

INDUSTRIAL AND SYSTEMS ENGINEERING

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# A Proximal-Gradient Method for Constrained Optimization

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ISE Technical Report 24T-006



# A Proximal-Gradient Method for Constrained Optimization \*

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**Abstract.** We present a new algorithm for solving optimization problems with objective functions that are the sum of a smooth function and a (potentially) nonsmooth regularization function, and nonlinear equality constraints. The algorithm may be viewed as an extension of the well-known proximal-gradient method that is applicable when constraints are not present. To account for nonlinear equality constraints, we combine a decomposition procedure for computing trial steps with an exact merit function for determining trial step acceptance. Under common assumptions, we show that both the proximal parameter and merit function parameter eventually remain fixed, and then prove a worst-case complexity result for the maximum number of iterations before an iterate satisfying approximate first-order optimality conditions for a given tolerance is computed. Our preliminary numerical results indicate that our approach has great promise, especially in terms of returning approximate solutions that are structured (e.g., sparse solutions when a one-norm regularizer is used).

**Key words.** nonlinear optimization, nonconvex optimization, worst-case iteration complexity, regularization methods, sequential quadratic programming, sequential quadratic optimization

**AMS subject classifications.** 49M37, 65K05, 65K10, 65Y20, 68Q25, 90C30, 90C60

**1. Introduction.** In this paper we consider the problem

$$(1.1) \quad \min_{x \in \mathbb{R}^n} f(x) + r(x) \quad \text{subject to (s.t.) } c(x) = 0,$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable,  $r : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex (possibly non-differentiable) and nonnegative valued, and  $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuously differentiable with  $m \leq n$ . The optimization problem (1.1) has applications in model predictive control [2], image processing [35], nonsmooth optimization on a Steifel manifold [34], and low rank matrix completion [6]. In addition, optimization problems such as sparse approximation, empirical risk minimization, and neural network modeling with mixed activations can be reformulated as (1.1); see [37] for additional details.

When the regularizer  $r$  is not present, the algorithms most commonly employed to solve problem (1.1) are penalty methods [9, 11, 12, 15, 28, 32, 36] and sequential quadratic optimization (SQO) methods [1, 19, 20, 22, 23, 24, 29, 31]. Penalty methods is to solve problem (1.1) by minimizing a sequence of unconstrained optimization subproblems defined in terms of  $f$ , a measure of constraint violation, and various parameters (e.g., Lagrange multiplier estimates and penalty parameters). After each minimization subproblem in the sequence is solved, the parameters are updated in a manner that allows for convergence guarantees. Since computing each subproblem minimizer may be expensive, and the number of subproblems solved may be nontrivial, penalty methods often require a significant amount of computation (e.g., numbers of iterations, function/derivative evaluations, and linear systems solved), which may be prohibitive. On the other hand, during each iteration of a line search SQO method, the main expense is the computation of a search direction, which is achieved by solving a single linear system of equations. Equivalently, the search direction is the minimizer of a certain quadratic approximation of  $f$  subject to a linearization of the constraints. SQO methods are generally viewed as the state-of-the-art because

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\*This work is supported by the U.S. Office of Naval Research under Award Number N00014-21-1-2532 and by the U.S. National Science Foundation, Division of Mathematical Sciences, Computational Mathematics Program under Award Number DMS-2012243.

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42 of their remarkable practical performance. The superior performance of line search  
43 SQO methods over penalty methods can be attributed to two main sources. First,  
44 line search SQO methods solve a sequence of linear systems rather than a sequence  
45 of general optimization subproblems. Second, the search directions for SQO methods  
46 are designed to find a solution of problem (1.1) (again, when  $r$  is not present), whereas  
47 penalty methods *indirectly* aim to find a solution of problem (1.1) (again, when  $r$  is  
48 not present) by adjusting its parameters after minimizing each subproblem.

49 When the constraint  $c(x) = 0$  is not present in problem (1.1), the algorithms most  
50 commonly employed are variants of the proximal-gradient (PG) method [4, 5, 7, 8,  
51 25, 26]. Each iterate of a basic PG method is the minimizer of a subproblem (i.e., the  
52 PG subproblem) formed by replacing  $f$  in (2.1) by the sum of its first-order Taylor  
53 expansion (expanded at the current point) and a simple quadratic-regularization term.  
54 For some commonly used regularizers, the PG subproblem has a closed-form solution,  
55 which is an attractive feature of such methods. Moreover, since the regularizer  $r$  is  
56 explicitly used in the definition of the PG subproblem (i.e., it is not approximated), the  
57 solutions generated by a PG method inherit the structure induced by the regularizer  
58 (e.g., if  $r(x) = \|x\|_1$ , then a PG method can produce sparse solution estimates). This  
59 *structure preserving* property is an important feature of PG methods when used to  
60 solve problem (1.1) (again, when the constraint  $c(x) = 0$  is not present).

61 The work in this paper is motivated by both SQO methods for solving (1.1) when  
62  $r$  is not present, and the structure preserving property of PG methods for solving (1.1)  
63 when the constraint  $c(x) = 0$  is not present. In particular, we design and analyze a  
64 method for solving problem (1.1) based on subproblems that linearize the constraints  
65 (like SQO methods) and explicitly use the regularizer (like PG methods).

66 **1.1. Literature review.** We are aware of four papers, namely [14, 17, 27, 37],  
67 that present algorithms for minimizing regularized optimization functions subject to  
68 nonlinear constraints. The algorithms in [14, 37] are penalty methods built upon  
69 the popular augmented Lagrangian function. Therefore, both approaches have a  
70 penalty parameter and a vector of Lagrange multiplier estimates that balance the  
71 objective and constraint functions, and must be updated throughout the optimization  
72 procedure. We note that [14] can solve regularized optimization problems with both  
73 equality and inequality constraints, whereas the algorithm in [37] can only handle  
74 special classes of regularized optimization problems with constraints. The algorithms  
75 presented in [17, 27] are of the SQO variety. In [17], subgradient information of  
76 the nonsmooth function is used to formulate a sequence of min-max subproblems.  
77 Since the regularizer is approximated in each subproblem, the structure preserving  
78 property of the iterates is lost. In contrast, [27] relies on a smoothing technique  
79 that approximates the nonsmooth term in the objective function and, thereafter,  
80 sequentially solves a convex quadratic problem with linear constraints. Unfortunately,  
81 in general, the smoothing technique ruins the structure of the composite optimization  
82 problem, and consequently the structure preserving property is lost.

83 **1.2. Contributions.** Our contributions relate to the proposal and analysis of a  
84 new algorithm for solving problem (1.1), as we summarize next.

- 85 • We propose a PG-based algorithm for solving problem (1.1) that uses sub-  
86 problems with linearized constraints (like SQO methods) and explicit regu-  
87 larization (like PG methods). The method that results from this combina-  
88 tion avoids the previously discussed challenges and weaknesses of augmented  
89 Lagrangian approaches, and provides solution estimates that are structure  
90 preserving. During each iteration, we compute a trial step as the sum of two

orthogonal directions called the normal and tangential steps. First, the normal step is computed from a trust region subproblem designed to reduce the constraint violation. Second, the tangential step is computed from a linearly constrained convex optimization subproblem with objective function reminiscent of PG methods (i.e.,  $r$  appears explicitly and a proximal term is used). Overall, the tangential step aims to reduce the objective function while maintaining the predicted progress in reducing infeasibility achieved by the normal step. The quality of the trial step, defined as the sum of the normal and tangential steps, is then determined by an  $\ell_2$  merit function that uses a merit parameter to weight the objective function relative to the two-norm of the constraint violation. The merit parameter and PG parameter (i.e., the weight on the proximal term) are reduced as the iterations proceed, if necessary, to promote convergence of the iterates to a solution of problem (1.1).

- Under minimal assumptions, we prove that a measure of first-order optimality for a feasibility problem converges to zero. Under additional commonly used assumptions, we prove that the merit parameter and PG parameter both remain uniformly bounded away from zero. These results allow us to then prove that our algorithm generates a sequence of iterates such that any limit point is a KKT point (see Theorem 3.17). Moreover, we provide a worst-case complexity result for the maximum number of iterations before a certain criticality measure will be less than a given tolerance (see Theorem 3.15).
- We present numerical experiments that verify our theoretical convergence results, and illustrate that our algorithm is capable of returning solutions that preserve the structure related to  $r$ . Specifically, we confirm that our method returns sparse solution estimates when  $r$  is chosen as the  $\ell_1$ -norm function, which is known to be a sparsity-inducing regularizer.

**1.3. Notation and assumptions.** We use  $\mathbb{R}$  to denote the set of real numbers (i.e., scalars),  $\mathbb{R}_{\geq 0}$  (resp.,  $\mathbb{R}_{> 0}$ ) to denote the set of nonnegative (resp., positive) real numbers,  $\mathbb{R}^n$  to denote the set of  $n$ -dimensional real vectors, and  $\mathbb{R}^{m \times n}$  to denote the set of  $m$ -by- $n$ -dimensional real matrices. The set of natural numbers is denoted as  $\mathbb{N} := \{0, 1, 2, \dots\}$ . Given a matrix  $M \in \mathbb{R}^{m \times n}$ , we let  $\sigma_{\min}(M)$  (resp.,  $\sigma_{\max}(M)$ ) denote the smallest (resp., largest) singular value of  $M$ . For  $v \in \mathbb{R}^n$ , we let  $\|v\|_2 := \sqrt{\sum_{i=1}^n v_i^2}$  denote its two norm. For a nonempty compact set  $\mathcal{R} \subset \mathbb{R}^n$ , we let  $\|\mathcal{R}\|_2 := \max\{\|s\|_2 : s \in \mathcal{R}\}$  denote its largest element measured in the two-norm.

The following assumption is used throughout the paper.

**ASSUMPTION 1.1.** *Let  $\mathcal{X} \subset \mathbb{R}^n$  be an open convex set that contains the iterates  $\{x_k\} \subset \mathbb{R}^n$  and trial steps  $\{x_k + s_k\} \subset \mathbb{R}^n$  generated by Algorithm 2.1. The function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable and bounded below over  $\mathcal{X}$  and its gradient function  $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is Lipschitz continuous and bounded over  $\mathcal{X}$ . Similarly, the function  $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuously differentiable and bounded over  $\mathcal{X}$  and its Jacobian  $\nabla c(x)^T$  is Lipschitz continuous and bounded over  $\mathcal{X}$ . Finally, the function  $r : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is convex and its subdifferential  $\partial r : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is bounded over  $\mathcal{X}$ .*

Under Assumption 1.1, there exist constants  $(f_{\inf}, \kappa_{\nabla f}, \kappa_{\partial r}, \kappa_c, \kappa_{\nabla c}, L_g, L_J) \in \mathbb{R} \times \mathbb{R}_{> 0} \times \mathbb{R}_{> 0} \times \mathbb{R}_{> 0} \times \mathbb{R}_{> 0} \times \mathbb{R}_{> 0} \times \mathbb{R}_{> 0}$  such that for all  $x \in \mathcal{X}$  one has

$$(1.2) \quad \begin{aligned} f(x) &\geq f_{\inf}, & \|\nabla f(x)\|_2 &\leq \kappa_{\nabla f}, & \|\partial r(x)\|_2 &\leq \kappa_{\partial r}, \\ \|c(x)\|_2 &\leq \kappa_c, & \|\nabla c(x)^T\|_2 &\leq \kappa_{\nabla c}, \end{aligned}$$

136 and for all  $(x, \bar{x}) \in \mathcal{X} \times \mathcal{X}$  one has

$$137 \quad (1.3) \quad \|\nabla f(x) - \nabla f(\bar{x})\|_2 \leq L_g \|x - \bar{x}\|_2 \quad \text{and} \quad \|\nabla c(x)^T - \nabla c(\bar{x})^T\|_2 \leq L_J \|x - \bar{x}\|_2.$$

138 For convenience, we define  $g(x) := \nabla f(x)$  and  $J(x) := \nabla c(x)^T$ . We append a  
 139 natural number as a subscript for a quantity to denote its value during an iteration  
 140 of an algorithm; i.e., we let  $f_k := f(x_k)$ ,  $g_k := g(x_k)$ ,  $c_k := c(x_k)$ , and  $J_k := J(x_k)$ .

141 **1.4. Organization.** In Section 2, we propose our algorithm for solving prob-  
 142 lem (1.1), and its convergence properties are presented in Section 3. In Section 4, we  
 143 discuss our numerical tests. Final conclusions are provided in Section 5.

