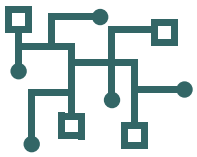

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# Quantum IPMs for Linear Optimization

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# Quantum IPMs for Linear Optimization

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**Abstract:** Quantum computing has potential to speed up some optimization methods, and one direction is using quantum linear algebra to solve linear systems arising in Interior Point Method (IPM). In this paper, we review state-of-the-art Quantum IPMs for solving linear optimization problems and analyze their complexities.

**Keywords:** Quantum Interior Point Methods, Quantum Linear Equation Solver, Quantum Tomography Algorithm, Inexact Interior Point Method, Iterative Refinement

**MSC codes:** 90C51, 90C05, 68Q12, 81P68

## 1 Introduction

Building efficient quantum computing algorithms is an active, emerging area of research. The Deutsch-Jozsa algorithm [8] was the first instance of a quantum algorithm outperforming the best classical algorithm. After this algorithm, many quantum algorithms are proposed with speed-up compared to algorithms on conventional computers to solve challenging mathematical problems, such as integer factorization [20] and unstructured search [10]. Due to various applications of mathematical optimization and the computational challenges on classical computers, several quantum computing optimization algorithms were developed such as the Quantum Approximation Optimization Algorithm (QAOA) for quadratic unconstrained binary optimization (QUBO) [9], and Quantum Interior Point Methods (QIPMs) for linear optimization problems (LOPs) [17].

QIPMs are analogous to classical Interior Points Methods (IPMs) that use Quantum Linear System Algorithms (QLSAs) to solve the Newton system at each iteration. Before reviewing the results related to QIPMs for linear optimization (or linear programming), let's define LOPs and some of their characteristics.

**Definition 1** (Linear Optimization Problem: Standard Form)**.** For $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and matrix $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = m$, the LOP is defined as

$$\begin{aligned} &\min\ c^T x, && &&\max\ b^T y, \\ \text{(P)}\quad &\text{s.t.}\ Ax = b, && \text{(D)}\quad &\text{s.t.}\ A^T y + s = c, \\ &\quad x \geq 0, && &\quad s \geq 0, \end{aligned}$$

where $x \in \mathbb{R}^n$ is the vector of primal variables, and $y \in \mathbb{R}^m$, $s \in \mathbb{R}^n$ are vectors of the dual variables. Problem $(P)$ is called the primal problem and problem $(D)$ is called the dual problem.

LOPs can also be presented in other forms, e.g., in canonical form.

**Definition 2** (Linear Optimization Problem: Canonical Form)**.** For $b \in \mathbb{R}^{m'}$, $c \in \mathbb{R}^{n'}$, and matrix $A \in \mathbb{R}^{m' \times n'}$, the canonical formulation of LOP is

$$\begin{aligned} &\min\ c'^T x, && &&\max\ b'^T y, \\ (\mathrm{P}')\quad &\text{s.t. } A'x \geq b', && (\mathrm{D}')\quad &\text{s.t. } A'^T y \leq c', \\ &\quad x \geq 0, && &\quad y \geq 0. \end{aligned}$$

The standard and canonical forms are equivalent, and one can derive both forms for any LOP. By choosing a basis in a standard-form problem, we can derive the equivalent canonical form. In this case, the canonical form has $n' = n - m$ variables and $m' = m$ constraints. An LOP in canonical form can be transformed into standard form just by adding slack variables. We are going to use the standard form, unless we notify that the canonical form is used. The set of primal-dual feasible solutions is defined as

$$\mathcal{PD} = \left\{(x, y, s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n :\ Ax = b,\ A^T y + s = c,\ (x, s) \geq 0\right\}.$$

Then, the set of interior feasible solutions is

$$\mathcal{PD}^0 = \left\{(x, y, s) \in \mathcal{PD}\ :\ (x, s) > 0\right\}.$$

By the Strong Duality Theorem [19], all optimal solutions, if there exist any, belong to the set $\mathcal{PD}^*$ defined as

$$\mathcal{PD}^* = \left\{(x, y, s) \in \mathcal{PD}\ :\ x^T s = 0\right\}.$$

For decades, Interior Point Methods (IPMs) have been widely employed to solve LOPs due to their polynomial complexity and fast convergence [23]. At each iteration of IPMs, a linear system of equations, the so-called Newton system, needs to be solved. The prevailing approach is using Cholesky factorization to solve the Normal Equation System (NES) [19], but some papers study inexact IPMs in which the Newton system is solved inexactly by iterative methods such as Conjugate Gradient methods [18]. Due to the development of quantum computing and its promise to solve linear systems fast but inexactly, inexact variants of IPMs have attracted greater interest than before. Most papers on inexact IPMs used Infeasible IPMs (I-IPMs), since inexact Newton steps may lead to infeasibility [16]. Recently, an Inexact Feasible IPM (IF-IPM) is proposed using a novel representation of the Newton system, the so-called Orthogonal Subspaces Systems (OSS) [17]. This IF-IPM attains the best iteration complexity of exact IPMs.

