2}{\lambda_4}\Big)^2 h^2 \\ \frac{x_2 h}{\lambda_4}\Big(\frac{\tilde{x}_1^4}{\lambda_4} + \frac{x_1 - \tilde{x}_1^4}{z_1 - \lambda_4}\Big) g_2 &= \frac{x_2 h}{\lambda_4}\Big(\frac{2\tilde{x}_1^4}{\lambda_4} g_2 - \frac{x_2}{\lambda_4} h\Big) \\ \frac{x_2}{\lambda_4} h &= \Big(\frac{\tilde{x}_1^4}{\lambda_4} - \frac{x_1 - \tilde{x}_1^4}{z_1 - \lambda_4}\Big) g_2, \end{aligned}$$

where the validity of the last equality follows from (57). We now examine the validity of inequality (58). First assume that $X_{12}\lambda_4 - \tilde{x}_1^4 x_2 \leq 0$ and $X_{12}x_2 - X_{22}\tilde{x}_1^4 \geq 0$: using the valid inequality $X_{22}z_2 \geq x_2^2$, it follows that $X_{12}x_2 \geq X_{22}\tilde{x}_1^4 \geq \frac{x_2^2 \tilde{x}_1^4}{z_2}$, implying $X_{12}z_2 \geq \tilde{x}_1^4 x_2$ which together with $\lambda_4 \leq z_2$ contradicts with the inequality $X_{12}\lambda_4 - \tilde{x}_1^4 x_2 \leq 0$. Hence this case is not possible. Therefore, using $h \neq 0$, inequality (58) is satisfied if and only if

$$X_{22}\tilde{x}_1^4 - X_{12}x_2 \geq 0, \tag{60}$$

and

$$X_{12}\lambda_4 - \tilde{x}_1^4 x_2 > 0. \tag{61}$$

By feasibility we must have $g_2 > 0$; i.e.,

$$X_{22} - \frac{x_2^2}{\lambda_4} > 0. \tag{62}$$

Next, we consider equation (57); by (62) and $\lambda_4 \le z_1$, we conclude that $X_{22}z_1 - x_2^2 > 0$. Then solving equation (57) for $\tilde{x}_1^4$, we obtain

$$\tilde{x}_1^4 = \frac{x_2(X_{12}z_1 - x_1x_2) + \lambda_4(X_{22}x_1 - X_{12}x_2)}{X_{22}z_1 - x_2^2}. \tag{63}$$

Substituting (63) in inequality (60), it follows that this inequality is satisfied if together with (62), we have

$$X_{22}x_1 - X_{12}x_2 \ge 0, \tag{64}$$

and inequality (61) is satisfied if together with (62), we have

$$X_{12}z_1 - x_1x_2 > 0. \tag{65}$$

It is then simple to check that by (63), (64) and (65), the constraint $0 \le \tilde{x}_1^4 \le x_1$ is always satisfied. Finally, to optimally determine λ_4, we consider two cases:

- $z_2 < z_1$: in this case we let $\lambda_4 = z_2$; then inequality (62) simplifies to $X_{22} > \frac{x_2^2}{z_2}$. Hence we conclude that over the region:

$$\mathcal{R}' = \{(x, X, z):\ z_2 < z_1,\ X_{12}z_1 > x_1x_2,\ X_{22}x_1 \ge X_{12}x_2\}, \tag{66}$$

  an optimal solution of Problem (P) is attained at

$$(\tilde{x}_1^4,\ \tilde{x}_2^4,\ \lambda_4) = \Big(\frac{x_1(X_{22}z_2 - x_2^2) + X_{12}x_2(z_1 - z_2)}{X_{22}z_1 - x_2^2},\ x_2,\ z_2\Big).$$

  Notice that $X_{22}z_1 - x_2^2 > 0$ since $X_{22}z_2 - x_2^2 \ge 0$ and $z_2 < z_1$. Therefore by Lemma 4, we get

$$\begin{aligned}\tilde{\mathcal{S}} \cap \mathcal{R}' = \Big\{(x, X, z):\ & X_{22} > \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_1}\Big)\Big(X_{22} - \frac{x_2^2}{z_1}\Big) \ge \Big(X_{12} - \frac{x_1x_2}{z_1}\Big)^2, \\ & x_1 > 0,\ x_2 > 0,\ X_{12} > 0,\ z_1 \le 1,\ z_2 \ge 0\Big\} \cap \mathcal{R}'. \end{aligned} \tag{67}$$

- $z_1 \le z_2$: as we discussed before, conditions (57) and (59) are defined for $\lambda_4 \in (0, z_1)$. Hence in this case we let $\lambda_4 = z_1 - \epsilon$ for some $\epsilon > 0$. Then inequality (62) simplifies to $X_{22} - \frac{x_2^2}{z_1 - \epsilon} > 0$. It then follows that over the region:

$$\Omega_\epsilon := \Big\{(x, X, z):\ z_1 \le z_2,\ X_{12}z_1 > x_1x_2,\ X_{22}x_1 \ge X_{12}x_2,\ X_{22} > \frac{x_2^2}{z_1 - \epsilon}\Big\},$$

  an optimal solution of Problem (P) is attained at

$$(\tilde{x}_1^4,\ \tilde{x}_2^4,\ \lambda_4) = \Big(x_1 - \epsilon\Big(\frac{X_{22}x_1 - X_{12}x_2}{X_{22}z_1 - x_2^2}\Big),\ x_2,\ z_1 - \epsilon\Big).$$