144 **2. Algorithm.** The algorithm that we propose for solving problem (1.1) is for-  
 145 mally stated as Algorithm 2.1. Given the  $k$ th iterate  $x_k$ , the  $k$ th PG parameter  $\alpha_k$ ,  
 146 and constant  $\kappa_v \in \mathbb{R}_{>0}$ , we compute a step  $v_k$  that aims to reduce the constraint  
 147 infeasibility at  $x_k$  as an approximate solution to the following problem:

$$148 \quad (2.1) \quad \min_{v \in \mathbb{R}^n} m_k(v) \quad \text{s.t.} \quad \|v\|_2 \leq \kappa_v \alpha_k \|J_k^T c_k\|_2, \quad \text{with} \quad m_k(v) := \frac{1}{2} \|c_k + J_k v\|_2^2.$$

149 The PG parameter  $\alpha_k$  is used to define the trust-region constraint so that  $\{v_k\} \rightarrow 0$   
 150 if  $\{\alpha_k\} \rightarrow 0$ . We consider a vector  $v_k$  to be an adequate approximate solution to  
 151 subproblem (2.1) if, for some  $\kappa_v \in \mathbb{R}_{>0}$ , it satisfies the following conditions:

$$152 \quad (2.2a) \quad v_k \in \text{Range}(J_k^T),$$

$$153 \quad (2.2b) \quad \|v_k\|_2 \leq \kappa_v \alpha_k \|J_k^T c_k\|_2, \quad \text{and}$$

$$154 \quad (2.2c) \quad \|c_k + J_k v_k\|_2 \leq \|c_k + J_k v_k^c\|_2$$

155 where  $v_k^c$  is the Cauchy point given by

$$156 \quad (2.3) \quad v_k^c := -\beta_k^c J_k^T c_k \quad \text{with} \quad \beta_k^c := \arg \min_{\beta \in \mathbb{R}} m_k(-\beta J_k^T c_k) \quad \text{s.t.} \quad 0 \leq \beta \leq \kappa_v \alpha_k.$$

157 In other words, the Cauchy point  $v_k^c$  minimizes  $m_k(v)$  along the direction  $-\nabla m_k(0) =$   
 158  $-J_k^T c_k$  and within  $\{v : \|v\|_2 \leq \kappa_v \alpha_k \|J_k^T c_k\|_2\}$ . It is known (see [10]) that  $v_k^c$  satisfies

$$159 \quad (2.4) \quad m_k(0) - m_k(v_k^c) \geq \frac{1}{2} \|J_k^T c_k\|_2^2 \min \left\{ \frac{1}{1 + \|J_k^T J_k\|_2}, \kappa_v \alpha_k \right\}.$$

160 We note that the conditions (2.2) are well-posed since they are satisfied by  $v_k = v_k^c$ .

161 Next, we compute a direction  $u_k$  that maintains the level of linearized infeasibility  
 162 achieved by  $v_k$  while also reducing a model of the objective function. Specifically, we  
 163 compute  $u_k$  as the unique solution to the strongly convex subproblem

$$\begin{aligned} u_k &:= \arg \min_{u \in \mathbb{R}^n} g_k^T(v_k + u) + \frac{1}{2\alpha_k} \|v_k + u\|_2^2 + r(x_k + v_k + u) \quad \text{s.t.} \quad J_k u = 0. \\ 164 \quad (2.5) \quad &= \arg \min_{u \in \mathbb{R}^n} (g_k + \frac{1}{\alpha_k} v_k)^T u + \frac{1}{2\alpha_k} \|u\|_2^2 + r(x_k + v_k + u) \quad \text{s.t.} \quad J_k u = 0 \\ &= \arg \min_{u \in \mathbb{R}^n} g_k^T u + \frac{1}{2\alpha_k} \|u\|_2^2 + r(x_k + v_k + u) \quad \text{s.t.} \quad J_k u = 0 \end{aligned}$$

165 where we used the fact that every  $u$  feasible for (2.5) satisfies  $v_k^T u = 0$  since  $v_k \in$   
 166  $\text{Range}(J_k^T)$  (see (2.2a)). The trial step  $s_k$  is then defined as

$$167 \quad (2.6) \quad s_k := v_k + u_k.$$

We adopt the  $\ell_2$  merit function, which for parameter  $\tau \in \mathbb{R}_{>0}$  is defined as

$$\Phi_\tau(x) := \tau(f(x) + r(x)) + \|c(x)\|_2.$$

During the  $k$ th iteration, we want to choose  $\tau_k$  such that  $\tau_k \leq \tau_{k-1}$  and  $s_k$  is a direction of sufficient descent for the merit function  $\Phi_{\tau_k}(\cdot)$  at  $x_k$ . To define an appropriate value for  $\tau_k$ , let us define the model of the merit function given by

$$q_k(s, \tau) := \tau(f_k + g_k^T s + \frac{1}{2\alpha_k} \|s\|_2^2 + r(x_k + s)) + \|c_k + J_k s\|_2,$$

168 as well as the change in the model

$$\begin{aligned} \Delta q_k(s, \tau) &:= q_k(0) - q_k(s) \\ 169 \quad (2.7) \quad &= -\tau(g_k^T s + \frac{1}{2\alpha_k} \|s\|_2^2 + r(x_k + s) - r_k) + \|c_k\|_2 - \|c_k + J_k s\|_2. \end{aligned}$$

170 Then, with parameters  $\sigma_c \in (0, 1)$  and  $\sigma_u \in (0, \frac{1}{2}]$ , we set  $\bar{\sigma}_u := \sigma_u + \frac{1}{2} \in (\frac{1}{2}, 1]$  and

$$171 \quad \tau_{k,\text{trial}} \leftarrow \begin{cases} \infty & \text{if } g_k^T s_k + \frac{\bar{\sigma}_u \|s_k\|_2^2}{\alpha_k} + r(x_k + s_k) - r_k \leq 0, \\ \frac{(1-\sigma_c)(\|c_k\|_2 - \|c_k + J_k v_k\|_2)}{g_k^T s_k + \frac{\bar{\sigma}_u \|s_k\|_2^2}{\alpha_k} + r(x_k + s_k) - r_k} & \text{otherwise,} \end{cases}$$

172 and then set, with  $\epsilon_\tau \in (0, 1)$ , the  $k$ th merit parameter value as

$$173 \quad (2.8) \quad \tau_k \leftarrow \begin{cases} \tau_{k-1} & \text{if } \tau_{k-1} \leq \tau_{k,\text{trial}}, \\ \min\{(1 - \epsilon_\tau)\tau_{k-1}, \tau_{k,\text{trial}}\} & \text{otherwise.} \end{cases}$$

174 This update ensures that if the merit parameter is decreased during the  $k$ th iteration,  
175 it is decreased by at least a fraction of its previous value. Moreover, the value for  
176  $\tau_{k,\text{trial}}$  ensures that  $\Delta q_k(s_k, \tau_k)$  is an upper bound for quantities related to measures  
177 of criticality for problem (1.1) (see Lemma 3.4). Moreover, Lemma 3.4 shows that  
178  $-\Delta q_k(s_k, \tau_k)$  is an upper bound for the directional derivative of  $\Phi_{\tau_k}(\cdot)$  at  $x_k$  in the  
179 direction  $s_k$  (this result holds regardless of the value of the merit parameter).

180 The  $k$ th iteration is completed by checking whether the merit function achieves  
181 sufficient decrease in Line 19, and then defining the next iterate and proximal pa-  
182 rameter. Specifically, if sufficient decrease is observed in the merit function, then the  
183 trial step  $s_k$  is accepted (i.e.,  $x_{k+1} \leftarrow x_k + s_k$ ) and the proximal parameter value is  
184 unchanged (i.e.,  $\alpha_{k+1} \leftarrow \alpha_k$ ); otherwise, the trial step is rejected (i.e.,  $x_{k+1} \leftarrow x_k$ )  
185 and the proximal parameter value is decreased (i.e.,  $\alpha_{k+1} \leftarrow \xi \alpha_k$  for some  $\xi \in (0, 1)$ ).  
186 This updating scheme motivates the definition of the following index set:

$$187 \quad (2.9) \quad \mathcal{S} := \{k : x_{k+1} = x_k + s_k\},$$

188 which contains the indices of the successful iterations associated with Algorithm 2.1.

189 **3. Analysis.** In this section, we prove convergence results for Algorithm 2.1.  
190 Our first result shows that the normal step  $v_k$  is zero if and only if  $J_k^T c_k$  is zero.

191 **LEMMA 3.1.** *For all  $k \in \mathbb{N}$ , it holds that  $v_k = 0$  if and only if  $J_k^T c_k = 0$ .*

192 *Proof.* If  $J_k^T c_k = 0$ , it follows from (2.2b) that  $v_k = 0$ . To prove the reverse  
193 implication, suppose that  $v_k = 0$ . Then it follows from (2.2c) that  $m_k(v_k) \leq m_k(v_k^c)$ ,  
194 which combined with (2.4) and  $v_k = 0$  shows that  $0 = m_k(0) - m_k(v_k) \geq m_k(0) -$   
195  $m_k(v_k^c) \geq \frac{1}{2} \|J_k^T c_k\|_2^2 \min \left\{ \frac{1}{1 + \|J_k^T J_k\|_2}, \kappa_v \alpha_k \right\}$ . Since  $\alpha_k > 0$  for all  $k \in \mathbb{N}$  and  $\kappa_v \in$   
196  $\mathbb{R}_{>0}$ , it follows that  $J_k^T c_k = 0$ , completing the proof.  $\square$

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**Algorithm 2.1** A proximal-gradient algorithm for problem (1.1).

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1: input:  $x_0 \in \mathbb{R}^n$ ,  $\alpha_0 \in \mathbb{R}_{>0}$ , and  $\tau_{-1} \in \mathbb{R}_{>0}$ .
2: constants:  $\kappa_v \in \mathbb{R}_{>0}$ ,  $\{\sigma_c, \epsilon_\tau, \xi, \eta, \} \subset (0, 1)$ , and  $\sigma_u \in (0, 1/2]$ .
3: for  $k = 0, 1, 2, \dots$  do
4:   if  $J_k^T c_k \neq 0$  then
5:     Compute  $v_k$  as an approximate solution to (2.1) satisfying (2.2).
6:   else
7:     Set  $v_k \leftarrow 0$ .
8:     if  $c_k \neq 0$  then
9:       return  $x_k$  (infeasible stationary point)
10:    end if
11:  end if
12:  Compute  $u_k$  as the unique solution to (2.5).
13:  Set  $s_k \leftarrow v_k + u_k$ .
14:  if  $s_k = 0$  then
15:    return  $x_k$  (first-order KKT point)
16:  end if
17:  Compute  $\tau_k$  using (2.8).
18:  Compute  $\Delta q_k(s_k, \tau_k)$  using (2.7).
19:  if  $\Phi_{\tau_k}(x_k + s_k) \leq \Phi_{\tau_k}(x_k) - \eta \Delta q_k(s_k, \tau_k)$  then
20:    Set  $x_{k+1} \leftarrow x_k + s_k$  and  $\alpha_{k+1} \leftarrow \alpha_k$ .
21:  else
22:    Set  $x_{k+1} \leftarrow x_k$  and  $\alpha_{k+1} \leftarrow \xi \alpha_k$ .
23:  end if
24: end for

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197 Concerning the computation of the tangential step  $u_k$ , it follows from the opti-  
198 mality conditions for the convex optimization problem (2.5) that  $u_k$  and the resulting  
199  $s_k = v_k + u_k$  satisfy, for some  $g_{r,k} \in \partial r(x_k + s_k)$  and  $y_k \in \mathbb{R}^m$ , the equalities

$$200 \quad (3.1) \quad g_k + \frac{1}{\alpha_k} u_k + g_{r,k} - J_k^T y_k = 0 \quad \text{and} \quad J_k u_k = 0.$$

201 Multiplying the first equality by  $u_k^T$  and using the second equality, it follows that

$$202 \quad (3.2) \quad (g_k + g_{r,k})^T u_k + \frac{1}{\alpha_k} \|u_k\|_2^2 = 0.$$

203 These equations related to the tangential step  $u_k$  will be useful in the analysis.

204 **3.1. Finite termination.** In this section we justify the finite termination con-  
205 ditions in Algorithm 2.1 given in line 9 and line 15. In particular, we show that if  
206 Algorithm 2.1 terminates in line 9 then  $x_k$  is an infeasible stationary point, and if  
207 termination occurs in line 15 then  $x_k$  is a first-order KKT point for problem (1.1).