Some recent papers investigate Quantum Interior Point Methods (QIPMs) for LOPs. First, Kerenidis and Prakash [12] used Block Encoding and Quantum RAM (QRAM) for finding a $\zeta$-optimal solution with $\mathcal{O}(\frac{n^2}{\epsilon^2}\bar{\kappa}^3 \log(\frac{1}{\zeta}))$ complexity, where $\bar{\kappa}$ is a bound for the condition number of the Newton systems, $\epsilon$ is the QLSA's error bound, and $\zeta$ is the final complementarity gap. Casares and Martin-Delgado [5] used QLSA and developed a Predictor-correcter QIPM with $\mathcal{O}(L\sqrt{n}(n+m)\|\bar{M}\|_F \frac{\bar{\kappa}^2}{\epsilon^2})$ complexity, where $L$ is the binary length of input data and $\|\bar{M}\|_F$ is an upper bound on the Frobenius norm of the coefficient matrix of Newton system. Both papers used exact IPMs, which is not a valid choice for inexact QLSAs. To find an exact optimal solution, $\frac{1}{\epsilon}$ and $\bar{\kappa}$ increase exponentially, leading to exponential complexities. To address these problems, Mohammadisiahroudi et al. [16] developed an II-QIPM using QLSA efficiently with $\mathcal{O}(n^4 L \kappa_A^2)$ time complexity, where the constant $\kappa_A$ is the condition number of matrix $A$. In [16], an Iterative Refinement method (IRM) is used that made possible to replace the condition number of Newton

systems with the condition number of input matrix $A$. To improve this time complexity, Mohammadisiahroudi et al. [17] proposed a short-step IF-QIPM for solving LOPs using QLSAs to solve the OSS system.

The structure of this article is as follows. Quantum linear algebra is reviewed in Section 2. Then, two convergent QIPMs, II-QIPM and IF-QIPM are presented in Sections 3 and 4, respectively. Section 5 presents an Iterative Refinement Method (IRM) that helps to use limited-precision quantum oracles to solve LOPs exactly. Finally, a discussion regarding the complexity of different QIPMs along with the directions for improving them is provided in Section 6.

## 2 Quantum Linear Algebra

The first quantum linear system algorithm (QSLA) was the algorithm of Harrow, Hassidim and Loyd [11], which takes as input a sparse, square, Hermitian matrix $M$, and prepares a state $|z\rangle = |M^{-1}\sigma\rangle$ that is proportional to the solution of the linear system $Mz = \sigma$. Let $\kappa_M$ denote the condition number of $M$. The complexity of HHL algorithm is $\tilde{\mathcal{O}}_d\left(\frac{\tau^2\kappa_M^2}{\epsilon}\right)$, where $d$ is the dimension of the problem, $\tau$ is the maximum number of non-zeros found in any row of $M$, $\epsilon$ is the target bound on the error, and the $\tilde{\mathcal{O}}$ notation suppresses the polylogarithmic factors in the "Big-O" notation in terms of the subscripts. This complexity bound shows a speed-up w.r.t dimension, although it depends on an upper bound for the condition number $\kappa_M$ of the coefficient matrix. Following a number of improvements to the HHL algorithm [1, 22, 21, 7], the current state-of-the-art QLSA is attributed to Charkraborty et al. [6], who use variable-time *amplitude estimation* and so-called *block-encoded* matrices, while HHL algorithm uses *sparse-encoding* model [11]. The block-encoding model was formalized in [14], and it assumes that one has access to unitaries that store the coefficient matrix in their top-left block:

$$U = \begin{pmatrix} M/\psi & \cdot \\ \cdot & \cdot \end{pmatrix},$$

where $\psi \geq \|M\|$ is a normalization factor chosen to ensure that $U$ has operator norm at most 1. With assuming the access to QRAM, the QLSA of [6] has $\tilde{\mathcal{O}}_{d,\kappa_M,\frac{1}{\epsilon}}(\kappa_M\psi)$ complexity.

QLSAs provide a quantum state proportional to the solution. We cannot extract the classical solution by a single measurement. We need Quantum Tomography Algorithms (QTAs) to extract the classical solution. There are several papers improving QTAs, and the best QTA [2] has $\mathcal{O}(\frac{d\varrho}{\epsilon})$ complexity, where $\varrho$ is a bound for the norm of the solution.

**Theorem 1.** *The QLSA of [6] and the QTA of [2] can finds an inexact solution $\tilde{z}$, with $\|\sigma - M\tilde{z}\| \leq \epsilon$, in $\tilde{\mathcal{O}}_{d,\kappa_M,\frac{1}{\epsilon}}(d\kappa_M\frac{\|\sigma\|}{\epsilon})$ time complexity.*

*Proof.* The proof follows from the analysis done in Section 2 of [16]. □

Recently, an IRM using a QLSA+QTA as a subroutine for solving linear systems achieved an exponential speedup w.r.t. precision compared to QLSA+QTA to extract a classical solution with the same precision [15]. The IRM+QLSA+QTA algorithm provides a classical solution accurately in the sparse-encoding model with $\tilde{\mathcal{O}}(d\tau\kappa_M\log(\frac{\|\sigma\|}{\epsilon}))$ complexity, where $\tau$ is the sparsity parameter of matrix $M$. While this complexity bound depends on sparsity $\tau$ compared to the direct use of QLSA+QTA, it has better dependence on precision and it is more suitable for noisy intermediate-scale quantum (NISQ) devices. For simplicity, we use QLSA+QTA directly and improve the precision by an IRM outside QIPMs [16, 17] as described in Section 5.