  Therefore, by Lemma 4 we get

$$\tilde{\mathcal{S}} \cap \Omega_\epsilon = \Big\{(x, X, z) : \Big(X_{11} - \frac{x_1^2}{z_1}\Big)\Big(X_{22} - \frac{x_2^2}{z_1}\Big) \ge \Big(X_{12} - \frac{x_1x_2}{z_1}\Big)^2,\ x_1 > 0,\ x_2 > 0,\ X_{12} > 0,$$

$$z_1 > 0,\ z_2 \leq 1\Big\} \cap \Omega_\epsilon.$$

We now utilize Lemma 6 to complete the projection in this region. Letting $g(x,\epsilon) := X_{22} - \frac{x_2^2}{z_1-\epsilon}$, $C = \{(x,X,z) : z_1 \leq z_2,\ X_{12}z_1 - x_1x_2 > 0,\ X_{22}x_1 - X_{12}x_2 \geq 0\}$,

$$\mathcal{R}'' := \Big\{(x,X,z):\ z_1 \leq z_2,\ X_{12}z_1 > x_1x_2,\ X_{22}x_1 \geq X_{12}x_2\Big\}, \tag{68}$$

and using the fact that $X_{12}z_1 > x_1x_2$ and $X_{22}x_1 \geq X_{12}x_2$, imply $X_{22} > \frac{x_2^2}{z_1}$, by Lemma 6, we get

$$\tilde{S} \cap \mathcal{R}'' = \Big\{(x,X,z):\ \Big(X_{11} - \frac{x_1^2}{z_1}\Big)\Big(X_{22} - \frac{x_2^2}{z_1}\Big) \geq \Big(X_{12} - \frac{x_1x_2}{z_1}\Big)^2, x_1 > 0,$$
$$x_2 > 0,\ X_{12} > 0,\ z_1 > 0,\ z_2 \leq 1\Big\} \cap \mathcal{R}''. \tag{69}$$

From (66), (67), (68), and (69), it follows that

$$\mathrm{cl}(\tilde{S} \cap \mathcal{R}_5) = \Big\{(x,X,z):\ X_{22} \geq \frac{x_2^2}{z_2},\ \Big(X_{11} - \frac{x_1^2}{z_1}\Big)\Big(X_{22} - \frac{x_2^2}{z_1}\Big) \geq \Big(X_{12} - \frac{x_1x_2}{z_1}\Big)^2,$$
$$x_1, x_2 \geq 0,\ X_{12} \geq 0,\ z_1, z_2 \in [0,1]\Big\} \cap \mathrm{cl}(\mathcal{R}_5), \tag{70}$$

where $\mathcal{R}_5 = \mathcal{R}' \cup \mathcal{R}''$ and $\mathrm{cl}(\mathcal{R}_5) = \{(x,X,z) : X_{12}z_1 \geq x_1x_2,\ X_{22}x_1 \geq X_{12}x_2\}$.

This concludes the proof of Part (III) of Theorem 1 for $i = 1$.

5.3 Convex hull characterization over Region (1)

We now show that the union of the regions defined by system of inequalities (42), (52), (55), (66), and (68) covers Region (1); i.e., $\mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \cup \mathcal{R}_5$ contains the set of points $(x,X,z) \in \overline{\mathrm{conv}}(\mathcal{S}_2)$ satisfying inequality (25). Two cases arise:

(i) $z_1 \leq z_2$: in this case by (25) defining Region (1), we have $X_{12}z_2 > x_1x_2$. Hence it suffices show that any point $(x,X,z) \in \overline{\mathrm{conv}}(\mathcal{S}_2)$ satisfying $X_{12}z_2 > x_1x_2$ is also in the set $\mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_5$. Let us define the sets $\mathcal{B} = \{(x,X,z) : x_1, x_2 \geq 0, X_{12} \geq 0, z_1, z_2 \in [0,1]\}$, $\mathcal{C}_1 = \{(x,X,z) : X_{12}z_1 \leq x_1x_2\}$, $\mathcal{C}_2 = \{(x,X,z) : X_{12}x_2 > x_1X_{22}\}$, $\mathcal{C}_3 = \{(x,X,z) : x_1^2(z_2-z_1)(X_{22}z_2 - x_2^2) \geq z_1(X_{12}z_2 - x_1x_2)^2\}$, which can also be written as $\mathcal{C}_3 = \{(x,X,z) : x_1^2(z_2-z_1)X_{22} \geq X_{12}^2 z_1 z_2 - 2X_{12}x_1x_2z_1 + (x_1x_2)^2\}$. Given a set $\mathcal{C}$, we denote by $\mathcal{C}^c$, the complement of $\mathcal{C}$. Then to prove the statement, it suffices to show that the following sets are empty:

- $\mathcal{C}_1 \cap \mathcal{C}_2^c \cap \mathcal{C}_3^c \cap \mathcal{B}$: by $(x,X,z) \in \mathcal{C}_2^c$ and $z_1 \leq z_2$, we have $x_1^2(z_2-z_1)X_{22} \geq x_1x_2X_{12}(z_2 - z_1)$. Moreover, since $X_{12}z_2 - x_1x_2 > 0$, it follows that $X_{12}x_1x_2(z_2-z_1) > X_{12}^2z_1z_2 - 2x_1x_2X_{12}z_1 + (x_1x_2)^2$, implying $x_1^2(z_2-z_1)X_{22} > X_{12}^2z_1z_2 - 2X_{12}x_1x_2z_1 + (x_1x_2)^2$, which is in contradiction with $(x,X,z) \in \mathcal{C}_3^c$.
- $\mathcal{C}_1^c \cap \mathcal{C}_2 \cap \mathcal{C}_3 \cap \mathcal{B}$: by $(x,X,z) \in \mathcal{C}_2$ and $z_1 \leq z_2$, we have $x_1^2X_{22}(z_2-z_1) \leq X_{12}x_1x_2(z_2-z_1)$. Since $(x,X,z) \in \mathcal{C}_1^c$; i.e., $X_{12}z_1 - x_1x_2 > 0$ and $X_{12}z_2 - x_1x_2 > 0$, we can multiply the left-hand side of the two inequalities to obtain $X_{12}x_1x_2(z_2-z_1) < X_{12}^2z_1z_2 - 2x_1x_2X_{12}z_1 + (x_1x_2)^2$. Combining these two inequalities we get a contradiction with $(x,X,z) \in \mathcal{C}_3$.

(ii) $z_2 \leq z_1$: in this case by (25) defining Region (1), we have $X_{12}z_1 > x_1x_2$. Hence it suffices to show that any point $(x,X,z) \in \overline{\mathrm{conv}}(\mathcal{S}_2)$ satisfying $X_{12}z_1 > x_1x_2$ is also in the set $\mathcal{R}_4 \cup \mathcal{R}_5$. To this end, we need to show that any point $(x,X,z) \in \overline{\mathrm{conv}}(\mathcal{S}_2)$ satisfying $X_{12}z_1 > x_1x_2$ does not satisfy the two inequalities: $X_{12}x_2 > x_1X_{22}$ and $X_{12}z_2 \leq x_1x_2$. Using the valid inequality $X_{22}z_2 \geq x_2^2$ together with $X_{12}x_2 > x_1X_{22}$ we get $X_{12}x_2 > x_1\frac{x_2^2}{z_2}$, which is in contradiction with $X_{12}z_2 \leq x_1x_2$.

6 Region (2): $\lambda_4 = z_1 + z_2 - 1$

As we detailed in Section 4, over Region (2) defined by (26), there always exists an optimal solution of Problem (P) with $\lambda_4 = z_1 + z_2 - 1$. Hence, in order to characterize the closure of the convex hull of $\mathcal{S}_2$ over Region (2), in this section, given any (x, X, z) satisfying inequality (26) together with $x_1, x_2, X_{12} > 0$, we solve the following convex optimization problem analytically:

$$\min_{\tilde{x}_1^4, \tilde{x}_2^4} \quad \frac{(\tilde{x}_1^4)^2}{z_1 + z_2 - 1} + \frac{(x_1 - \tilde{x}_1^4)^2}{1 - z_2} + \frac{\left(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{z_1+z_2-1}\right)^2}{X_{22} - \frac{(\tilde{x}_2^4)^2}{z_1+z_2-1} - \frac{(x_2-\tilde{x}_2^4)^2}{1-z_1}}, \tag{BP}$$

$$\text{s.t.} \quad X_{22} - \frac{(\tilde{x}_2^4)^2}{z_1 + z_2 - 1} - \frac{(x_2 - \tilde{x}_2^4)^2}{1 - z_1} \geq 0, \tag{71}$$

$$0 \leq \tilde{x}_1^4 \leq x_1, \quad 0 \leq \tilde{x}_2^4 \leq x_2. \tag{72}$$

By (26) and $X_{12} > 0$, we get $z_1 + z_2 > 1$. We start by characterizing all points of zero-gradient of the objective function f of Problem (BP). Subsequently, we identify conditions under which each point of zero-gradient is feasible, which by convexity of Problem (BP) implies its optimality. It can be checked that

$$\frac{\partial f}{\partial \tilde{x}_1^4} = 0 \quad \Leftrightarrow \quad \frac{\tilde{x}_1^4}{z_1 + z_2 - 1} - \frac{x_1 - \tilde{x}_1^4}{1 - z_2} = \frac{\tilde{x}_2^4}{z_1 + z_2 - 1} \frac{h}{g_2}, \tag{73}$$

$$\frac{\partial f}{\partial \tilde{x}_2^4} = 0 \quad \Leftrightarrow \quad \frac{\tilde{x}_1^4}{z_1 + z_2 - 1} h g_2 = \left(\frac{\tilde{x}_2^4}{z_1 + z_2 - 1} - \frac{x_2 - \tilde{x}_2^4}{1 - z_1}\right) h^2, \tag{74}$$

where as before we define

$$h = X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{z_1 + z_2 - 1}, \qquad g_2 = X_{22} - \frac{(\tilde{x}_2^4)^2}{z_1 + z_2 - 1} - \frac{(x_2 - \tilde{x}_2^4)^2}{1 - z_1}$$

Notice that conditions (73) and (74) are defined for $z_1, z_2 \in (0, 1)$. Two cases arise:

- $h = 0$: in this case, by (73) and (74), the point of zero-gradient is given by:

  $$\tilde{x}_1^4 = (z_1 + z_2 - 1)\frac{x_1}{z_1}, \qquad \tilde{x}_2^4 = X_{12} \frac{z_1}{x_1}. \tag{75}$$