208 **THEOREM 3.2.** *The following finite termination results hold for Algorithm 2.1.*

209 (i) *If termination occurs in line 9 then  $x_k$  is an infeasible stationary point, i.e.,*  
210  *$x_k$  satisfies  $c_k \neq 0$  and  $J_k^T c_k = 0$ .*

211 (ii) *If termination occurs in line 15 then  $x_k$  is a first-order KKT point for (1.1).*

212 *Proof.* To prove part (i), suppose that termination occurs in line 9 so that  $c_k \neq 0$   
213 and  $v_k = 0$ . It follows from  $v_k = 0$  and Lemma 3.1 that  $J_k^T c_k = 0$ , as claimed.

214 To prove part (ii), suppose that termination occurs in line 15 so that  $s_k = 0$ . Since  
215 by construction  $v_k^T u_k = 0$ , it also follows that  $v_k = u_k = 0$ . It follows from  $v_k = 0$

216 and Lemma 3.1 that  $J_k^T c_k = 0$ . Since termination must not have occurred in line 9,  
 217 we also know that  $c_k = 0$ . It follows from  $v_k = u_k = 0$  and (3.1) that there exists  
 218  $g_{r,k} \in \partial r(x_k + s_k) \equiv \partial r(x_k)$  and  $y_k \in \mathbb{R}^m$  so that  $g_k + g_{r,k} - J_k^T y_k = 0$ . Combining  
 219 this equality with  $c_k = 0$  shows that  $x_k$  is a first-order KKT point for problem (1.1).  $\square$

220 Theorem 3.2 shows that if Algorithm 2.1 finitely terminates, then the vector  $x_k$   
 221 returned has favorable properties. Admittedly, although finite termination in line 9  
 222 is not ideal, the existence of infeasible stationary points is something that every algo-  
 223 rithm must contend with unless an appropriate constraint qualification is assumed.

224 **3.2. Non-finite termination.** In this section, we study the convergence prop-  
 225 erties of Algorithm 2.1 when finite termination does not occur. Therefore, given how  
 226 Algorithm 2.1 is constructed, we know in this section that, for all  $k \in \mathbb{N}$ , it holds that  
 227

$$228 \quad (3.3) \quad (i) \ s_k \neq 0 \quad \text{and} \quad (ii) \ J_k^T c_k = 0 \text{ if and only if } c_k = 0.$$

229 Our first goal is to prove a bound on the directional derivative of  $\Phi_\tau(\cdot)$  at  $x_k$   
 230 in the direction  $s_k$ . Given the Lipschitz constants  $L_g$  and  $L_J$  in Assumption 1.1, it  
 231 follows for all  $t \in \mathbb{R}_{>0}$  from [13, equation (19)] that

$$232 \quad (3.4) \quad \begin{aligned} f(x_k + ts_k) &\leq f_k + tg_k^T s_k + \frac{L_g}{2} t^2 \|s_k\|_2^2 \quad \text{and} \\ \|c(x_k + ts_k)\|_2 &\leq \|c_k + tJ_k s_k\|_2 + \frac{L_J}{2} t^2 \|s_k\|_2^2. \end{aligned}$$

233 The next result gives an upper bound on the quantity  $D_{\Phi_\tau}(x_k, s_k)$ , which we use to  
 234 denote the directional derivative of  $\Phi_\tau(\cdot)$  at  $x_k$  in the direction  $s_k$ .

LEMMA 3.3. *The directional derivative of the merit function satisfies*

$$D_{\Phi_\tau}(x_k, s_k) \leq \tau(g_k^T s_k + r(x_k + s_k) - r_k) + \|c_k + J_k s_k\|_2 - \|c_k\|_2.$$

235 *Proof.* For all  $t \in \mathbb{R}_{>0}$ , it follows from (3.4) and the triangle inequality that

$$236 \quad \begin{aligned} \|c(x_k + ts_k)\|_2 - \|c_k\|_2 &\leq \|c_k + tJ_k s_k\|_2 - \|c_k\|_2 + \frac{L_J}{2} t^2 \|s_k\|_2^2 \\ 237 &\leq t\|c_k + J_k s_k\|_2 + (1-t)\|c_k\|_2 - \|c_k\|_2 + \frac{L_J}{2} t^2 \|s_k\|_2^2 \\ 238 &= t\|c_k + J_k s_k\|_2 - t\|c_k\|_2 + \frac{L_J}{2} t^2 \|s_k\|_2^2. \end{aligned}$$

239 On the other hand, it follows from [3, Theorem 2.25] that  $D_r(x_k, s_k) \leq r(x_k + s_k) -$   
 240  $r(x_k)$ . The conclusion follows from this result, the previous displayed equation after  
 241 dividing by  $t$  and taking the limit  $t \searrow 0$ , and the fact that  $f$  is differentiable.  $\square$

242 Combining the previous lemma with how the merit parameter  $\tau_k$  is defined, allows  
 243 us to prove that the change in the model  $q_k(s_k, \tau_k)$  is an upper bound for quantities  
 244 used in our ultimate convergence result.

245 LEMMA 3.4. *The choice of  $\tau_k$  in (2.8) ensures that the direction  $s_k$  satisfies*

$$246 \quad \begin{aligned} \Delta q_k(s_k, \tau_k) &\geq \frac{\sigma_u \tau_k}{\alpha_k} \|s_k\|_2^2 + \sigma_c (\|c_k\|_2 - \|c_k + J_k v_k\|_2) > 0 \quad \text{and} \\ 247 \quad D_{\Phi_{\tau_k}}(x_k, s_k) &\leq -\frac{\sigma_u \tau_k}{\alpha_k} \|s_k\|_2^2 - \sigma_c (\|c_k\|_2 - \|c_k + J_k v_k\|_2) < 0. \end{aligned}$$

248 *Proof.* The first result follows from (2.8), definition of  $\tau_{k,\text{trial}}$ , and  $J_k s_k = J_k v_k$   
 249 (recall that  $J_k u_k = 0$  because of the constraint in (2.5)). The second result follows  
 250 from Lemma 3.3,  $\frac{1}{\alpha_k} \|s_k\|_2^2 \geq 0$ , and the first result of this lemma.  $\square$



251 We now give a sufficient condition for a successful iteration (see (2.9)) to occur.

252 LEMMA 3.5. *If  $(1 - \eta)\Delta q_k(s_k, \tau_k) \geq \frac{1}{2}(-\frac{\tau_k}{\alpha_k} + \tau_k L_g + L_J)\|s_k\|_2^2$ , then  $k \in \mathcal{S}$ .*

253 *Proof.* It follows from (3.4), (2.7), and the assumed inequality in this lemma that

$$\begin{aligned}
254 & \phi_{\tau_k}(x_k + s_k) - \phi_{\tau_k}(x_k) \\
255 &= \tau_k(f(x_k + s_k) + r(x_k + s_k)) + \|c(x_k + s_k)\|_2 - \tau_k(f_k + r_k) - \|c_k\|_2. \\
256 &\leq \tau_k g_k^T s_k + \tau_k(r(x_k + s_k) - r_k) + \|c_k + J_k s_k\|_2 - \|c_k\|_2 + \frac{1}{2}(\tau_k L_g + L_J)\|s_k\|_2^2 \\
257 &= -\Delta q_k(s_k, \tau_k) - \frac{\tau_k}{2\alpha_k}\|s_k\|_2^2 + \frac{1}{2}(\tau_k L_g + L_J)\|s_k\|_2^2 \\
258 &= -\Delta q_k(s_k, \tau_k) + \frac{1}{2}\left(-\frac{\tau_k}{\alpha_k} + \tau_k L_g + L_J\right)\|s_k\|_2^2 \leq -\eta\Delta q_k(s_k, \tau_k).
\end{aligned}$$

259 Therefore, it follows from Line 19 of Algorithm 2.1 that  $k \in \mathcal{S}$ , as claimed.  $\square$

260 The following result gives a bound on the decrease in linearized feasibility achieved  
261 by  $s_k$  that is similar to that achieved by the Cauchy point in (2.4).

LEMMA 3.6. *The step  $s_k = v_k + u_k$  satisfies*

$$\|c_k\|_2 - \|c_k + J_k s_k\|_2 = \|c_k\|_2 - \|c_k + J_k v_k\|_2 \geq \frac{1}{2\kappa_c}\|J_k^T c_k\|_2^2 \min\left\{\frac{1}{1+\kappa_{\nabla c}^2}, \kappa_v \alpha_k\right\}.$$

262 *Proof.* From (1.2), (2.4), (2.2c), and the constraint in (2.5), we have

$$\begin{aligned}
& \frac{1}{2}\|J_k^T c_k\|_2^2 \min\left\{\frac{1}{1+\kappa_{\nabla c}^2}, \kappa_v \alpha_k\right\} \\
& \leq \frac{1}{2}\|J_k^T c_k\|_2^2 \min\left\{\frac{1}{1+\|J_k^T J_k\|_2}, \kappa_v \alpha_k\right\} \\
263 & \leq m_k(0) - m_k(v_k^c) = \frac{1}{2}(\|c_k\|_2^2 - \|c_k + J_k v_k^c\|_2^2) \\
& = \frac{1}{2}(\|c_k\|_2 + \|c_k + J_k v_k^c\|_2)(\|c_k\|_2 - \|c_k + J_k v_k^c\|_2) \\
& \leq \|c_k\|_2(\|c_k\|_2 - \|c_k + J_k v_k^c\|_2) \\
& \leq \kappa_c(\|c_k\|_2 - \|c_k + J_k v_k\|_2) = \kappa_c(\|c_k\|_2 - \|c_k + J_k s_k\|_2),
\end{aligned}$$

264 from which the desired result follows.  $\square$

265 We now begin investigating quantities related to the merit parameter. The fol-  
266 lowing result bounds the denominator in the definition of  $\tau_{k,\text{trial}}$ .

LEMMA 3.7. *For all  $k \in \mathbb{N}$ , it follows that*

$$g_k^T s_k + \frac{\bar{\sigma}_u \|s_k\|_2^2}{\alpha_k} + r(x_k + s_k) - r_k \leq (\kappa_{\nabla f} + \kappa_{\partial r})\|v_k\|_2 + \frac{\bar{\sigma}_u \|v_k\|_2^2}{\alpha_k}.$$

267 *Proof.* With  $g_{r,k}$  defined as in (3.1), it follows from convexity of  $r$  that  $r_k \geq$   
268  $r(x_k + s_k) + g_{r,k}^T(-s_k)$ . Combining this inequality with  $s_k = v_k + u_k$ ,  $v_k^T u_k = 0$ ,  
269  $\bar{\sigma}_u \in (\frac{1}{2}, 1]$ , (3.2), the Cauchy-Schwartz inequality, and (1.2) it follows that

$$\begin{aligned}
270 & g_k^T s_k + \frac{\bar{\sigma}_u \|s_k\|_2^2}{\alpha_k} + r(x_k + s_k) - r_k \\
271 & \leq (g_k + g_{r,k})^T s_k + \frac{\bar{\sigma}_u \|s_k\|_2^2}{\alpha_k} \\
272 & = (g_k + g_{r,k})^T v_k + \frac{\bar{\sigma}_u \|v_k\|_2^2}{\alpha_k} + (g_k + g_{r,k})^T u_k + \frac{\bar{\sigma}_u \|u_k\|_2^2}{\alpha_k} \\
273 & \leq (g_k + g_{r,k})^T v_k + \frac{\bar{\sigma}_u \|v_k\|_2^2}{\alpha_k} + (g_k + g_{r,k})^T u_k + \frac{\|u_k\|_2^2}{\alpha_k} \\
274 & \leq \|g_k + g_{r,k}\|_2 \|v_k\|_2 + \frac{\bar{\sigma}_u \|v_k\|_2^2}{\alpha_k}
\end{aligned}$$

$$275 \quad \leq (\kappa_{\nabla f} + \kappa_{\partial r}) \|v_k\|_2 + \frac{\bar{\sigma}_u \|v_k\|_2^2}{\alpha_k},$$

276 which completes the proof.  $\square$

277 We next show that the merit sequence is positive and monotonically decreasing.

278 LEMMA 3.8. *For all  $k \geq 1$ , it holds that  $0 < \tau_k \leq \tau_{k-1}$ .*

279 *Proof.* It is clear from  $\tau_0 > 0$  and the update (2.8) that  $\{\tau_k\}$  is monotonically  
 280 decreasing, and therefore all that remains is to prove that  $\tau_k > 0$  for all  $k \in \mathbb{N}$ . It  
 281 follows from Lemma 3.7 and the definition of  $\tau_{k,\text{trial}}$  that  $\tau_{k,\text{trial}} = \infty$  if  $v_k = 0$ , and so  
 282 for such  $k$  we have  $\tau_k \leftarrow \tau_{k-1}$ . Therefore, for the remainder we only need to consider  
 283  $k \in \mathbb{N}$  such that  $v_k \neq 0$ . For such  $k \in \mathbb{N}$ , we know from Lemma 3.1 that  $J_k^T c_k \neq 0$ .  
 284 The result  $\tau_k > 0$  follows from this observation, (2.8),  $\alpha_k > 0$ , and Lemma 3.6.  $\square$

285 The first part of the next lemma shows that the merit parameter never needs to  
 286 be decreased for iterations  $k \in \mathbb{N}$  such that  $J_k^T c_k = 0$ . On the other hand, for all  
 287  $k \in \mathbb{N}$  such that  $J_k^T c_k \neq 0$ , the second part of the result gives a lower bound on how  
 288 small the previous merit parameter could have been.