## 3 Inexact-Infeasible QIPM

To speed up IPMs, Mohammadisiahroudi et al. [16] used QLSAs to solve the Newton system at each iteration of IPMs. QLSAs inherently produce inexact solutions. Thus, one approach for using QLSA efficiently is to develop an Inexact Infeasible QIPM (II-QIPM). In [16], they utilized the IPM proposed by Kojima et al. [13] with inexact Newton steps calculated by a QLSA. For any $0 < \gamma_1 < 1$ and $0 < \gamma_2$, the authors defined the neighborhood of the central path as

$$\mathcal{N}(\gamma_1, \gamma_2) = \big\{(x, y, s) \in \mathbb{R}^{n+m+n} \ : (x, s) \geq 0, \|(R_P, R_D)\| \leq \gamma_2 \frac{x^T s}{n}, \\ x_i s_i \geq \gamma_1 \frac{x^T s}{n} \text{ for } i \in \{1, \ldots, n\}\big\},$$

where $R_P = b - Ax$, and $R_D = c - A^T y - s$. The II-QIPM is designed to find a $\zeta$-optimal solution which belongs to set

$$\mathcal{PD}_I(\zeta) = \big\{(x, y, s) \in \mathbb{R}^{n+m+n} : (x, s) \geq 0, \|(R_P, R_D)\| \leq \zeta, \frac{x^T s}{n} \leq \zeta\big\}.$$

Given $(x^k, y^k, s^k) \in \mathcal{N}(\gamma_1, \gamma_2)$ and $0 < \beta_1 < 1$, let $\mu^k = \frac{(x^k)^T s^k}{n}$, then the Newton system is defined as

$$\begin{aligned} A\Delta x^k &= b - Ax^k, \\ A^T \Delta y^k + \Delta s^k &= c - A^T y^k - s^k, \\ X^k \Delta s^k + S^k \Delta x^k &= \beta_1 \mu^k e - X^k s^k, \end{aligned} \tag{FNS}$$

where $e$ is the all-one vector with appropriate dimension, $X^k = \text{diag}(x^k)$, and $S^k = \text{diag}(s^k)$. Instead of solving the full Newton system (FNS), we may solve the Augmented system or the Normal Equation System (NES). In the following, we use the NES formulated as

$$M^k \Delta y^k = \sigma^k, \tag{NES}$$

where

$$\begin{aligned} D^k &= (X^k)^{1/2}(S^k)^{-1/2}, \\ M^k &= A(D^k)^2 A^T, \\ \sigma^k &= A(D^k)^2 c - A(D^k)^2 A^T y^k - \beta_1 \mu^k A(S^k)^{-1} e + R_P^k = b - \beta_1 \mu^k A(S^k)^{-1} e + A(D^k)^2 R_D^k. \end{aligned}$$

The NES has a smaller size, $m$, than the FNS. Further, the matrix of the NES is symmetric and positive definite, thus Hermitian. Consequently, QLSAs can solve the NES efficiently. By its nature, a QLSA generates an inexact solution $\widetilde{\Delta y}^k$ with error bound $\|\Delta y^k - \widetilde{\Delta y}^k\| \leq \epsilon^k$. This error leads to residual $r^k$, thus we have

$$M^k \widetilde{\Delta y}^k = \sigma^k + r^k,$$

where $r^k = M^k(\widetilde{\Delta y}^k - \Delta y^k)$. After finding $\widetilde{\Delta y}^k$, the inexact $\widetilde{\Delta x}^k$ and $\widetilde{\Delta s}^k$ are computed classically as

$$\begin{aligned} \widetilde{\Delta s}^k &= c - A^T y^k - s^k - A^T \widetilde{\Delta y}^k, \\ \widetilde{\Delta x}^k &= \beta_1 \mu^k (S^k)^{-1} e - x^k - (S^k)^{-1} X^k \widetilde{\Delta s}^k. \end{aligned} \tag{1}$$

Since $\widetilde{\Delta s}^k$ and $\widetilde{\Delta x}^k$ are directly calculated by equations (1), one can verify that $(\widetilde{\Delta x}^k, \widetilde{\Delta s}^k, \widetilde{\Delta y}^k)$ satisfies

$$\begin{aligned} A\widetilde{\Delta x}^k &= b - Ax^k + r^k, \\ A^T\widetilde{\Delta y}^k + \widetilde{\Delta s}^k &= c - A^T y^k - s^k, \\ X^k\widetilde{\Delta s}^k + S^k\widetilde{\Delta x}^k &= \beta_1\mu^k e - X^k s^k. \end{aligned} \tag{2}$$

To have an II-QIPM using (NES) with iteration complexity $\mathcal{O}(n^2L)$, the residual norm must decrease at least $\mathcal{O}(\lambda_{\min}(A)\sqrt{n}\log n)$ time faster than $(x^k)^T s^k$, where $\lambda_{\min}(A)$ is the smallest singular value of $A$ [25]. Since tight residual bound leads to the high complexity of QLSA+QTA, in [16], the authors used a modification of the NES which leads to $\mathcal{O}(n^2L)$ iteration complexity of the proposed II-QIPM, where the residual is decreasing with the rate of $\mathcal{O}(\sqrt{\mu^k})$.