  It can be checked that over Region (2), the above point satisfies constraints (72). Hence, if inequality (71) is satisfied; i.e.,

  $$(1 - z_1)(z_1 + z_2 - 1)x_1^2(X_{22}z_2 - x_2^2) \geq \Big(X_{12}z_1z_2 - x_1x_2(z_1 + z_2 - 1)\Big)^2, \tag{76}$$

  then $(\tilde{x}_1^4, \tilde{x}_2^4)$ given by (75) is an optimal solution of Problem (BP). Let us denote by $\mathcal{R}_6$, the region defined by inequalities (26) and (76). Then by Lemma 4, over $\mathcal{R}_6$, the projection of $\tilde{\Sigma}$ defined by (20) onto the space (x, X, z) is given by:

  $$\tilde{\mathcal{S}} \cap \mathcal{R}_6 = \Big\{(x, X, z) : \ X_{11} \geq \frac{x_1^2}{z_1}, \ x_1 > 0, \ x_2 > 0, \ X_{12} > 0, \ z_1, z_2 \in (0, 1)\Big\} \cap \mathcal{R}_6.$$

  which in turn implies that

  $$\text{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_6) = \Big\{(x, X, z) : \ X_{11} \geq \frac{x_1^2}{z_1}, \ X_{22} \geq \frac{x_2^2}{z_2}, \ x_1 \geq 0, \ x_2 \geq 0, \ X_{12} \geq 0, \ z_1, z_2 \in [0, 1]\Big\} \cap \text{cl}(\mathcal{R}_6), \tag{77}$$

  where $\text{cl}(\mathcal{R}_6)$ consists of the set of points satisfying inequality (76) together with inequality $X_{12}z_1z_2 \leq x_1x_2(z_1 + z_2 - 1)$.

  This concludes the proof of Part (I) of Theorem 1 for $i = 6$.

- $h \neq 0$: in this case at any minimizer we must have $g_2 \neq 0$; then by (73) and (74), a point of zero-gradient satisfies the following:

$$\frac{\tilde{x}_1^4}{z_1+z_2-1} - \frac{x_1-\tilde{x}_1^4}{1-z_2} = \frac{\tilde{x}_2^4}{z_1+z_2-1}\frac{h}{g_2}, \tag{78}$$

$$\frac{\tilde{x}_1^4}{z_1+z_2-1} = \Big(\frac{\tilde{x}_2^4}{z_1+z_2-1} - \frac{x_2-\tilde{x}_2^4}{1-z_1}\Big)\frac{h}{g_2}, \tag{79}$$

From (78) and (79), it follows that

$$\frac{\tilde{x}_1^4}{z_1+z_2-1}\frac{(x_2-\tilde{x}_2^4)}{1-z_1} + \frac{\tilde{x}_2^4}{z_1+z_2-1}\frac{(x_1-\tilde{x}_1^4)}{1-z_2} = \frac{(x_1-\tilde{x}_1^4)(x_2-\tilde{x}_2^4)}{(1-z_1)(1-z_2)},$$

which (since $\tilde{x}_2^4 = x_2 z_1$ does not satisfy the equation) can be equivalently written as:

$$\tilde{x}_1^4 = x_1\Big(\frac{\tilde{x}_2^4 z_2 - x_2(z_1+z_2-1)}{\tilde{x}_2^4 - x_2 z_1}\Big). \tag{80}$$

Substituting (80) in (78) and (79), we find that the only point of zero-gradient of the objective function of Problem (BP) in this case is given by:

$$\tilde{x}_1^4 = \frac{X_{12}x_2(1-z_2) + x_1(X_{22}z_2 - x_2^2)}{X_{22}-x_2^2}, \tag{81}$$

$$\tilde{x}_2^4 = \frac{x_1(x_2^2 - X_{22}(1-z_1)) - X_{12}x_2z_1}{x_1x_2 - X_{12}}. \tag{82}$$

By feasibility we must have $g_2 > 0$, which by (79), (81) and using the valid inequality $X_{22}z_2 \geq x_2^2$ is satisfied if and only if:

$$\Big(\frac{\tilde{x}_2^4}{z_1+z_2-1} - \frac{x_2-\tilde{x}_2^4}{1-z_1}\Big)h > 0.$$

Moreover, by (26) we have

$$\begin{aligned}
&\frac{\tilde{x}_2^4}{z_1+z_2-1} - \frac{x_2-\tilde{x}_2^4}{1-z_1} < 0 \quad \Leftrightarrow \quad \tilde{x}_2^4 < (z_1+z_2-1)\frac{x_2}{z_2}\\
\Leftrightarrow \quad & x_1z_2(x_2^2 - X_{22}(1-z_1)) - X_{12}x_2z_1z_2 < x_2(z_1+z_2-1)(x_1x_2 - X_{12})\\
\Leftrightarrow \quad & x_1(X_{22}z_2 - x_2^2) + (1-z_2)x_2X_{12} > 0,
\end{aligned}$$

where the validity of the last inequality follows since for all $(x, X, z) \in \overline{\text{conv}}(\mathcal{S}_2)$ we have $X_{22}z_2 - x_2^2 \geq 0$. Hence we conclude that $g_2 > 0$, if and only if $h < 0$; i.e.,

$$\tilde{x}_1^4\tilde{x}_2^4 > (z_1+z_2-1)X_{12}. \tag{83}$$

Moreover, over Region (2), the above inequality can be equivalently written as:

$$\begin{aligned}
x_1^2(x_2^2 - X_{22}(1-z_1))(X_{22}z_2 - x_2^2) >& 2x_1x_2X_{12}z_1(X_{22}z_2 - x_2^2)\\
&-X_{12}^2(X_{22}(z_1+z_2-1) + x_2^2(1-2z_1-z_2(1-z_1))).
\end{aligned} \tag{84}$$