289 LEMMA 3.9. *The following merit parameter update holds for each  $k \in \mathbb{N} \setminus \{0\}$ .*

- 290 (i) *If  $J_k^T c_k = 0$ , then  $\tau_{k,\text{trial}} = \infty$  and  $\tau_k \leftarrow \tau_{k-1}$ .*  
 291 (ii) *There exists a constant  $\epsilon_\tau > 0$  such that, for all  $k \in \mathbb{N}$  satisfying  $J_k^T c_k \neq 0$   
 292 and  $\tau_k < \tau_{k-1}$ , it holds that  $\tau_{k-1} \geq \epsilon_\tau \|J_k^T c_k\|_2$ .*

293 *Proof.* For part (i), it follows from  $J_k^T c_k = 0$  and Lemma 3.1 that  $v_k = 0$ . This  
 294 fact, Lemma 3.7, and the definition of  $\tau_{k,\text{trial}}$  show that  $\tau_{k,\text{trial}} = \infty$ , so that  $\tau_k \leftarrow \tau_{k-1}$ .

295 For part (ii), it follows from (2.8), Lemma 3.6, Lemma 3.7, the trust-region con-  
 296 straint in problem (2.1), and (1.2) that if  $\tau_k < \tau_{k-1}$ , then

$$297 \quad \begin{aligned} \tau_{k-1} &> \frac{(1 - \sigma_c)(\|c_k\|_2 - \|c_k + J_k v_k\|_2)}{g_k^T s_k + \frac{\bar{\sigma}_u \|s_k\|_2^2}{\alpha_k} + r(x_k + s_k) - r_k} \\ &\geq \frac{\frac{(1 - \sigma_c)}{2\kappa_c} \|J_k^T c_k\|_2^2 \min\{\frac{1}{1 + \kappa_{\nabla c}^2}, \kappa_v \alpha_k\}}{(\kappa_{\nabla f} + \kappa_{\partial r}) \|v_k\|_2 + \frac{\bar{\sigma}_u \|v_k\|_2^2}{\alpha_k}} \\ &\geq \frac{(1 - \sigma_c) \|J_k^T c_k\|_2^2 \min\{\frac{1}{1 + \kappa_{\nabla c}^2}, \kappa_v \alpha_k\}}{2\kappa_c (\kappa_{\nabla f} + \kappa_{\partial r}) \kappa_v \alpha_k \|J_k^T c_k\|_2 + \frac{\bar{\sigma}_u \kappa_v^2 \alpha_k^2 \|J_k^T c_k\|_2^2}{\alpha_k}} \\ &= \frac{(1 - \sigma_c) \|J_k^T c_k\|_2 \min\{\frac{1}{1 + \kappa_{\nabla c}^2}, \kappa_v \alpha_k\}}{2\kappa_c (\kappa_{\nabla f} + \kappa_{\partial r}) \kappa_v \alpha_k + \bar{\sigma}_u \kappa_v^2 \alpha_k \|J_k^T c_k\|_2} \\ &\geq \frac{(1 - \sigma_c) \|J_k^T c_k\|_2 \min\{\frac{1}{1 + \kappa_{\nabla c}^2}, \kappa_v \alpha_k\}}{2\kappa_c (\kappa_{\nabla f} + \kappa_{\partial r}) \kappa_v \alpha_k + \bar{\sigma}_u \kappa_v^2 \alpha_k \kappa_{\nabla c} \kappa_c}. \end{aligned}$$

298 It follows from (3.5) and the fact that  $\{\alpha_k\}$  is monotonically nonincreasing that

$$299 \quad \tau_{k-1} \geq \begin{cases} \frac{(1 - \sigma_c) \|J_k^T c_k\|_2}{2\kappa_c (\kappa_{\nabla f} + \kappa_{\partial r}) + \bar{\sigma}_u \kappa_v \kappa_{\nabla c} \kappa_c} & \text{if } \kappa_v \alpha_k \leq 1/(1 + \kappa_{\nabla c}^2), \\ \frac{(1 - \sigma_c) \|J_k^T c_k\|_2}{2\kappa_c (1 + \kappa_{\nabla c}^2) (\kappa_{\nabla f} + \kappa_{\partial r}) \kappa_v \alpha_0 + \bar{\sigma}_u \kappa_v^2 \alpha_0 \kappa_{\nabla c} \kappa_c} & \text{otherwise,} \end{cases}$$

300 which completes the proof.  $\square$

301 We now prove our first key convergence result. In particular, we prove that  
 302 there must exist a subsequence of the set of successful iterations over which  $\{J_k^T c_k\}$

303 converges to zero. This conclusion is relevant to our setting because, under a suitable  
 304 constraint qualification, if  $\bar{x}$  is a local minimizer of  $\frac{1}{2}\|c(x)\|_2^2$ , then  $J(\bar{x})^T c(\bar{x}) = 0$ .

305 **THEOREM 3.10.** *Let Assumption 1.1 hold. Then, there exists a subsequence of*  
 306 *the iterations  $\mathcal{K} \subseteq \mathbb{N}$  such that  $\lim_{k \in \mathcal{K}} J_k^T c_k = 0$ .*

307 *Proof.* For a proof by contradiction, suppose that there exists a  $\bar{k}_1 \in \mathbb{N}$  and  $\epsilon > 0$   
 308 such that  $\|J_k^T c_k\|_2 \geq \epsilon$  for all  $k \geq \bar{k}_1$ . Then, it follows from Lemma 3.9 and  $\tau_0 \in \mathbb{R}_{>0}$   
 309 that there exists  $\bar{\tau}_1 > 0$  such that, for all  $k \in \mathbb{N}$ , it holds that  $\tau_k \geq \bar{\tau}_1$ . Moreover, since  
 310  $\{\tau_k\}$  is monotonically nonincreasing and when  $\tau_k < \tau_{k-1}$  the reduction is by at least  
 311 a constant factor (see (2.8)), we know that there exists  $\bar{k}_2 \geq \bar{k}_1$  and  $\bar{\tau}_2 \geq \bar{\tau}_1$  such  
 312 that  $\tau_k = \bar{\tau}_2$  for all  $k \geq \bar{k}_2$ . Combining this with  $\Delta q_k(s_k, \tau_k) > 0$  (see Lemma 3.4)  
 313 and Lemma 3.5 it follows that for all  $k \geq \bar{k}_2$  such that  $\alpha_k \leq \bar{\tau}_2/(\bar{\tau}_2 L_g + L_J)$  it must  
 314 also hold that  $k \in \mathcal{S}$ . Since  $\alpha_{k+1} < \alpha_k$  only when  $k \notin \mathcal{S}$ , it follows that there must  
 315 exist  $\bar{\alpha} \in \mathbb{R}_{>0}$  and  $\bar{k}_3 \geq \bar{k}_2$  such that  $\alpha_k = \bar{\alpha}$  for all  $k \geq \bar{k}_3$ . To summarize, we have  
 316 proved that for all  $k \geq \bar{k}_3$  it holds that  $\alpha_k = \bar{\alpha}$ ,  $\tau_k = \bar{\tau}_2$ , and  $k \in \mathcal{S}$ . It now follows  
 317 from line 19 of Algorithm 2.1 that  $\Phi_{\bar{\tau}_2}(x_{k+1}) \leq \Phi_{\bar{\tau}_2}(x_k) - \eta \Delta q_k(s_k, \bar{\tau}_2)$  for all  $k \geq \bar{k}_3$ .  
 318 Summing over all  $k \geq \bar{k}_3$  and using (1.2) and Lemma 3.4 we have

$$\begin{aligned}
 319 \quad \Phi_{\bar{\tau}_2}(x_{\bar{k}_3}) - \bar{\tau}_2 f_{\text{inf}} &\geq \sum_{k \geq \bar{k}_3} (\Phi_{\bar{\tau}_2}(x_k) - \Phi_{\bar{\tau}_2}(x_{k+1})) \\
 320 &\geq \eta \sum_{k \geq \bar{k}_3} \Delta q_k(s_k, \bar{\tau}_2) \\
 321 &\geq \eta \sum_{k \geq \bar{k}_3} \frac{\sigma_u \tau_k}{\alpha_k} \|s_k\|_2^2 + \sigma_c (\|c_k\|_2 - \|c_k + J_k v_k\|_2) \\
 322 &> \eta \sigma_c \sum_{k \geq \bar{k}_3} (\|c_k\|_2 - \|c_k + J_k v_k\|_2).
 \end{aligned}$$

Since the summation of nonnegative terms is finite, we know that

$$\lim_{k \rightarrow \infty} (\|c_k\|_2 - \|c_k + J_k v_k\|_2) = 0.$$

323 This fact, Lemma 3.6, and  $\alpha_k = \bar{\alpha}$  for all  $k \geq \bar{k}_3$  imply that  $\lim_{k \rightarrow \infty} J_k^T c_k = 0$ , which  
 324 contradicts our earlier assumption that  $\|J_k^T c_k\|_2 \geq \epsilon$  for all  $k \geq \bar{k}_1$ .  $\square$

325 The remainder of the analysis considers two settings that are characterized by  
 326 whether a certain constraint qualification holds or not.

327 **3.2.1. Strong LICQ.** In this section we make the following assumption, which  
 328 is closely related to the linear independence constraint qualification (LICQ).

329 **ASSUMPTION 3.1.** *The smallest singular values of  $\{J_k\}$  are uniformly bounded*  
 330 *away from zero, i.e., there exists  $\sigma_{\min} > 0$  such that, for all  $k \in \mathbb{N}$ ,  $\sigma_{\min}(J_k) \geq \sigma_{\min}$ .*

331 We can now prove a nontrivial bound on the improvement in linearized infeasibility  
 332 achieved by the trial step  $s_k$  relative to the actual infeasibility. This result is critical  
 333 when we prove a uniform lower bound on the sequence of merit parameters.