Since $A$ has full row rank, one can choose an arbitrary basis $\hat{B}$, and calculate $A_{\hat{B}}^{-1}$, $\hat{A} = A_{\hat{B}}^{-1}A$, and $\hat{b} = A_{\hat{B}}^{-1}b$. This calculation needs $\mathcal{O}(m^2n)$ arithmetic operations and happens just one time as preprocessing before the iterations of IPMs. The cost of this preprocessing is dominated by the cost of II-QIPM, but it can be reduced by using the structure of $A$. For example, if the problem is in the canonical form, there is no need for this preprocessing. In the rest of this paper, all methodology is applied to the preprocessed problem with input data $(\hat{A}, \hat{b}, c)$, where $\hat{A}$ includes an identity matrix. Now, the (NES) can be modified to

$$\hat{M}^k z^k = \hat{\sigma}^k, \tag{MNES}$$

where

$$\begin{aligned} \hat{M}^k &= (D_{\hat{B}}^k)^{-1}A_{\hat{B}}^{-1}M^k((D_{\hat{B}}^k)^{-1}A_{\hat{B}}^{-1})^T = (D_{\hat{B}}^k)^{-1}\hat{A}(D^k)^2((D_{\hat{B}}^k)^{-1}\hat{A})^T, \\ \hat{\sigma}^k &= (D_{\hat{B}}^k)^{-1}A_{\hat{B}}^{-1}\sigma^k = (D_{\hat{B}}^k)^{-1}\hat{b} - \beta_1\mu^k(D_{\hat{B}}^k)^{-1}\hat{A}(S^k)^{-1}e + (D_{\hat{B}}^k)^{-1}\hat{A}(D^k)^2(c - A^T y^k - s^k). \end{aligned} \tag{3}$$

By the following procedure, one can find the Newton direction by solving (MNES) inexactly with QLSA+QTA.

**Step 1.** Find $\tilde{z}^k$ such that $\hat{M}^k\tilde{z}^k = \hat{\sigma}^k + \hat{r}^k$ and $\|\hat{r}^k\| \leq \eta\frac{\sqrt{\mu^k}}{\sqrt{n}}$.

**Step 2.** Calculate $\widetilde{\Delta y}^k = ((D_{\hat{B}}^k)^{-1}A_{\hat{B}}^{-1})^T\tilde{z}^k$.

**Step 3.** Calculate $v^k = (v_{\hat{B}}^k, v_{\hat{N}}^k) = (D_{\hat{B}}^k\hat{r}^k, 0)$.

**Step 4.** Calculate $\widetilde{\Delta s}^k = c - A^T y^k - s^k - A^T\widetilde{\Delta y}^k$.

**Step 5.** Calculate $\widetilde{\Delta x}^k = \beta_1\mu^k(S^k)^{-1}e - x^k - (D^k)^2\widetilde{\Delta s}^k - v^k$.

The following Lemma shows how the inexact solution of (MNES) leads to residual only in the last equation of the Newton system.

**Lemma 1** (Lemma 4.1 in [16]). *For the Newton direction $(\widetilde{\Delta x}^k, \widetilde{\Delta y}^k, \widetilde{\Delta s}^k)$, we have*

$$\begin{aligned} A\widetilde{\Delta x}^k &= b - Ax^k, \\ A^T\widetilde{\Delta y}^k + \widetilde{\Delta s}^k &= c - A^T y^k - s^k, \\ X^k\widetilde{\Delta s}^k + S^k\widetilde{\Delta x}^k &= \beta_1\mu^k e - X^k s^k - S^k v^k. \end{aligned} \tag{4}$$

To have a convergent IPM, we need $\|S^k v^k\|_\infty \leq \eta \mu^k$, where $0 \leq \eta < 1$ is an enforcing parameter.

**Lemma 2** (Lemma 4.2 in [16]). *For the Newton direction* $(\widetilde{\Delta x}^k, \widetilde{\Delta y}^k, \widetilde{\Delta s}^k)$, *if the residual* $\|\hat{r}^k\| \leq \eta \frac{\sqrt{\mu^k}}{\sqrt{n}}$, *then* $\|S^k v^k\|_\infty \leq \eta \mu^k$.

To solve the MNES, $\hat{M}^k$ and $\hat{\sigma}^k$ need to be calculated. Calculating $\hat{M}^k$ and $\hat{\sigma}^k$ classically costs $\mathcal{O}(n^3)$ and $\mathcal{O}(n^2)$ arithmetic operations, respectively. To reduce the cost per iteration, $\hat{M}^k$ can be calculated in the quantum machine as $\hat{M}^k = (E^k)^T E^k$ efficiently [6], where $E^k = (D^k_{\hat{B}})^{-1} \hat{A} D^k$. Calculating $E^k$ classically costs just $\mathcal{O}(n^2)$ arithmetic operations. Based on Theorem 1, the following theorem presents the complexity of solving the (MNES) system by utilizing the QLSA of [6] and QTA of [2].

**Lemma 3** (Lemma 4.3 in [16]). *The QLSA by [6] and the QTA by [2] can build the* (MNES) *system, and produce a solution* $\widetilde{z}^k$ *for the* (MNES) *system satisfying* $\|\hat{r}^k\| \leq \eta \frac{\sqrt{\mu^k}}{\sqrt{n}}$ *with* $\tilde{\mathcal{O}}_{n, \kappa^k_E, \frac{1}{\mu^k}}(m\sqrt{n} \frac{\kappa^k_E \|\hat{\sigma}^k\|}{\sqrt{\mu^k}})$ *complexity, where* $\kappa^k_E$ *is the condition number of* $E^k$.

For solving LOPs, the II-QIPM of [16] is presented as Algorithm 1. In this algorithm, we use QLSA and QTA to solve the MNES.