Now we examine the bounds constraints (72). First notice that since by (81) we have $\tilde{x}_1^4 \geq 0$, and inequality (83) implies that at any point with $g_2 \geq 0$, we also have $\tilde{x}_2^4 \geq 0$. Moreover, it can be checked that over Region (2), the constraints $\tilde{x}_1^4 \leq x_1$ and $\tilde{x}_2^4 \leq x_2$ are satisfied if and only if $x_1X_{22} \geq X_{12}x_2$. To see the validity of the latter inequality, suppose that $X_{12}x_2 > x_1X_{22}$. Then using the valid inequality $X_{22}z_2 \geq x_2^2$, we get $X_{12}x_2 > x_1X_{22} \geq x_1\frac{x_2^2}{z_2}$, implying $X_{12}z_2 > x_1x_2$, which is in contradiction with inequality (26) defining Region (2). Therefore, the point defined by (81) and (82) satisfies the bound constraints (72).

Let us denote by $\mathcal{R}_7$ the region defined by inequalities (26) and (84). We then conclude that over $\mathcal{R}_7$, the optimal solution of Problem (BP) and as a result Problem (P) is given by (81) and (82). Therefore, by Lemma 4, over $\mathcal{R}_7$, the projection of $\tilde{\Sigma}$ defined by (20) onto the space (x, X, z) is given by:

$$\tilde{\mathcal{S}} \cap \mathcal{R}_7 = \Big\{(x, X, z) : X_{22} \geq \frac{x_2^2}{z_2},\ (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1x_2)^2,\ x_1 > 0,\ x_2 > 0,\ X_{12} > 0,\ z_1, z_2 \in (0, 1)\Big\} \cap \mathcal{R}_7,$$

which in turn implies that

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_7) = \Big\{(x, X, z) : X_{11} \geq x_1^2,\ X_{22} \geq \frac{x_2^2}{z_2},\ (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1x_2)^2,\ x_1 \geq 0,\ x_2 \geq 0,\ X_{12} \geq 0,\ z_1, z_2 \in [0, 1]\Big\} \cap \mathrm{cl}(\mathcal{R}_7), \tag{85}$$

where $\mathrm{cl}(\mathcal{R}_7)$ is obtained from $\mathcal{R}_7$ by replacing the strict inequalities by nonstrict inequalities.

This concludes the proof of Part (IV) of Theorem 1 for $i = 7$.

The boundary $g_2 = 0$. By the above analysis, at any (x, X, z) in Region (2) that does not satisfy inequalities (77) and (85), the following convex optimization problem has no point of zero-gradient:

$$\min_{\tilde{x}_1^4, \tilde{x}_2^4} \quad \frac{(\tilde{x}_1^4)^2}{z_1 + z_2 - 1} + \frac{(x_1 - \tilde{x}_1^4)^2}{1 - z_2} + \frac{\left(X_{12} - \frac{\tilde{x}_1^4 \tilde{x}_2^4}{z_1 + z_2 - 1}\right)^2}{X_{22} - \frac{(\tilde{x}_2^4)^2}{z_1 + z_2 - 1} - \frac{(x_2 - \tilde{x}_2^4)^2}{1 - z_1}}, \tag{UBP}$$

$$\text{s.t.} \quad X_{22} - \frac{(\tilde{x}_2^4)^2}{z_1 + z_2 - 1} - \frac{(x_2 - \tilde{x}_2^4)^2}{1 - z_1} \geq 0. \tag{86}$$

Inequality (86) can be equivalently written as:

$$\lambda_4 \frac{x_2}{z_2} - \frac{\sqrt{(X_{22}z_2 - x_2^2)(1 - z_1)\lambda_4}}{z_2} \leq \tilde{x}_2^4 \leq \lambda_4 \frac{x_2}{z_2} + \frac{\sqrt{(X_{22}z_2 - x_2^2)(1 - z_1)\lambda_4}}{z_2}, \tag{87}$$

where as before $\lambda_4 = z_1 + z_2 - 1$. Over Region (2), all points of zero-gradient of the objective function of Problem (UBP) defined by (75), and (81)-(82) satisfy $\tilde{x}_4^2 \leq \lambda_4 \frac{x_2}{z_2}$. To see this, first consider (75); i.e., $\tilde{x}_2^4 = X_{12}\frac{z_1}{x_1}$. In this case $\tilde{x}_2^4 \leq \lambda_4 \frac{x_2}{z_2}$ follows directly from (26) defining Region (2). Next, consider (82); in this case $\tilde{x}_2^4 \leq \lambda_4 \frac{x_2}{z_2}$, can be equivalently written as

$$(1 - z_1)(X_{12}x_2(1 - z_2) + x_1(X_{22}z_2 - x_2^2)) \geq 0,$$

whose validity follows the fact that $X_{22}z_2 - x_2^2 \geq 0$ for all $(x, X, z) \in \overline{\mathrm{conv}}(\mathcal{S}_2)$.