334 **LEMMA 3.11.** *If  $J_k^T c_k \neq 0$ , then  $s_k$  satisfies  $\|c_k + J_k s_k\|_2 \leq \rho_k \|c_k\|_2$  where*

$$335 \quad \rho_k := \sqrt{\max \left\{ 1 - \kappa_v \alpha_k \sigma_{\min}^2, 1 - \sigma_{\min}^2 / \kappa_{\nabla c}^2 \right\}} \in [0, 1).$$

336 *Proof.* It follows from [30, Section 4.1] that the Cauchy step  $v_k^c$  in (2.3) satisfies

$$337 \quad (3.6) \quad v_k^c = -\beta_k^c J_k^T c_k \text{ with } \beta_k^c = \min \left\{ \frac{\|J_k^T c_k\|_2^2}{\|J_k J_k^T c_k\|_2^2}, \kappa_v \alpha_k \right\}.$$

338 We now consider two cases.

339 **Case 1:**  $\|J_k^T c_k\|_2^2 \leq \kappa_v \alpha_k \|J_k J_k^T c_k\|_2^2$ . In this case, the minimum in (3.6) is the first  
 340 term, and  $J_k J_k^T c_k \neq 0$  since  $J_k^T c_k \neq 0$ . These facts, the inequality that defines this  
 341 case, the Cauchy-Schwartz inequality, definition of  $m_k(0)$ , and Assumption 3.1 give

$$342 \quad (3.7) \quad m_k(v_k^c) = m_k(0) - \frac{1}{2} \frac{\|J_k^T c_k\|_2^4}{\|J_k J_k^T c_k\|_2^2} \leq \frac{1}{2} \|c_k\|_2^2 - \frac{1}{2} \frac{\|J_k^T c_k\|_2^2}{\kappa_v^2 \alpha_k}$$

$$343 \quad (3.8) \quad \leq \frac{1}{2} \|c_k\|_2^2 - \frac{1}{2} \frac{\sigma_{\min}^2(J_k)}{\kappa_v^2 \alpha_k} \|c_k\|_2^2 \leq \frac{1}{2} \left(1 - \frac{\sigma_{\min}^2}{\kappa_v^2 \alpha_k}\right) \|c_k\|_2^2.$$

344 **Case 2:**  $\|J_k^T c_k\|_2^2 > \kappa_v \alpha_k \|J_k J_k^T c_k\|_2^2$ . In this case, the minimum in (3.6) is the second  
 345 term. This fact, the previous inequality, definition of  $m_k(0)$ , and Assumption 3.1 give

$$346 \quad m_k(v_k^c) = m_k(0) - \kappa_v \alpha_k \|J_k^T c_k\|_2^2 + \frac{1}{2} \kappa_v^2 \alpha_k^2 \|J_k J_k^T c_k\|_2^2$$

$$347 \quad \leq m_k(0) - \kappa_v \alpha_k \|J_k^T c_k\|_2^2 + \frac{1}{2} \kappa_v \alpha_k \|J_k^T c_k\|_2^2$$

$$348 \quad = m_k(0) - \frac{1}{2} \kappa_v \alpha_k \|J_k^T c_k\|_2^2 \leq \frac{1}{2} (1 - \kappa_v \alpha_k \sigma_{\min}^2) \|c_k\|_2^2.$$

By combining the final result for the two cases, we find that  $m_k(v_k^c) \leq \frac{1}{2} \rho_k^2 \|c_k\|_2^2$ .  
 Multiplying both sides of this inequality by two, taking the square root, and using (2.2c) and the fact that  $c_k + J_k s_k = c_k + J_k v_k$  since  $J_k u_k = 0$ , it follows that

$$\|c_k + J_k s_k\|_2 = \|c_k + J_k v_k\|_2 \leq \|c_k + J_k v_k^c\|_2 \leq \rho_k \|c_k\|_2,$$

349 which completes the proof.  $\square$

350 We may now prove that  $\{\tau_k\}$  is bounded away from zero.

351 **LEMMA 3.12.** *For all  $k \in \mathbb{N}$ , it holds that  $\tau_{k, \text{trial}} \geq \tau_{\min, \text{trial}}$  with*

$$352 \quad \tau_{\min, \text{trial}} :=$$

$$353 \quad \min \left\{ \frac{(1 - \sigma_c) \kappa_v \sigma_{\min}^2}{2 \kappa_v \kappa_{\nabla c} (\kappa_{\nabla f} + \kappa_{\partial r} + \bar{\sigma}_u \kappa_c \kappa_v \kappa_{\nabla c})}, \frac{(1 - \sigma_c) (\sigma_{\min} / \kappa_{\nabla c})^2}{2 \kappa_v \kappa_{\nabla c} (\kappa_{\nabla f} + \kappa_{\partial r} + \bar{\sigma}_u \kappa_c \kappa_v \kappa_{\nabla c}) \alpha_0} \right\},$$

354 which when combined with (3.9) gives  $\tau_k \geq \tau_{\min} := \min\{\tau_0, (1 - \epsilon_\tau) \tau_{\min, \text{trial}}\}$ .

355 *Proof.* We first prove a lower bound on  $\tau_{k, \text{trial}}$ . Since it follows that  $\tau_{k, \text{trial}} = \infty$   
 356 for all  $k \in \mathbb{N}$  satisfying  $J_k^T c_k = 0$  (see Lemma 3.9(i)), we may assume without loss  
 357 of generality that each  $k \in \mathbb{N}$  satisfies  $J_k^T c_k \neq 0$ . Next, we see from Lemma 3.7, the  
 358 trust-region constraint, and (1.2) that

$$359 \quad g_k^T s_k + \frac{\bar{\sigma}_u \|s_k\|_2^2}{\alpha_k} + r(x_k + s_k) - r_k$$

$$360 \quad \leq (\kappa_{\nabla f} + \kappa_{\partial r}) \|v_k\|_2 + \frac{\bar{\sigma}_u \|v_k\|_2^2}{\alpha_k}$$

$$361 \quad = (\kappa_{\nabla f} + \kappa_{\partial r}) \kappa_v \alpha_k \|J_k^T c_k\|_2 + \bar{\sigma}_u \kappa_v^2 \alpha_k \|J_k^T c_k\|_2^2$$

$$362 \quad \leq (\kappa_{\nabla f} + \kappa_{\partial r}) \kappa_v \alpha_k \kappa_{\nabla c} \|c_k\|_2 + \bar{\sigma}_u \kappa_v^2 \alpha_k \kappa_{\nabla c}^2 \|c_k\|_2^2$$

$$363 \quad \leq (\kappa_{\nabla f} + \kappa_{\partial r}) \kappa_v \alpha_k \kappa_{\nabla c} \|c_k\|_2 + \bar{\sigma}_u \kappa_c \kappa_v^2 \alpha_k \kappa_{\nabla c}^2 \|c_k\|_2$$

$$364 \quad = \kappa_v \kappa_{\nabla c} (\kappa_{\nabla f} + \kappa_{\partial r} + \bar{\sigma}_u \kappa_c \kappa_v \kappa_{\nabla c}) \alpha_k \|c_k\|_2 \text{ for all } k \in \mathbb{N}.$$

365 On the other hand, we may use Lemma 3.11 to obtain

$$366 \quad \|c_k\|_2 - \|c_k + J_k v_k\|_2 \geq \|c_k\|_2 - \rho_k \|c_k\|_2 = (1 - \rho_k) \|c_k\|_2 \quad \text{for all } k \in \mathbb{N}.$$

367 Using the above two bounds and the definition of  $\tau_{k,\text{trial}}$ , it follows that

$$368 \quad \tau_{k,\text{trial}} \geq \frac{(1 - \sigma_c)(1 - \rho_k) \|c_k\|_2}{\kappa_v \kappa_{\nabla c} (\kappa_{\nabla f} + \kappa_{\partial r} + \bar{\sigma}_u \kappa_c \kappa_v \kappa_{\nabla c}) \alpha_k \|c_k\|_2}$$

$$369 \quad = \frac{(1 - \sigma_c)(1 - \rho_k)}{\kappa_v \kappa_{\nabla c} (\kappa_{\nabla f} + \kappa_{\partial r} + \bar{\sigma}_u \kappa_c \kappa_v \kappa_{\nabla c}) \alpha_k} \quad \text{for all } k \in \mathbb{N}.$$

370 Next, notice that it follows from the definition of  $\rho_k$  that

$$371 \quad 1 - \rho_k = \frac{1 - \rho_k^2}{1 + \rho_k} \geq \frac{1 - \max\{1 - \kappa_v \alpha_k \sigma_{\min}^2, 1 - \sigma_{\min}^2 / \kappa_{\nabla c}^2\}}{2}$$

$$372 \quad = \frac{1 - (1 - \min\{\kappa_v \alpha_k \sigma_{\min}^2, \sigma_{\min}^2 / \kappa_{\nabla c}^2\})}{2}$$

$$373 \quad = \frac{\min\{\kappa_v \alpha_k \sigma_{\min}^2, \sigma_{\min}^2 / \kappa_{\nabla c}^2\}}{2} \quad \text{for all } k \in \mathbb{N}.$$

Combining this result with the previous displayed equation shows that

$$\tau_{k,\text{trial}} \geq \frac{(1 - \sigma_c) \min\{\kappa_v \alpha_k \sigma_{\min}^2, (\sigma_{\min} / \kappa_{\nabla c})^2\}}{2 \kappa_v \kappa_{\nabla c} (\kappa_{\nabla f} + \kappa_{\partial r} + \bar{\sigma}_u \kappa_c \kappa_v \kappa_{\nabla c}) \alpha_k} \quad \text{for all } k \in \mathbb{N}.$$

It follows from this inequality and the fact that  $\alpha_k \leq \alpha_0$  for all  $k \in \mathbb{N}$  that

$$\tau_{k,\text{trial}} \geq \begin{cases} \frac{(1 - \sigma_c) \kappa_v \sigma_{\min}^2}{2 \kappa_v \kappa_{\nabla c} (\kappa_{\nabla f} + \kappa_{\partial r} + \bar{\sigma}_u \kappa_c \kappa_v \kappa_{\nabla c})} & \text{if } \kappa_v \alpha_k \sigma_{\min}^2 \leq (\sigma_{\min} / \kappa_{\nabla c})^2, \\ \frac{(1 - \sigma_c) (\sigma_{\min} / \kappa_{\nabla c})^2}{2 \kappa_v \kappa_{\nabla c} (\kappa_{\nabla f} + \kappa_{\partial r} + \bar{\sigma}_u \kappa_c \kappa_v \kappa_{\nabla c}) \alpha_0} & \text{otherwise,} \end{cases}$$

374 for all  $k \in \mathbb{N}$ , which proves our first result.

375 The second result, namely the positive lower bound on  $\{\tau_k\}$ , follows from the first  
376 result,  $\tau_0 \in \mathbb{R}_{>0}$ , and (2.8), which completes the proof.  $\square$

377 The positive lower bound on  $\{\tau_k\}$  lets us prove a positive lower bound on  $\{\alpha_k\}$ .

378 LEMMA 3.13. *If  $\alpha_k \leq \tau_{\min} / (\tau_{\min} L_g + L_J)$ , then  $k \in \mathcal{S}$ . Therefore,*

$$379 \quad (3.9) \quad \alpha_k \geq \alpha_{\min} := \min\{\alpha_0, \xi \tau_{\min} / (\tau_{\min} L_g + L_J)\} > 0 \quad \text{for all } k \in \mathbb{N},$$

380 and a bound on the number of unsuccessful iterations is given by

$$381 \quad (3.10) \quad |\{k : x_k \notin \mathcal{S}\}| \leq \max \left( 0, \left\lceil \frac{\log \left( \frac{\tau_{\min}}{\alpha_0 (\tau_{\min} L_g + L_J)} \right)}{\log(\xi)} \right\rceil \right).$$

*Proof.* Suppose that  $k \in \mathbb{N}$  satisfies  $\alpha_k \leq \tau_{\min} / (\tau_{\min} L_g + L_J)$ . Then it follows from the definition of  $\alpha_{\min}$ , Lemma 3.12, and the fact that  $\tau / (\tau L_g + L_J)$  is a monotonically increasing function on the nonnegative real line as a function of  $\tau$  that

$$\alpha_k \leq \tau_{\min} / (\tau_{\min} L_g + L_J) \leq \tau_k / (\tau_k L_g + L_J),$$

which after rearrangement shows that  $-\tau_k / \alpha_k + \tau_k L_g + L_J \leq 0$ . It follows from this inequality, Lemma 3.4, and  $\eta \in (0, 1)$  that

$$(1 - \eta) \Delta q_k(s_k, \tau_k) > 0 \geq \frac{1}{2} \left( -\frac{\tau_k}{\alpha_k} + \tau_k L_g + L_J \right) \|s_k\|_2^2,$$

382 which together with Lemma 3.5 shows that  $k \in \mathcal{S}$ , as claimed. We know from the  
 383 result we just proved and the update strategy for  $\{\alpha_k\}$  that the bound in (3.9) holds.

384 Finally, the first result we proved in this lemma and the updating strategy for  $\{\alpha_k\}$   
 385 shows that the maximum number of unsuccessful iterations is the smallest nonnegative  
 386 integer  $n_u$  such that  $\xi^{n_u} \alpha_0 \leq \tau_{\min}/(\tau_{\min} L_g + L_J)$ , which gives the final result.  $\square$

387 Our worst-case complexity result uses the KKT-residual measure

$$388 \quad (3.11) \quad \chi_k := \max\{\|g_k + g_{r,k} - J_k^T y_k\|_2, \|c_k\|_2\},$$

where we remind the reader that  $g_{r,k}$  is given in (3.1). In proving our complexity  
 result, it will be convenient to define the shifted merit function

$$\bar{\phi}_\tau(x) := \tau(f(x) - f_{\inf} + r(x)) + \|c(x)\|_2,$$

389 where  $f_{\inf}$  is defined in Assumption 1.2. We stress that the (typically) unknown  
 390 value  $f_{\inf}$  is never used in the algorithm statement or its implementation, only in our  
 391 analysis. The following results pertain to the shifted merit function.