**Algorithm 1** II-QIPM

Choose $\zeta > 0$, $\gamma_1 \in (0,1)$,$0 < \eta < \beta_1 < \beta_2 < 1$,
Choose $\omega \geq \max\{1, \|x^*, s^*\|_\infty\}$.
$k \leftarrow 0$, $(x^0, y^0, s^0) \leftarrow (\omega e, 0e, \omega e)$, $\mu^0 = \omega^2$, and $\gamma_2 \leftarrow \frac{\|(R^0_P, R^0_D)\|}{\mu^0}$
**while** $(x^k, y^k, s^k) \notin \mathcal{PD}_I(\zeta)$ **do**
    $\mu^k \leftarrow \frac{(x^k)^T s^k}{n}$
    Calculate $E^k \leftarrow (D^k_{\hat{B}})^{-1} \hat{A} D^k$ and $\hat{\sigma}^k$ by (3)
    $\epsilon^k_{QLSA} \leftarrow \eta \frac{\sqrt{\mu^k}}{2\sqrt{n}\|\sigma^k\|}$ and $\epsilon^k_{QTA} \leftarrow \eta \frac{\sqrt{\mu^k}}{2\sqrt{n}\|\sigma^k\|}$
    $(\Delta x^k, \Delta y^k, \Delta s^k) \leftarrow$ **solve** MNES$(\beta_1)$ by QLSA+QTA with precision $\epsilon^k_{QLSA}$ and $\epsilon^k_{QTA}$
    $\hat{\alpha}^k \leftarrow \max\Big\{\bar{\alpha} \in [0,1] \mid$ for all $\alpha \in [0, \bar{\alpha}]$ we have
$$\big((x^k, y^k, s^k) + \alpha(\Delta x^k, \Delta y^k, \Delta s^k)\big) \in \mathcal{N}(\gamma_1, \gamma_2) \text{ and}$$
$$(x^k + \alpha \Delta x^k)^T (s^k + \alpha \Delta s^k) \leq \big(1 - \alpha(1 - \beta_2)\big)(x^k)^T s^k \Big\}$$
    $(x^{k+1}, y^{k+1}, s^{k+1}) \leftarrow (x^k, y^k, s^k) + \hat{\alpha}^k(\Delta x^k, \Delta y^k, \Delta s^k)$
    **if** $\|x_{k+1}, s_{k+1}\|_\infty > \omega$ **then**
        **return** Primal or dual is infeasible.
    $k \leftarrow k + 1$
**return** $(x^k, y^k, s^k)$

It can be verified easily that e.g., $\beta_1 = 0.5$, $\beta_2 = 0.9995$, $\eta = 0.4$ and $\gamma_1 = 0.5$ yield a valid choice in Algorithm 1. The general scheme of II-QIPM is similar to the classical long-step I-IPM in which at each iteration we solve the (MNES) system with QLSA and QTA along with an Armijo line search to ensure the next step remains in the $\mathcal{N}(\gamma_1, \gamma_2)$ neighborhood of the central path.

**Theorem 2** (Theorem 4.1 in [16]). *If Algorithm 1 does not terminate in line 11, then it reaches a solution in $\mathcal{PD}_I(\zeta)$ at most $\mathcal{O}(n^2 \log \frac{\omega}{\zeta})$ iterations.*

The most expensive classical operations at each step of II-QIPM are some metarix-vector multiplications, which cost $\mathcal{O}(n^2)$ arithmetic operations. The cost of quantum operations of an iteration is given by Lemma 3, and it is bounded by $\tilde{\mathcal{O}}_{n,\kappa_{\hat{A}},\frac{1}{\zeta}}(n^2 \frac{\kappa_{\hat{A}}}{\zeta^4})$ throughout the II-QIPM using bounds on $\kappa_E^k$ and $\|\hat{\sigma}^k\|$ analyzed in [16]. Thus, the total time complexity of the II-QIPM [16] as presented in Algorithm 1 is

$$\tilde{\mathcal{O}}_{n,\kappa_{\hat{A}},\frac{1}{\zeta}}\left(n^4 \frac{\kappa_{\hat{A}}}{\zeta^4}\right).$$

As we can see the complexity of classical feasible IPMs (F-IPMs) is better, w.r.t both dimension and final precision, than the one of an II-QIPM. In the next section, an IF-QIPM is reviewed which has better complexity than other classical and quantum IPMs.

# 4 Inexact-Feasible QIPM

Small-neighborhood short-step F-IPMs have the best computational complexity, which can be further enhanced by solving the Newton system with QLSA+QTAs at each iteration. In order to investigate this opportunity, [17] proposed a novel IF-IPM. In each step of IPMs, there are three choices of linear systems to calculate the Newton step: Augmented system (AS), Normal Equation System (NES), and Full Newton System (FNS). Solving any of these three systems inexactly leads to residuals in the primal and/or dual feasibility equations. In [17], an inexact feasible IPM is developed by constructing a new system that offers a primal-dual feasible step based on a basis from orthogonal subspaces, avoiding the additional cost associated with II-IPMs. With this motivation, the authors utilized a short-step feasible IPM with inexact Newton steps.