Therefore, in constraint (87) the upper bound on $\tilde{x}_2^4$ is redundant and hence in Problem (UBP), we can replace constraint (86) by the following inequality:

$$\tilde{x}_2^4 \geq \lambda_4 \frac{x_2}{z_2} - \frac{\sqrt{(X_{22}z_2 - x_2^2)(1 - z_1)\lambda_4}}{z_2}. \tag{88}$$

Since Problem (UBP) has no point of zero-gradient inside its domain; it has a minimizer at the boundary of its domain; i.e., $g_2 = 0$, which in turn implies $h = 0$; that is, the minimizer of Problem (UBP) is given by:

$$\tilde{x}_1^4 = \frac{X_{12} z_2}{x_2 - \sqrt{\frac{(X_{22} z_2 - x_2^2)(1-z_1)}{z_1+z_2-1}}} \tag{89}$$

$$\tilde{x}_2^4 = (z_1 + z_2 - 1)\frac{x_2}{z_2} - \frac{\sqrt{(X_{22} z_2 - x_2^2)(1 - z_1)(z_1 + z_2 - 1)}}{z_2}. \tag{90}$$

To prove that the above point is also a minimizer of Problem (BP), we must show that it satisfies the bounds constraints (72). It can be checked $\tilde{x}_1^4 \geq 0$ and $\tilde{x}_2^4 \geq 0$, if

$$x_2^2 > X_{22}(1 - z_1). \tag{91}$$

We now show that (91) is valid since by assumption inequality (76) is not satisfied; i.e.,

$$\begin{aligned} &\Big(\frac{1 - z_1}{z_1 + z_2 - 1}\Big) x_1^2 (X_{22} z_2 - x_2^2) \leq \Big(x_1 x_2 - X_{12}\frac{z_1 z_2}{z_1 + z_2 - 1}\Big)^2, \\ \Leftrightarrow \quad &(X_{22}(1 - z_1) - x_2^2) x_1^2 (z_1 + z_2 - 1) \leq X_{12} z_1 (X_{12} z_1 z_2 - 2x_1 x_2 (z_1 + z_2 - 1)). \end{aligned} \tag{92}$$

The proof then follows since by (26) we have $X_{12} z_1 z_2 - 2x_1 x_2 (z_1 + z_2 - 1) < 0$, and hence $X_{22}(1 - z_1) - x_2^2 < 0$; i.e., inequality (91) holds. Hence we have $\tilde{x}_1^4 \geq 0$ and $\tilde{x}_2^4 \geq 0$.

Next, we prove $\tilde{x}_1^4 \leq x_1$; by (26) and (89), this inequality can be equivalently written as:

$$\Big(\frac{1 - z_1}{z_1 + z_2 - 1}\Big) x_1^2 (X_{22} z_2 - x_2^2) \leq (x_1 x_2 - X_{12} z_2)^2. \tag{93}$$

Over Region (2) defined by (26) we have

$$\Big(x_1 x_2 - X_{12}\frac{z_1 z_2}{z_1 + z_2 - 1}\Big)^2 < (x_1 x_2 - X_{12} z_2)^2.$$

The validity of inequality (93) then follows from (92). Finally, by (90) we have $\tilde{x}_2^4 \leq (z_1 + z_2 - 1)\frac{x_2}{z_2} \leq x_2$. We conclude that over the region defined by:

$$\begin{aligned} \mathcal{R}_8 := \Big\{ &(x, X, z) : \; X_{12} z_1 z_2 < x_1 x_2 (z_1 + z_2 - 1), \\ &(1 - z_1)(z_1 + z_2 - 1) x_1^2 (X_{22} z_2 - x_2^2) < \Big(X_{12} z_1 z_2 - x_1 x_2 (z_1 + z_2 - 1)\Big)^2, \\ &x_1^2 (x_2^2 - X_{22}(1 - z_1))(X_{22} z_2 - x_2^2) \leq 2x_1 x_2 X_{12} z_1 (X_{22} z_2 - x_2^2) \\ &\quad - X_{12}^2 (X_{22}(z_1 + z_2 - 1) + x_2^2 (1 - 2z_1 - z_2(1 - z_1)))\Big\}, \end{aligned} \tag{94}$$

the optimal solution of Problem (BP) is given by (89) and (90). Therefore, by Lemma 4, over $\mathcal{R}_8$, the projection of the set $\tilde{\Sigma}$ defined by (20) onto the space (x, X, z) is given by:

$$\begin{aligned} \tilde{\mathcal{S}} \cap \mathcal{R}_8 = \Big\{ &(x, X, z) : \; z_1 (1 - z_2)\Big(X_{11} - \frac{x_1^2}{z_1}\Big) x_2^2 \geq (z_1 + z_2 - 1)\Big(X_{12}\frac{z_1 z_2}{W} - x_1 x_2\Big)^2, \; x_1 > 0, \\ &x_2 > 0, \; X_{12} > 0, \; z_1, z_2 \in (0, 1)\Big\} \cap \mathcal{R}_8, \end{aligned}$$

where we define

$$W := (z_1 + z_2 - 1) - \frac{1}{x_2}\sqrt{(X_{22} z_2 - x_2^2)(1 - z_1)(z_1 + z_2 - 1)}.$$

It can be checked that by (91), we have $W > 0$. Moreover, it can be shown that

$$\mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_8) = \Big\{(x, X, z) : \ X_{11} \geq \frac{x_1^2}{z_1}, \ X_{22} \geq \frac{x_2^2}{z_2}, \ x_1 \geq 0, \ x_2 \geq 0, \ X_{12} \geq 0, \ z_1, z_2 \in [0, 1]$$
$$z_1(1 - z_2)\Big(X_{11} - \frac{x_1^2}{z_1}\Big)x_2^2 \geq (z_1 + z_2 - 1)\Big(X_{12}\frac{z_1 z_2}{W} - x_1 x_2\Big)^2\Big\} \cap \mathrm{cl}(\mathcal{R}_8), \quad (95)$$

where $\mathrm{cl}(\mathcal{R}_8)$ is obtained by replacing strict inequalities in (94) by non-strict inequalities.