392 LEMMA 3.14. *The following properties hold for the shifted merit function  $\bar{\phi}_\tau$ :*

- 393 (i) *For all  $\{x, y\} \subset \mathbb{R}^n$  and  $\tau \in \mathbb{R}_{>0}$ , it holds that  $\bar{\phi}_\tau(x) - \bar{\phi}_\tau(y) = \phi_\tau(x) - \phi_\tau(y)$ .*
- 394 (ii) *For all  $x \in \mathbb{R}^n$  and  $0 < \tau_2 \leq \tau_1$ , it holds that  $\bar{\phi}_{\tau_2}(x) \leq \bar{\phi}_{\tau_1}(x)$ .*
- 395 (iii) *The sequence  $\{\bar{\phi}_{\tau_k}(x_k)\}$  is monotonically decreasing.*

396 *Proof.* For part (i), it follows from the definitions of  $\bar{\phi}_\tau$  and  $\phi_\tau$  that

$$\begin{aligned} 397 \quad \bar{\phi}_\tau(x) - \bar{\phi}_\tau(y) &= \tau(f(x) - f_{\inf} + r(x)) + \|c(x)\|_2 - \tau(f(y) - f_{\inf} + r(y)) - \|c(y)\|_2 \\ 398 \quad &= \tau(f(x) + r(x)) + \|c(x)\|_2 - \tau(f(y) + r(y)) - \|c(y)\|_2 \\ 399 \quad &= \phi_\tau(x) - \phi_\tau(y), \end{aligned}$$

which proves part (i). For (ii), the definition of  $f_{\inf}$  and nonnegativity of  $r$  imply that

$$\bar{\phi}_{\tau_2}(x) = \tau_2(f(x) - f_{\inf} + r(x)) + \|c(x)\|_2 \leq \tau_1(f(x) - f_{\inf} + r(x)) + \|c(x)\|_2 = \bar{\phi}_{\tau_1}(x),$$

which proves (ii). Finally, for each  $k \in \mathbb{N}$ , it follows from Lemma 3.8, parts (i) and  
 (ii) of the current lemma, and how  $x_{k+1}$  is updated in Algorithm 2.1 that

$$\bar{\phi}_{\tau_k}(x_k) - \bar{\phi}_{\tau_{k+1}}(x_{k+1}) \geq \bar{\phi}_{\tau_k}(x_k) - \bar{\phi}_{\tau_k}(x_{k+1}) = \phi_{\tau_k}(x_k) - \phi_{\tau_k}(x_{k+1}) \geq 0,$$

400 which completes the proof of this theorem.  $\square$

401 We may now state our worst-case complexity result for Algorithm 2.1.

402 THEOREM 3.15. *Suppose that Assumption 1.1 and Assumption 3.1 hold, and let*  
 403  *$\epsilon \in \mathbb{R}_{>0}$  be given. If  $\{k_1, k_2\} \subset \mathbb{N}$  are two iterations with  $k_1 < k_2$  such that  $k \in \mathcal{S}$  and*  
 404  *$\chi_k > \epsilon$  for all iterations  $k_1 \leq k < k_2$ , then it follows that*

$$405 \quad (3.12) \quad k_2 - k_1 \leq \left\lceil \frac{\tau_0(f(x_0) - f_{\inf} + r(x_0)) + \|c(x_0)\|_2}{\kappa_\Phi \epsilon^2} \right\rceil$$

406 with  $\kappa_\Phi := \eta \min\{\sigma_u \tau_{\min} \alpha_{\min}, \frac{\sigma_c \sigma_{\min}^2}{2\kappa_c(1+\kappa_c^2)}, \frac{\sigma_c \sigma_{\min}^2 \kappa_v \alpha_{\min}}{2\kappa_c}\}$ . Moreover, the maximum  
 407 number of iterations before  $\chi_k \leq \epsilon$  for some iteration  $k \in \mathbb{N}$  is

$$408 \quad (3.13) \quad \left( \max \left\{ 0, \left\lceil \frac{\log \left( \frac{\tau_{\min}}{\alpha_0(\tau_{\min} L_g + L_J)} \right)}{\log(\xi)} \right\rceil \right\} + 1 \right) \left\lceil \frac{\tau_0(f(x_0) - f_{\inf} + r(x_0)) + \|c(x_0)\|_2}{\kappa_\Phi \epsilon^2} \right\rceil.$$

409 *Proof.* Let  $\{k_1, k_2\} \subset \mathbb{N}$  be as described in the theorem statement. Then, it  
 410 follows from Lemma 3.8, Lemma 3.14(i-ii), Line 19 of Algorithm 2.1, Lemma 3.4,  
 411 Lemma 3.12, and (3.9) that the following inequalities hold for all  $k_1 \leq k < k_2$ :

$$\begin{aligned}
 412 \quad \bar{\phi}_{\tau_k}(x_k) - \bar{\phi}_{\tau_{k+1}}(x_{k+1}) &\geq \bar{\phi}_{\tau_k}(x_k) - \bar{\phi}_{\tau_k}(x_{k+1}) \\
 413 \quad &= \Phi_{\tau_k}(x_k) - \Phi_{\tau_k}(x_{k+1}) \\
 414 \quad &\geq \eta \Delta q_k(s_k, \tau_k) \\
 415 \quad &\geq \eta \frac{\sigma_u \tau_k}{\alpha_k} \|s_k\|_2^2 + \eta \sigma_c (\|c_k\|_2 - \|c_k + J_k v_k\|_2) \\
 416 \quad &\geq \eta \sigma_u \tau_{\min} \alpha_{\min} \left( \frac{\|s_k\|_2}{\alpha_k} \right)^2 + \eta \sigma_c (\|c_k\|_2 - \|c_k + J_k v_k\|_2).
 \end{aligned}$$

417 Combining this inequality with  $s_k = v_k + u_k$  and  $v_k^T u_k = 0$  for all  $k \in \mathbb{N}$ , Lemma 3.6,  
 418 Lemma 3.13, (3.1), and Assumption 3.1 it follows, for all  $k_1 \leq k < k_2$ , that

$$\begin{aligned}
 419 \quad \bar{\phi}_{\tau_k}(x_k) - \bar{\phi}_{\tau_{k+1}}(x_{k+1}) \\
 420 \quad &\geq \eta \sigma_u \tau_{\min} \alpha_{\min} \left( \frac{\|u_k\|_2}{\alpha_k} \right)^2 + \eta \sigma_c \frac{1}{2\kappa_c} \|J_k^T c_k\|_2^2 \min\{1/(1 + \kappa_{\nabla c}^2), \kappa_v \alpha_{\min}\} \\
 421 \quad &\geq \eta \sigma_u \tau_{\min} \alpha_{\min} \|g_k + g_{r,k} - J_k^T y_k\|_2^2 + \eta \sigma_c \frac{\sigma_{\min}^2}{2\kappa_c} \|c_k\|_2^2 \min\{1/(1 + \kappa_{\nabla c}^2), \kappa_v \alpha_{\min}\} \\
 422 \quad &\geq \kappa_{\Phi} \chi_k^2,
 \end{aligned}$$

423 where  $\kappa_{\Phi}$  is defined in the theorem statement. Using this inequality, Lemma 3.14(iii),  
 424 and nonnegativity of  $\bar{\phi}_{\tau}$  for all  $\tau \in \mathbb{R}_{>0}$ , we find that

$$\begin{aligned}
 425 \quad \bar{\phi}_{\tau_0}(x_0) &\geq \bar{\phi}_{\tau_{k_1}}(x_{k_1}) \geq \bar{\phi}_{\tau_{k_1}}(x_{k_1}) - \bar{\phi}_{\tau_{k_2}}(x_{k_2}) \\
 426 \quad &\geq \sum_{k=k_1}^{k_2-1} (\bar{\phi}_{\tau_k}(x_k) - \bar{\phi}_{\tau_{k+1}}(x_{k+1})) \geq \sum_{k=k_1}^{k_2-1} \kappa_{\Phi} \chi_k^2,
 \end{aligned}$$

which may then be combined with the fact that  $\chi_k > \epsilon$  for all iterations  $k_1 \leq k \leq k_2$   
 (see the assumptions of the current theorem) to conclude that

$$\bar{\phi}_{\tau_0}(x_0) \geq (k_2 - k_1) \kappa_{\Phi} \epsilon^2,$$

427 from which (3.12) follows.

428 The final result in the theorem, namely the claimed upper bound on the maximum  
 429 iterations before  $\chi_k \leq \epsilon$ , follows from what we just proved and the fact that maximum  
 430 number of unsuccessful iterations is bounded as in (3.10).  $\square$

431 Before proving a result concerning convergence to a KKT point, we need to prove  
 432 that the Lagrange multiplier estimates generated by subproblem (2.5) are bounded.

433 **LEMMA 3.16.** *The Lagrange multiplier estimate sequence  $\{y_k\}$  is bounded.*

*Proof.* Note from (3.2) and the Cauchy-Schwarz and triangle inequalities that

$$\frac{1}{\alpha_k} \|u_k\|_2^2 = -(g_k + g_{r,k})^T u_k \leq \|g_k + g_{r,k}\|_2 \|u_k\|_2 \leq (\|g_k\|_2 + \|g_{r,k}\|_2) \|u_k\|_2,$$

434 which when combined with (1.2) shows that

$$435 \quad (3.14) \quad \frac{1}{\alpha_k} \|u_k\|_2 \leq \kappa_{\nabla f} + \kappa_{\partial r}.$$

436 Also observe that it follows from (3.1) and Assumption 3.1 that

$$437 \quad (3.15) \quad J_k^T y_k = g_k + \frac{1}{\alpha_k} u_k + g_{r,k} \iff y_k = (J_k J_k^T)^{-1} J_k (g_k + \frac{1}{\alpha_k} u_k + g_{r,k}).$$

438 Combining (3.15), Assumption 3.1, the triangle inequality, and (3.14) it follows that

$$\begin{aligned}
439 \quad \|y_k\|_2 &\leq \frac{1}{\sigma_{\min}} \|g_k + \frac{1}{\alpha_k} u_k + g_{r,k}\|_2 \\
440 \quad &\leq \frac{1}{\sigma_{\min}} \left( \kappa_{\nabla f} + \kappa_{\partial r} + \frac{1}{\alpha_k} \|u_k\|_2 \right) \\
441 \quad &\leq \frac{2}{\sigma_{\min}} \left( \kappa_{\nabla f} + \kappa_{\partial r} \right).
\end{aligned}$$

442 Since this result holds for arbitrary  $k \in \mathbb{N}$ , we have proved the result.  $\square$

443 We can now prove that limit points of the primal sequence are KKT points.

444 **THEOREM 3.17.** *Let Assumption 1.1 and Assumption 3.1 hold. Any limit point  $x_*$*   
445 *of the sequence  $\{x_k\}$  is a first-order KKT point for problem (1.1), i.e.,  $c(x_*) = 0$  and*  
446 *there exist vectors  $y_* \in \mathbb{R}^m$  and  $g_{r,*} \in \partial r(x_*)$  such that  $g(x_*) + g_{r,*} - J(x_*)^T y_* = 0$ .*

447 *Proof.* Let  $x_*$  be a limit point of  $\{x_k\}$ , i.e., there exists  $\mathcal{K}_1$  so that  $\{x_k\}_{k \in \mathcal{K}_1} \rightarrow x_*$ .  
448 Theorem 3.15 allows us to conclude that there exists a subsequence  $\mathcal{K}_2 \subseteq \mathcal{K}_1$  so that

$$449 \quad (3.16) \quad 0 = \lim_{k \in \mathcal{K}_2} \chi_k = \lim_{k \in \mathcal{K}_2} \max\{\|g_k + g_{r,k} - J_k^T y_k\|_2, \|c(x_k)\|_2\}.$$

Lemma 3.16 allows us to assert the existence of a vector  $y_* \in \mathbb{R}^m$  and subsequence  $\mathcal{K}_3 \subseteq \mathcal{K}_2$  such that  $\{y_k\}_{k \in \mathcal{K}_3} = y_*$ . It follows from this limit,  $\{x_k\}_{k \in \mathcal{K}_3} \rightarrow x_*$ , continuity of  $g$  and  $J$ , and (3.16) that

$$\lim_{k \in \mathcal{K}_3} g_{r,k} = \lim_{k \in \mathcal{K}_3} (-g_k + J_k^T y_k) = -g(x_*) + J(x_*)^T y_* =: g_{r,*}.$$

450 Finally, combining this equality with  $\{x_k\}_{k \in \mathcal{K}_3} \rightarrow x_*$ , continuity of  $c$ , and (3.16) it  
451 follows that  $g(x_*) + g_{r,*} - J(x_*)^T y_* = 0$  and  $c(x_*) = 0$ , which completes the proof.  $\square$

452 **3.2.2. Strong LICQ fails.** In this section we prove properties of the iterate  
453 sequence  $\{x_k\}$  in Algorithm 2.1 when the strong LICQ assumption used in the previous  
454 section (see Assumption 3.1) does not hold. In such a setting, we should expect to  
455 prove weaker results since, for example, Lagrange multipliers may not even exist.