For a feasible interior solution $(x^k, y^k, s^k) \in \mathcal{PD}^0$, the (FNS) system can be simplified as

$$\begin{aligned} A\Delta x^k &= 0, \\ A^T \Delta y^k + \Delta s^k &= 0, \\ X^k \Delta s^k + S^k \Delta x^k &= \beta \mu^k e - X^k s^k. \end{aligned} \qquad \text{(Feasible-FNS)}$$

Let $a_i$ be the $i$'th column of the matrix $A$. We define set $B \subseteq \{1, \ldots, n\}$ as the index set of $m$ linearly independent columns of $A$, and $A_B = [a_i]_{i \in B}$. Since $A$ has full row rank, $m$ linearly independent columns of $A$ exist. Thus, the matrix $A_B$ is non-singular, and $A_B^{-1}$, as the inverse of $A_B$ exits. For ease of exposition, it may be assumed w.l.g. that matrix $A_B$ is formed by the first $m$ columns of matrix $A$. By pivoting on matrix $A = \begin{bmatrix} A_B & A_N \end{bmatrix}$, we can construct $\hat{A} = \begin{bmatrix} I & A_B^{-1} A_N \end{bmatrix}$.

We also construct matrices $V \in \mathbb{R}^{n \times (n-m)}$ and $W \in \mathbb{R}^{n \times m}$ as follows

$$V = \begin{bmatrix} A_B^{-1} A_N \\ -I \end{bmatrix}, \qquad W = A^T.$$

Calculating $V$ needs $\mathcal{O}(mn^2)$ arithmetic operations. If the LOP is defined in canonical formulation, then there is no need to pay this cost. In practice, most of the constraints are inequalities and their

slack variables can be used to form basis $A_B$ which reduces this preprocessing cost. However, by the following reformulation, one can avoid this preprocessing cost for standard formulation:

$$\begin{aligned} \min\ & c^T x, \\ \text{s.t.}\ & Ax + x' = b, \\ & -Ax + x'' = -b, \\ & x, x', x'' \geq 0, \end{aligned} \qquad V = \begin{pmatrix} A \\ -A \\ -I \end{pmatrix},$$

Here, we have a basis given by $x'$ and $x''$ without extra computation. The price we pay is that the number of equality constraints doubles and the number of variables becomes $n+2m$, i.e., triples in the worst case. Vector $w_j$ is the $j$'th column of matrix $W$ (or the $j$'th row of matrix $A$), and vector $v_i$ denotes the $i$'th column of matrix $V$.

**Lemma 4** (Lemma 2.1 in [17]). *Vectors $w_j$ form a basis for the row space of $A$ and vectors $v_i$ form a basis for the null space of $A$. Consequently, for any $j \in \{1, \ldots, m\}$ and any $i \in \{1, \ldots, n-m\}$, we have $w_j^T v_i = 0$.*

Based on Lemma 4, using $(\lambda^k)^T = (\lambda_1^k, \ldots, \lambda_{n-m}^k)$, we reformulate (Feasible-FNS) as

$$\Delta x^k = V\lambda^k, \tag{5a}$$
$$\Delta s^k = -A^T \Delta y^k, \tag{5b}$$
$$X^k \Delta s^k + S^k \Delta x^k = \beta\mu^k e - X^k s^k. \tag{5c}$$

Substituting $\Delta x^k$ and $\Delta s^k$ in equation (5c) leads to

$$-X^k A^T \Delta y^k + S^k V \lambda^k = \beta\mu^k e - X^k s^k, \tag{OSS}$$

which has $n$ equations, $n-m$ variables in $\lambda^k$, and $m$ variables in $\Delta y^k$. After solving the (OSS) equation, $\Delta x^k$ and $\Delta s^k$ are calculated by equations (5a) and (5b).

**Lemma 5.** *The linear systems* (Feasible-FNS) *and* (OSS) *are equivalent.*

Due to the construction of the (OSS), the proof of Lemma 5 is obvious. Consequently, the system (OSS) has a unique solution because system (Feasible-FNS) has a unique solution [19].

Let $(\widetilde{\lambda}^k, \widetilde{\Delta y}^k)$ be an inexact solution of the system (OSS). Then, $\widetilde{\Delta x}^k$ and $\widetilde{\Delta s}^k$ can be calculated by equations (5a) and (5b). For the solution $(\widetilde{\Delta x}^k, \widetilde{\Delta s}^k, \widetilde{\Delta y}^k)$, we have

$$\begin{aligned} \widetilde{\Delta x}^k &= V\widetilde{\lambda}^k, \\ \widetilde{\Delta s} &= W\widetilde{\Delta y}^k, \\ X^k \widetilde{\Delta s}^k + S^k \widetilde{\Delta x}^k &= \beta\mu^k e - X^k s^k + r^k, \end{aligned} \tag{6}$$

where $r^k = S^k V(\widetilde{\lambda}^k - \lambda^k) - XA^T(\widetilde{\Delta y}^k - \Delta y^k)$, and $(\lambda^k, \Delta y^k)$ stands for the exact solution of (OSS). Regardless of the error in solving the (OSS), we have $\widetilde{\Delta x}^k \in \text{Null}(A)$ and $\widetilde{\Delta s}^k \in \text{Row}(A)$. Thus, for any step length $\alpha \in (0, 1]$, we have

$$\begin{aligned} A(x^k + \alpha\widetilde{\Delta x}^k) &= 0, \\ A^T(y^k + \alpha\widetilde{\Delta y}^k) + (s^k + \alpha\widetilde{\Delta s}^k) &= 0, \end{aligned} \tag{7}$$

which means that the inexact Newton step calculated by solving the (OSS) preserves both primal and dual feasibility. This feature of the (OSS) enables us to develop an IF-QIPM. As the coefficient matrix of the (OSS) is not symmetric, to solve the (OSS) by QLSA+QTAs, we must build the system $M^k z'^k = \sigma'^k$ where