This concludes the proof of Part (V) of Theorem 1 for $i = 8$.

7 A simple separation algorithm

In this section, we present a simple separation algorithm over the closure of the convex hull of $\mathcal{S}_2$. This algorithm can readily be incorporated in branch-and-cut based global solvers [18] to improve the quality of existing relaxations for nonconvex problems containing quadratic sets with indicator variables. Define

$$\tilde{\mathcal{C}} := \Big\{(x, X, z) : X_{11} \geq \frac{x_1^2}{z_1}, \ X_{22} \geq \frac{x_2^2}{z_2}, \ (X_{11} - x_1^2)(X_{22} - x_2^2) \geq (X_{12} - x_1 x_2)^2, \ X_{12} \geq 0,$$
$$x_1, x_2 \geq 0, \ z_1, z_2 \in [0, 1]\Big\}. \quad (96)$$

From Proposition 1 and Lemma 2 it follows that the set $\tilde{\mathcal{C}}$ is a convex relaxation of $\mathcal{S}_2$. We now define the separation problem:

The separation problem: Given a point $(\tilde{x}, \tilde{X}, \tilde{z}) \in \tilde{\mathcal{C}}$ with $\tilde{z}_1, \tilde{z}_2 > 0$, decide whether $(\tilde{x}, \tilde{X}, \tilde{z})$ is in the closure of the convex hull of $\mathcal{S}_2$ or not, and in the latter case, find a supporting hyperplane to $\overline{\mathrm{conv}}(\mathcal{S}_2)$ that is violated by $(\tilde{x}, \tilde{X}, \tilde{z})$.

In the following, given a set $\mathcal{C}$, we denote by $\partial\mathcal{C}$ the boundary of $\mathcal{C}$. In order to construct supporting hyperplanes of $\overline{\mathrm{conv}}(\mathcal{S}_2)$, we make use of the following lemma:

Lemma 7. *Given two sets $\mathcal{C}$, $\mathcal{R}$ such that*

$$\mathcal{R} := \{x : f_1(x) > 0, f_2(x) \geq 0\}, \quad \mathcal{C} \cap \mathcal{R} := \{x : g_1(x) > 0, g_2(x) \geq 0, f_1(x) > 0, f_2(x) \geq 0\},$$

then

$$\mathrm{cl}(\{g_1 = 0, g_2 > 0, f_1 > 0, f_2 > 0\}) \subset \partial\mathcal{C}.$$

Proof. Denote by $\mathcal{B}_r(x)$ a ball of radius r centered at x. By definition of $\mathcal{C} \cap \mathcal{R}$, if $\{g_1 = 0, g_2 > 0, f_1 > 0, f_2 > 0\} \subset \partial(\mathcal{C} \cap \mathcal{R})$, then $\{g_1 = 0, g_2 > 0, f_1 > 0, f_2 > 0\} \subset \partial(\mathcal{C} \cap \mathcal{R}) \setminus \partial\mathcal{R} \subset \partial\mathcal{C}$. The last inclusion can be proved as follows. Fix $x \in \partial(\mathcal{C} \cap \mathcal{R})$, by definition for every $n \in \mathbb{N}$ there exists $y_n \in \mathcal{B}_{1/n}(x) \cap (\mathcal{C} \cap \mathcal{R})$ and $z_n \in \mathcal{B}_{1/n}(x) \cap (\mathbb{R}^n \setminus (\mathcal{C} \cap \mathcal{R})) = (\mathcal{B}_{1/n}(x) \cap (\mathbb{R}^n \setminus \mathcal{C})) \cup (\mathcal{B}_{1/n}(x) \cap (\mathbb{R}^n \setminus \mathcal{R}))$. Hence, up to choosing a subsequence n_k, either $z_{n_k} \in \mathcal{B}_{1/n_k}(x) \cap (\mathbb{R}^n \setminus \mathcal{C})$ for every $k \in \mathbb{N}$, or $z_{n_k} \in \mathcal{B}_{1/n_k}(x) \cap (\mathbb{R}^n \setminus \mathcal{R})$ for every $k \in \mathbb{N}$. We deduce that $x \in \partial\mathcal{C} \cup \partial\mathcal{R}$. Hence $\partial(\mathcal{C} \cap \mathcal{R}) \subset \partial\mathcal{C} \cup \partial\mathcal{R}$. Since $\partial\mathcal{C}$ is closed, we conclude the proof of the lemma. □

Notice that in Lemma 7, if in addition we assume that $\mathcal{C}$ is a convex set, we get $\mathrm{cl}(\{g_1 = 0, g_2 > 0, f_1 > 0, f_2 > 0\}) \subset \partial\mathrm{cl}(\mathcal{C})$, since for a convex set we have $\partial\mathcal{C} = \partial\mathrm{cl}(\mathcal{C})$. We are now ready to present our separation algorithm; this algorithm consists of three steps outlined below:

Step 1. Choose the region: Find the region $\mathcal{R}_i$, $i \in \{1, \ldots, 8\}$ to which $(\tilde{x}, \tilde{X}, \tilde{z})$ belongs. From the proof of Theorem 1 it follows that $\cup_i \mathcal{R}_i \supseteq \tilde{\mathcal{C}}$. Moreover, since by construction $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ for all $i \neq j$, we conclude that there exists exactly one $\mathcal{R}_k$ for some $k \in \{1, \ldots, 8\}$ for which we have $(\tilde{x}, \tilde{X}, \tilde{z}) \in \mathcal{R}_k$.