456 Our main theorem of this section uses the quantity

$$457 \quad (3.17) \quad \bar{\chi}_k := \max\{\|g_k + g_{r,k} - J_k^T y_k\|_2, \|J_k^T c_k\|_2\},$$

458 which is related to the quantity  $\chi_k$  used in the previous section (see (3.11)).

459 **THEOREM 3.18.** *Let Assumption 1.1 hold. One of the following two cases occurs.*

460 (i) *There exists  $\bar{\tau}_{\min} > 0$  such that  $\tau_k \geq \bar{\tau}_{\min}$  for all  $k \in \mathbb{N}$ . In this case, it also*  
461 *follows that  $\alpha_k \geq \bar{\alpha}_{\min} := \min\{\alpha_0, \xi \bar{\tau}_{\min} / (\bar{\tau}_{\min} L_g + L_J)\}$  for all  $k \in \mathbb{N}$  and,*  
462 *for a given  $\epsilon > 0$ , the maximum number of iterations before  $\bar{\chi}_k \leq \epsilon$  is*

$$463 \quad \left( \max \left\{ 0, \left\lceil \frac{\log \left( \frac{\bar{\tau}_{\min}}{\alpha_0 (\bar{\tau}_{\min} L_g + L_J)} \right)}{\log(\xi)} \right\rceil \right\} + 1 \right) \left\lceil \frac{\tau_0 (f(x_0) - f_{\inf} + r(x_0)) + \|c(x_0)\|_2}{\bar{\kappa}_{\Phi} \epsilon^2} \right\rceil$$

464 *where  $\bar{\kappa}_{\Phi} := \eta \min\{\sigma_u \bar{\tau}_{\min} \bar{\alpha}_{\min}, \frac{\sigma_c}{2\kappa_c(1+\kappa_{\nabla}^2 c)}, \frac{\sigma_c \kappa_u \bar{\alpha}_{\min}}{2\kappa_c}\}$ .*

465 (ii) *The merit parameter values converge to zero, i.e.,  $\lim_{k \rightarrow \infty} \tau_k = 0$ . In this*  
466 *case, there exists a subsequence  $\mathcal{K} \subseteq \mathbb{N}$  such that  $\lim_{k \in \mathcal{K}} \|J_k^T c_k\|_2 = 0$ .*

467 *Proof.* Let us start by considering part (i), in which case we know that there  
468 exists  $\bar{\tau}_{\min} > 0$  such that  $\tau_k \geq \bar{\tau}_{\min}$  for all  $k \in \mathbb{N}$ . Using this lower bound on  $\{\tau_k\}$ ,



469 the proof of Lemma 3.13 still holds (with  $\tau_{\min}$  replaced by  $\bar{\tau}_{\min}$ ), so that both (3.9)  
470 and (3.10) hold (with  $\tau_{\min}$  replaced by  $\bar{\tau}_{\min}$ ), thus proving the first claim on  $\bar{\alpha}_{\min}$ .  
471 Using (3.9) and (3.10) (with  $\tau_{\min}$  replaced by  $\bar{\tau}_{\min}$ ), the proof of Theorem 3.15 holds  
472 almost exactly as written. In particular, the proof holds as written until the middle  
473 of the second displayed equation, where we have (now with  $\tau_{\min}$  and  $\alpha_{\min}$  replace by  
474  $\bar{\tau}_{\min}$  and  $\bar{\alpha}_{\min}$ , respectively) that

$$475 \quad \bar{\phi}_{\tau_k}(x_k) - \bar{\phi}_{\tau_{k+1}}(x_{k+1}) \\
476 \quad \geq \eta\sigma_u \bar{\tau}_{\min} \bar{\alpha}_{\min} \|g_k + g_{r,k} - J_k^T y_k\|_2^2 + \eta\sigma_c \frac{1}{2\bar{\kappa}_c} \|J_k^T c_k\|_2^2 \min\{(1/(1 + \kappa_{\nabla c}^2)), \kappa_v \bar{\alpha}_{\min}\}.$$

If we now use the definitions of  $\bar{\kappa}_{\Phi}$  and  $\bar{\chi}_k$  we find that

$$\bar{\phi}_{\tau_k}(x_k) - \bar{\phi}_{\tau_{k+1}}(x_{k+1}) \geq \bar{\kappa}_{\Phi} \bar{\chi}_k^2.$$

477 The remainder of the proof of Theorem 3.15 now follows exactly as written but with  
478  $\bar{\chi}_k$  and  $\bar{\kappa}_{\Phi}$  in place of  $\chi_k$  and  $\kappa_{\Phi}$ , respectively. This completes the proof of part (i).  
479 Part (ii) follows from Theorem 3.10.  $\square$

480 A discussion on Theorem 3.18(i) is of interest. In particular, the result in Theo-  
481 rem 3.18(i) is of the same form as the result Theorem 3.15, with the only difference  
482 being the values of the constants  $(\tau_{\min}, \alpha_{\min}, \kappa_{\Phi})$  versus  $(\bar{\tau}_{\min}, \bar{\alpha}_{\min}, \bar{\kappa}_{\Phi})$ . A conse-  
483 quence of Assumption 3.1 used in Section 3.2.1 is that we have an explicit definition  
484 for  $\tau_{\min}$  (see Lemma 3.12), which implies an explicit lower bound on  $\alpha_{\min}$  and  $\kappa_{\Phi}$   
485 (see Lemma 3.13 and Theorem 3.15). On the other hand, no explicit lower bound on  
486  $\bar{\tau}_{\min}$  is possible (in general) when Assumption 3.1 does not hold (in fact, it is even  
487 possible that  $\{\tau_k\} \rightarrow 0$ ), and therefore the values for the constants  $(\bar{\tau}_{\min}, \bar{\alpha}_{\min}, \bar{\kappa}_{\Phi})$  in  
488 Theorem 3.18(i) will depend on the particular value of  $\bar{\tau}_{\min}$  for that given problem. In  
489 this respect, the complexity result of Theorem 3.15 is stronger than Theorem 3.18(i),  
490 which is not surprising since Theorem 3.15 is proved under Assumption 3.1.

491 **4. Numerical Results.** In this section, we present results of numerical ex-  
492 periments performed with our Python implementation of Algorithm 2.1. The test  
493 problems are formulated with an  $\ell_1$  regularizer, which is a common choice in many  
494 applications since it is known to induce sparse solutions. The goal of our numerical  
495 tests is to validate the overall performance of our method using standard optimization  
496 metrics and to evaluate its ability to correctly identify the zero-nonzero structure of  
497 a solution. For comparison purposes, we use the solver Bazinga [14], which is a safe-  
498 guarded augmented Lagrangian method. The details concerning the test problems,  
499 our implementation, and the test results are given in the remainder of this section.

500 **4.1. Test problems.** We considered a special instance of an  $\ell_1$ -regularized ob-  
501 jective function with equality constraints that can be written in the form

$$502 \quad (4.1) \quad \min_{x \in \mathbb{R}^n, a \in \mathbb{R}^m} f(x) + \lambda \|a\|_1 \quad \text{s.t.} \quad c(x) + a = 0$$

503 for some chosen regularization parameter  $\lambda \in \mathbb{R}_{>0}$ . The functions  $f$  and  $c$  were chosen  
504 as a subset of the CUTEst [18] test problems, and we used PyCUTEst [16] to evaluate  
505 these functions in our Python code. Our *initial* test problems were chosen as the  
506 subset of CUTEst problems that satisfied the following properties: (i) the objective  
507 function was non-constant; (ii) the problem had at least one equality constraint, no  
508 inequality constraints, and no bound constraints on variables; and (iii) the number  
509 of equality constraints and variables satisfied  $1 \leq m < n \leq 1000$ . The restriction

510  $m < n$  rules out problems that essentially reduce to finding a feasible point for the  
511 constraints, while the restriction  $n < 1000$  is used to keep the computational cost to a  
512 manageable level. As for the choice of  $\lambda$ , one can show that if  $\bar{x}$  is a first-order KKT  
513 point with Lagrange multiplier vector  $\bar{y}$  to the problem

$$514 \quad (4.2) \quad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad c(x) = 0,$$

515 then  $(\bar{x}, 0)$  is a first-order KKT point to problem (4.1) with Lagrange multiplier  $\bar{y}$  as  
516 long as  $\lambda \geq \|\bar{y}\|_\infty$ . Therefore, in our tests, we set  $\lambda = \|\bar{y}\|_\infty + 10$  where  $\bar{y}$  is computed  
517 by solving problem (4.2) using IPOPT [33]. Since problems MSS1, MSS2, and CHAIN  
518 were not successfully solved by IPOPT, they were removed from the *initial* test set,  
519 thus resulting in the *final* set of 46 test problems found in Table A.1–Table A.2.  
520 Although the problem formulation 4.1 is somewhat contrived, this particular formu-  
521 lation allows us to better evaluate the structure identifying properties of the iterates  
522 produced by Algorithm 2.1 and Bazinga.

523 **4.2. Implementation details.** The parameter and input values used are pre-  
524 sented in Table 4.1 (no fine-tuning was performed). As for the starting point  $(x_0, a_0)$   
525 for problem (4.1), the vector  $x_0$  is set to the default value supplied by CUTEst and  
526 the vector  $a_0$  is set as  $-c(x_0)$  so that the initial point  $(x_0, a_0)$  is feasible.

TABLE 4.1  
Parameters and inputs used by Algorithm 2.1, with  $x_0$  set to the value supplied by CUTEst.

$\alpha_0$	$\tau_{-1}$	$\kappa_v$	$\sigma_c$	$\epsilon_\tau$	$\xi$	$\eta$	$\sigma_u$
10	1	1000	0.1	0.1	0.5	$10^{-4}$	0.1

To approximately solve the trust-region subproblem (2.1), as needed in Line 5 of Algorithm 2.1, we used a Newton-like method. In particular, assuming for now that  $J_k$  had full row-rank, we first computed the minimizer of  $m_k(v)$  over all  $v \in \text{Range}(J_k^T)$ . Using the relationship  $v = J_k^T w$ , this problem may be written as

$$\min_{w \in \mathbb{R}^m} \frac{1}{2} \|c_k\|_2^2 + w^T J_k J_k^T c_k + \frac{1}{2} w^T J_k J_k^T J_k J_k^T w.$$

It follows from the first-order optimality conditions and the full rank assumption on  $J_k$  that the unique solution, call it  $w_n$ , satisfies

$$J_k J_k^T J_k J_k^T w_n = -J_k J_k^T c_k \iff J_k J_k^T w_n = -c_k.$$

527 After solving this linear system for  $w_n$ , we have that  $v_n = J_k^T w_n$ . Next, we project  
528 this Newton step  $v_n$  onto the trust-region constraint by defining

$$529 \quad \bar{v}_n := \min\{\|v_n\|_2, \kappa_v \alpha_k \|J_k^T c_k\|_2\} \frac{v_n}{\|v_n\|_2}.$$

Also accounting for the possibility that  $J_k$  may be rank deficient, we define  $v_k$  as

$$v_k \leftarrow \begin{cases} v_k^c & \text{if } J_k \text{ does not have full rank or } m_k(v_k^c) < m_k(\bar{v}_n) \\ \bar{v}_n & \text{otherwise,} \end{cases}$$

530 which by construction ensures that  $v_k$  satisfies conditions (2.2a)–(2.2c), as needed.

531 Next, to solve subproblem (2.5) (as needed in Line 12 of Algorithm 2.1) we exploit  
 532 the structure of the  $\ell_1$ -norm. By introducing variables  $(p, q) \in \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{\geq 0}^n$  and using  
 533  $e$  to denote the vector of all ones, we can consider the equivalent problem

$$\begin{aligned}
 & \min_{u,p,q} g_k^T u + \frac{1}{2\alpha_k} \|u\|_2^2 + \lambda e^T (p + q) \\
 & \text{s.t. } J_k u = 0, \quad x_k + v_k + u = p - q, \quad p \geq 0, \quad q \geq 0,
 \end{aligned}
 \tag{4.3}$$

535 which is a convex quadratic program (QP). To solve subproblem (4.3) we use the  
 536 primal active-set QP solver in the state-of-the-art software Gurobi version 11.0.0 [21].  
 537 Note that only a subset of the optimization variables receive  $\ell_1$  regularization in the  
 538 test problem formulation (see (4.1)). This setting is handled using the above scheme  
 539 by introducing  $p$  and  $q$  variables only for those variables appearing in the  $\ell_1$  norm.