$$M'^k = \frac{1}{\|M^k\|}\begin{bmatrix} 0 & M^k \\ (M^k)^T & 0 \end{bmatrix}, z'^k = \begin{pmatrix} 0 \\ z^k \end{pmatrix}, \text{ and } \sigma'^k(\beta) = \frac{1}{\|M^k\|}\begin{pmatrix} \sigma^k(\beta) \\ 0 \end{pmatrix}, \tag{8}$$

where $M^k = \begin{bmatrix} -X^k A^T & S^k V \end{bmatrix}$ and $\sigma^k(\beta) = \beta\mu^k e - X^k s^k$. The new system can be implemented in a quantum setting and solved by QLSA since $M'^k$ is a Hermitian matrix and $\|M'^k\| = 1$. Similar to II-QIPMs, we use the QTA of [2] to extract the classical solution. Theorem 3 presents the complexity bound for QLSA by [6] to solve the OSS system.

**Theorem 3** (Theorem 4.1 of [17]). *Given the linear system (OSS), QLSA of [6] and QTA of [2] provide a solution $(\widetilde{\lambda^k}, \widetilde{\Delta y^k})$ with residual $r^k$, where $\|r^k\| \leq \eta\mu^k$, in at most $\tilde{\mathcal{O}}_{n,\kappa_A,\frac{1}{\mu^k}}(n^2\frac{\kappa_A}{(\mu^k)^2})$ total time.*

Algorithm 2 is the IF-QIPM of [17] with a classical short-step IPM structure, where QLSA and QTA is used to solve the (OSS) system. Similar to other F-IPMs, the authors used a small neighborhood of the central path, with $\theta \in [0, 1)$, defined as

$$\mathcal{N}(\theta) = \Big\{(x, y, s) \in \mathcal{PD}^0 \ : \|XSe - \frac{x^T s}{n}e\|_2 \leq \theta\frac{x^T s}{n}\Big\}.$$

For IF-QIPM, the set of $\zeta$-optimal solution is definded as $\mathcal{PD}_F(\zeta) = \{(x, y, s) \in \mathcal{PD} : \frac{x^T s}{n} \leq \zeta\}$. Theorem 4 shows that the IF-QIPM attains the best iteration complexity of exact IPMs.

**Algorithm 2** IF-QIPM using QLSA

Choose $\zeta > 0$, $\eta = 0.1$, $\theta = 0.3$ and $\beta = (1 - \frac{0.11}{\sqrt{n}})$.
$k \leftarrow 0$
Choose initial feasible interior solution $(x^0, y^0, s^0) \in \mathcal{N}(\theta)$
**while** $(x^k, y^k, s^k) \notin \mathcal{PD}_F(\zeta)$ **do**
    $\mu^k \leftarrow \frac{(x^k)^T s^k}{n}$
    $(M^k, \sigma^k(\beta)) \leftarrow$**build** system (8)
    $\epsilon_{QLSA} \leftarrow \frac{\eta\mu^k}{2\|\sigma^k(\beta)\|}$
    $\epsilon_{QTA} \leftarrow \frac{\eta\mu^k}{2\|\sigma^k(\beta)\|}$
    $(\lambda^k, \Delta y^k) \leftarrow$ **solve** the $(M^k, \sigma^k(\beta))$ using QLSA and QTA
    $\Delta x^k = V\lambda^k$ and $\Delta s^k = -A^T\Delta y^k$
    $(x^{k+1}, y^{k+1}, s^{k+1}) \leftarrow (x^k, y^k, s^k) + (\Delta x^k, \Delta y^k, \Delta s^k)$
    $k \leftarrow k + 1$
**return** $(x^k, y^k, s^k)$

**Theorem 4** (Theorem 2.6 in [17]). *The IF-QIPM produces a solution in $\mathcal{PD}_F(\zeta)$ after at most $\mathcal{O}(\sqrt{n}\log(\frac{\mu^0}{\zeta}))$ iterations.*

Detailed complexity of IF-QIPM in [17] shows that the cost of classical operations at each iteration is $\mathcal{O}(n^2)$, the cost of quantum operations is $\tilde{\mathcal{O}}_{n,\kappa_A,\frac{1}{\zeta}}(n^2\frac{\kappa_A}{\zeta^2})$, and the total time complexity of the IF-QIPM is

$$\tilde{\mathcal{O}}_{n,\kappa_A,\frac{1}{\zeta}}(n^{2.5}\frac{\kappa_A}{\zeta^2}).$$

For details of the complexity analysis of this IF-QIPM, the reader is referred to [17]. This time complexity includes $\frac{1}{\zeta^2}$, which indicates the exponential time for finding an exact solution. In the next section, we discuss how an IRM can address this issue.