Step 2. Check if the point is in or out: if all inequalities defining $\mathrm{cl}(\tilde{S} \cap \mathcal{R}_k) \setminus \mathrm{cl}(\mathcal{R}_k)$ are satisfied, then $(\tilde{x}, \tilde{X}, \tilde{z}) \in \overline{\mathrm{conv}}(\mathcal{S}_2)$ and we are done. Otherwise, we choose an inequality in $\mathrm{cl}(\tilde{S} \cap \mathcal{R}_k) \setminus \mathrm{cl}(\mathcal{R}_k)$ that is violated by $(\tilde{x}, \tilde{X}, \tilde{z})$ and go to the next step. Since by assumption $(\tilde{x}, \tilde{X}, \tilde{z}) \in \tilde{C}$, it follows that if $(\tilde{x}, \tilde{X}, \tilde{z}) \in \mathcal{R}_i$ for $i \in \{1, 2, 6, 7\}$, then $(\tilde{x}, \tilde{X}, \tilde{z}) \in \mathrm{cl}(\tilde{\mathcal{S}} \cap \mathcal{R}_i)$ and hence $(\tilde{x}, \tilde{X}, \tilde{z}) \in \overline{\mathrm{conv}}(\mathcal{S}_2)$. Therefore, in the next step, it suffices to consider regions $\mathcal{R}_i$, $i \in \{3, 4, 5, 8\}$.

Step 3. Generate a separating inequality: Since $(\tilde{x}, \tilde{X}, \tilde{z}) \in \tilde{\mathcal{C}}$, it follows that if $k \in \{3, 4\}$, then the inequality

$$\Big(X_{11} - \frac{x_1^2}{z_2}\Big)\Big(X_{22} - \frac{x_2^2}{z_2}\Big) \geq \Big(X_{12} - \frac{x_1 x_2}{z_2}\Big)^2, \tag{97}$$

must be violated by $(\tilde{x}, \tilde{X}, \tilde{z})$. Let us denote the above inequality by $q(x, X, z) \geq 0$. Now consider a point $(\hat{x}, \hat{X}, \hat{z})$ whose components are equal to $(\tilde{x}, \tilde{X}, \tilde{z})$ except for $\hat{X}_{11}$ which is chosen so that $g(\hat{x}, \hat{X}, \hat{z}) = 0$. Notice that since by assumption $(\tilde{x}, \tilde{X}, \tilde{z}) \in \mathcal{R}_k$ and since the inequalities defining $\mathcal{R}_k$ do not contain the variable X_{11}, we conclude that $(\hat{x}, \hat{X}, \hat{z}) \in \mathcal{R}_k$. Hence by Lemma 7, the point $(\hat{x}, \hat{X}, \hat{z})$ belongs to the boundary of $\overline{\mathrm{conv}}(\mathcal{S}_2)$. Denote by $h(x, X, z)$ the first-order Taylor expansion of $g(x, X, z)$ at $(x, X, z) = (\hat{x}, \hat{X}, \hat{z})$. Then the inequality $h(x, X, z) \geq 0$ is a supporting hyperplane of $\overline{\mathrm{conv}}(\mathcal{S}_2)$ that is violated by $(\tilde{x}, \tilde{X}, \tilde{z})$.

Similarly, if $k = 5$, then the inequality

$$\Big(X_{11} - \frac{x_1^2}{z_1}\Big)\Big(X_{22} - \frac{x_2^2}{z_1}\Big) \geq \Big(X_{12} - \frac{x_1 x_2}{z_1}\Big)^2,$$

must be violated and if $k = 8$, then the inequality

$$z_1(1 - z_2)\Big(X_{11} - \frac{x_1^2}{z_1}\Big)x_2^2 \geq (z_1 + z_2 - 1)\Big(X_{12}\frac{z_1 z_2}{W} - x_1 x_2\Big)^2,$$

must be violated. Since none of inequalities defining $\mathcal{R}_5$ or $\mathcal{R}_8$ contain the variable X_{11}, we can use the method described above to generate supporting hyperplanes of $\overline{\mathrm{conv}}(\mathcal{S}_2)$ corresponding to these inequalities that are violated by $(\tilde{x}, \tilde{X}, \tilde{z})$.

We conclude this section by providing the proof of Corollary 1, which follows from our separation algorithm.

Proof of Corollary 1. Let us consider Part (i) of the corollary; first note that by $X_{22}x_1 \geq X_{12}x_2$ and $X_{12}z_1 > x_1 x_2$, it follows that over the region defined by (4), we have $X_{22}z_1 \geq x_2^2$. The proof then follows from Part (III) of Theorem 1 together with Proposition 1, Lemma 2, and Lemma 7. The proof of Part (ii) follows by symmetry.

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