# 5 Application of the Iterative Refinement Method

As first used in the complexity analysis of Ellipsoid Method, one can find the exact optimal solution by using a rounding procedure on solution of an IPM with $\frac{1}{\zeta} = 2^{\mathcal{O}(L)}$ [19], where

$$L = mn + m + n + \sum_{i,j}\lceil\log(|a_{ij}|+1)\rceil + \sum_{i}\lceil\log(|c_i|+1)\rceil + \sum_{j}\lceil\log(|b_j|+1)\rceil.$$

To get an exact optimal solution, the time complexity of both II-QIPM and IF-QIPM contain an exponential term $\mathcal{O}(\frac{1}{\zeta}) = \mathcal{O}(2^L)$. To address this problem, we can fix $\zeta = 10^{-2}$ and improve the precision by an Iterative Refinement method (IRM) in $\mathcal{O}(L)$ iterations [16]. Here, we discuss the IR-IF-QIPM of [17]. The reader can also find an IR-II-QIPM in [16] with a partially different IRM. Let us consider an LOP in standard form with $(A, b, c)$. Let $\nabla > 1$ be a scaling factor. Given a feasible solution $(x, y, s) \in \mathcal{PD}$, the refining problem is defined as

$$\begin{aligned} \min_{\hat{x}} \ & \nabla s^T\hat{x}, & \max_{\hat{y},\hat{s}} \ & -\nabla x^T\hat{s}, \\ \text{s.t.} \ & A\hat{x} = 0, & \text{s.t.} \ & A^T\hat{y}+\hat{s} = \nabla s, \\ & \hat{x} \geq -\nabla x, & & \hat{s} \geq 0, \end{aligned} \tag{9}$$

One can easily reformulate this problem to a standard LOP by changing variables.

**Lemma 6** (Lemma 4.4 in [17]). *If $(\hat{x}, \hat{y}, \hat{s})$ is a $\zeta$-optimal solution for the refining problem (9) with $(x, y, s) \in \mathcal{PD}$ and $x^r = x + \frac{1}{\nabla}\hat{x}$, $y^r = y + \frac{1}{\nabla}\hat{y}$, and $s^r = c - A^Ty^r$, then $(x^r, y^r, s^r) \in \mathcal{PD}_F(\frac{\zeta}{\nabla^2})$, i.e. $(x^r, y^r, s^r)$ is a $\frac{\zeta}{\nabla^2}$-optimal solution for the LOP $(A, b, c)$.*

Based on the idea of Lemma 6, an IR-IF-QIPM is developed as Algorithm 3.

**Algorithm 3** IR-IF-QIPM

**Require:** $\left(A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, \hat{\zeta} < \zeta < 1\right)$

1: $k \leftarrow 0$ and $\nabla_0 \leftarrow 1$
2: $(x^0, y^0, s^0) \leftarrow$ **solve** $(A, b, c)$ using IF-QIPM with $\zeta$ precision
3: **while** $(x^k, y^k, s^k) \notin \mathcal{PD}_F(\hat{\zeta})$ **do**
4: $\quad \nabla^k \leftarrow \frac{1}{(x^k)^T s^k}$
5: $\quad (\hat{x}^k, \hat{y}^k, \hat{s}^k) \leftarrow$ **solve** (9) with $(A, \nabla^k x^k, \nabla^k s^k)$ using IF-QIPM with $\zeta$ precision
6: $\quad x^{k+1} \leftarrow x^k + \frac{1}{\nabla^k}\hat{x}^k$, $y^{k+1} \leftarrow y^k + \frac{1}{\nabla^k}\hat{y}^k$, and $s^{k+1} = c - A^Ty^{k+1}$
7: $\quad k \leftarrow k + 1$

The following theorem shows that, considering precision, the number of iterations of IR is logarithmic. One can use $\zeta = 10^{-2}$ and $\hat{\zeta} = 2^{-2L}$ to get an exact solution of an LOP.

**Theorem 5** (Theorem 4.5 of [17]). *The IR-IF-QIPM finds a $\hat{\zeta}$-optimal solution using at most $\mathcal{O}(\frac{\log(\hat{\zeta})}{\log(\zeta)})$ inquiry to IF-QIPM with precision $\zeta$.*

The total time complexity of finding an exact optimal solution with IR-IF-QIPM is

$$\mathcal{O}(n^{2.5}L\kappa_A).$$

For more details of the complexity analysis of IR-IF-QIPM, please consult [17].

# 6 Conclusions

As Table 1 shows, using the result of [17], the best theoretical bound for solving LOPs is improved w.r.t dimension for the first time, while the complexity still depends on the constant condition number of $A$, $\kappa_A$. In addition, the IR-IF-QIPM has much better time complexity than IR-II-QIPM for solving LOPs.

Table 1: Complexity of classical and quantum IPMs

| **Algorithm** | **System** | **LS Solver** | **Complexity** |
|---|---|---|---|
| Best Theoretical bound | NES | Partial Update | $\mathcal{O}(n^3L)$ |
| Feasible IPM | NES | Cholesky | $\mathcal{O}(n^{3.5}L)$ |
| IR-II-QIPM | NES | QTA+QLSA | $\mathcal{O}\left(n^4L\kappa_A\right)$ |
| IR-IF-QIPM | OSS | QTA+QLSA | $\mathcal{O}\left(n^{2.5}L\kappa_A\right)$ |

The IRM that uses IF-QIPM with low precision reduces dependence on both inexactness of QLSA+QTAs and the growing condition number of the Newton systems. The IRM+QLSA+QTA of [15] can also be used to solve the Newton system accurately; however, an outer IRM is still needed to address the exponential complexity caused by the condition number of the Newton system. Another beneficial direction is to study preconditioning or regularizing the Newton systems to mitigate the impact of the condition number in QIPMs. The IF-QIPM of [17] was the first attempt to have an inexact but feasible IPM to solve LOPs by modifying the Newton system to guarantee the feasibility of the inexact solution. An IF-QIPM and an II-QIPM were also developed for other optimization problems, such as Linearly Constrained Quadratic Optimization (LCQO) [24], Semidefinite Optimization (SDO) [3], and Second Order Cone Optimization (SOCO) [4]. Further developments may include QIPMs for other optimization problems.

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