540 Algorithm 2.1 was terminated when one of the following conditions was satisfied.

- 541 • **Approximate KKT point.** Algorithm 2.1 was terminated during the  $k$ th  
 542 iteration with  $x_k$  considered an approximate KKT point if  $\|c_k\|_2 \leq 10^{-6}$  and  
 543  $\|g_k + g_{r,k} - J_k^T y_k\|_2 \leq 10^{-6}$ , as motivated by (3.11) and Theorem 3.15.
- 544 • **Approximate infeasible stationary point.** Algorithm 2.1 was terminated  
 545 during the  $k$ th iteration with  $x_k$  considered an approximate infeasible station-  
 546 ary point if  $\|c_k\|_2 \geq 10^{-2}$  and  $\|J_k^T c_k\|_2 \leq 10^{-12}$ .
- 547 • **Gurobi error.** Algorithm 2.1 was terminated during the  $k$ th iteration if the  
 548 Gurobi solver failed to solve subproblem (4.3) using its default tolerances.
- 549 • **Maximum iterations.** Algorithm 2.1 was terminated if 1000 iterations was  
 550 completed without terminating for any of the previous reasons.

551 For comparison purposes, we solve the same test problems using the Bazinga  
 552 method. Bazinga is a safeguarded augmented Lagrangian framework that uses an inner  
 553 subproblem solver called PANOC<sup>+</sup>, which is a linesearch algorithm that combines  
 554 a forward-backward iteration and a quasi-Newton step over the forward-backward en-  
 555 velop of the objective function; see the Bazinga paper [14] for more details.<sup>1</sup> The  
 556 Bazinga algorithm was terminated when one of the following conditions was satisfied.

- 557 • **Approximate KKT point.** Bazinga was terminated if certain primal fea-  
 558 sibility and dual stationarity measures were less than  $10^{-6}$ . Further details  
 559 on the termination conditions of Bazinga can be found in [14, Section 3.3].
- 560 • **Not a number.** Bazinga was terminated if a NaN occurred.
- 561 • **Maximum iterations.** Bazinga was terminated if 100 iterations was com-  
 562 pleted without terminating for any of the previous reasons. Fewer maximum  
 563 iterations was allowed for Bazinga compared to Algorithm 2.1 because each it-  
 564 eration of Bazinga is significantly more expensive compared to Algorithm 2.1.  
 565 See the end of Section 4.3 and Appendix A for a discussion and table of results  
 566 concerning computational times, respectively.

567 **4.3. Test results.** In this subsection, we present the results of using our Algo-  
 568 rithm 2.1 and Bazinga to solve problems of the form (4.1) with test functions chosen  
 569 as described in Section 4.1. To see detailed results for each test problem, see Table A.1  
 570 and Table A.2 in Appendix A. In the remainder of this section, we discuss the key  
 571 results and observations summarized in Table 4.2.

572 We begin by describing the meanings of the columns of Table 4.2, and discuss their  
 573 corresponding values to compare the performances of Algorithm 2.1 and Bazinga.

- 574 • **Method.** The name of the method.

<sup>1</sup>The code package of Bazinga is downloaded from <https://github.com/aldma/Bazinga.jl>.

- 575 • **Feasible.** The number of test problems for which the corresponding method  
576 terminated at a point with constraint violation no larger than  $10^{-6}$ . For this  
577 metric we see that the two methods behaved similarly, with Bazinga achieving  
578 approximate feasibility on one more test problem.
- 579 • **Feasible, Better Objective.** To understand the meaning of this column,  
580 let  $f_{\text{Algorithm 2.1}}$  denote the final objective value returned by Algorithm 2.1  
581 and  $f_{\text{Bazinga}}$  denote the final objective value returned by Bazinga. We can  
582 then define the relative difference in the returned objective function values as  
583

$$(4.4) \quad f_{\text{diff}} := \frac{f_{\text{Bazinga}} - f_{\text{Algorithm 2.1}}}{\max(1, |\min(f_{\text{Bazinga}}, f_{\text{Algorithm 2.1}})|)}.$$

585 We indicate that Algorithm 2.1 (resp., Bazinga) had a better relative objective  
586 value if  $f_{\text{diff}} \geq 10^{-5}$  (resp.,  $f_{\text{diff}} \leq -10^{-5}$ ). Using this terminology, column  
587 “Feasible, Better Objective” gives the number of test problems for which both  
588 algorithms terminated at a point with constraint violation less than  $10^{-6}$  and  
589 the corresponding method had a better relative objective value. For this  
590 metric we see that Algorithm 2.1 significantly outperforms Bazinga in terms  
591 of final objective function values when both algorithms return vectors that  
592 satisfy the constraint violation tolerance.

- 593 • **Performs Better.** The number of test problems for which the corresponding  
594 method either (i) met the constraint violation tolerance and the other method  
595 did not, or (ii) both methods reached the constraint violation tolerance and  
596 the corresponding method had a better relative objective value (see (4.4)).  
597 Algorithm 2.1 significantly outperforms Bazinga on this metric.
- 598 •  **$a$  is Zero** The number of test problems for which the corresponding method  
599 returned  $a = 0$ . Algorithm 2.1 significantly outperformed Bazinga in this  
600 metric, with Algorithm 2.1 (resp., Bazinga) returning  $a = 0$  on 36 (resp.,  
601 13) of the test problems. We conjecture that Bazinga’s poor performance  
602 on this metric is due to its inner subproblem solver, which sacrifices solution  
603 sparsity for faster convergence of its iterates by combining proximal-gradient  
604 calculations with quasi-Newton ideas (see [14]). We investigated the test  
605 problems that Algorithm 2.1 did not return  $a = 0$  and a Gurobi error was  
606 not encountered, and found that by increasing the regularization parameter,  
607 Algorithm 2.1 would return solutions satisfying  $a = 0$ .
- 608 •  **$a$  is Small.** The number of test problems for which the corresponding method  
609 returned  $\|a\|_{\infty} \leq 10^{-5}$ , thus indicating that  $a$  was small (possibly equal to  
610 zero). When comparing this column with column “ $a$  is Zero”, we see that the  
611 only difference is that Algorithm 2.1 returned a small (nonzero) value for  $a$   
612 on one additional test problem; the results for Bazinga were unchanged.
- 613 • **KKT Found.** The number of test problems for which the corresponding  
614 method terminated with an approximate KKT point, as discussed in Sec-  
615 tion 4.2. Algorithm 2.1 computed an approximate KKT point on 33 of the 46  
616 test problems. Algorithm 2.1 encountered Gurobi errors (see Section 4.2) on  
617 test problems BT4 and HS56 that were related to large constraint violation  
618 values, which were caused by too large of an initial value for the merit pa-  
619 rameter. These failures can be avoided by decreasing the initial value for the  
620 merit parameter, but we did not do that for the numerical tests presented.

621 Overall, we are pleased with the results of Table 4.2. We believe that they in-  
622 dicate that there is significant merit to our proposed algorithm, especially in terms

TABLE 4.2

Algorithm 2.1 versus Bazinga on various performance metrics related to solving problem (4.1) with test functions given in Table A.1–Table A.2; see Section 4.3 for the meaning of the columns.

Method	Feasible	Feasible, Better Objective	Performs Better	$a$ is Zero	$a$ is Small	KKT Found
Algorithm 2.1	40	23	23	36	37	33
Bazinga	41	2	7	13	13	21

623 of computing structured approximate solutions. It is worth noting that we have not  
 624 discussed computational time since comparing our Python implementation of Algo-  
 625 rithm 2.1 with the Julia implementation of Bazinga gives an advantage to Bazinga  
 626 (purely because of the programming language used). Even still, one can observe from  
 627 Table A.1 and Table A.2 that Algorithm 2.1 requires less (often significantly less)  
 628 computing time compared to Bazinga on nearly every test problem.

629 **5. Conclusion.** We have presented one of the first proximal-gradient type meth-  
 630 ods that can handle nonlinear equality constraints, and effectively return structured  
 631 solutions where the structure is determined by the choice of regularization function. In  
 632 the future, it would be interesting to address inequality constraints, establish conver-  
 633 gence results under weaker assumptions, and accelerate convergence by incorporating  
 634 Nesterov acceleration or subspace acceleration.

635

## REFERENCES

- 636 [1] M. ANITESCU, *A superlinearly convergent sequential quadratically constrained quadratic pro-*  
 637 *gramming algorithm for degenerate nonlinear programming*, SIAM J. Optim., 12 (2002),  
 638 pp. 949–978.
- 639 [2] M. ANNERGREN, A. HANSSON, AND B. WAHLBERG, *An admm algorithm for solving l-1 regu-*  
 640 *larized mpc*, in 2012 IEEE 51st IEEE Conference on Decision and Control (CDC), IEEE,  
 641 2012, pp. 4486–4491.
- 642 [3] A. BAGIROV, N. KARMITSA, AND M. M. MÁKELÄ, *Introduction to Nonsmooth Optimization:*  
 643 *theory, practice and software*, vol. 12, Springer, 2014.
- 644 [4] A. BECK, *First-order methods in optimization*, SIAM, 2017.
- 645 [5] A. BECK AND M. TEOULLE, *A fast iterative shrinkage-thresholding algorithm for linear inverse*  
 646 *problems*, SIAM Journal on Imaging Sciences, 2 (2009), pp. 183–202.
- 647 [6] J.-F. CAI, E. J. CANDÈS, AND Z. SHEN, *A singular value thresholding algorithm for matrix*  
 648 *completion*, SIAM Journal on optimization, 20 (2010), pp. 1956–1982.
- 649 [7] T. CHEN, F. E. CURTIS, AND D. P. ROBINSON, *A reduced-space algorithm for minimizing  $\ell_1$ -*  
 650 *regularized convex functions*, SIAM Journal on Optimization, 27 (2017), pp. 1583–1610.
- 651 [8] T. CHEN, F. E. CURTIS, AND D. P. ROBINSON, *FaRSA for  $\ell_1$ -regularized convex optimiza-*  
 652 *tion: local convergence and numerical experience*, Optimization Methods and Software, 33  
 653 (2018), pp. 396–415.
- 654 [9] A. R. CONN, N. I. M. GOULD, AND PH. L. TOINT, *A globally convergent augmented Lagrangian*  
 655 *algorithm for optimization with general constraints and simple bounds*, SIAM J. Numer.  
 656 Anal., 28 (1991), pp. 545–572.
- 657 [10] A. R. CONN, N. I. M. GOULD, AND PH. L. TOINT, *Trust-Region Methods*, Society for Industrial  
 658 and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
- 659 [11] A. R. CONN AND T. PIETRZYKOWSKI, *A penalty function method converging directly to a con-*  
 660 *strained optimum*, SIAM J. Numer. Anal., 14 (1977), pp. 348–375.
- 661 [12] F. E. CURTIS, H. JIANG, AND D. P. ROBINSON, *An adaptive augmented lagrangian method for*  
 662 *large-scale constrained optimization*, Math. Program., (2013), pp. 1–45.
- 663 [13] F. E. CURTIS, D. P. ROBINSON, AND B. ZHOU, *Inexact sequential quadratic optimization for*  
 664 *minimizing a stochastic objective function subject to deterministic nonlinear equality con-*  
 665 *straints*, arXiv preprint arXiv:2107.03512, (2021).
- 666 [14] A. DE MARCHI, X. JIA, C. KANZOW, AND P. MEHLITZ, *Constrained composite optimization*  
 667 *and augmented lagrangian methods*, Mathematical Programming, (2023), pp. 1–34.

- 668 [15] A. V. FIACCO AND G. P. MCCORMICK, *Nonlinear Programming*, Classics in Applied Mathematics,  
669 Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second ed.,  
670 1990. Reprint of the 1968 original.
- 671 [16] J. FOWKES, L. ROBERTS, AND Á. BÚRMEN, *Pycutest: an open source python package of opti-*  
672 *mization test problems*, Journal of Open Source Software, 7 (2022), p. 4377.
- 673 [17] M. FUKUSHIMA, *A successive quadratic programming method for a class of constrained nons-*  
674 *mooth optimization problems*, Mathematical programming, 49 (1990), pp. 231–251.
- 675 [18] N. I. M. GOULD, D. ORBAN, AND PH. L. TOINT, *CUTEst: a constrained and unconstrained*  
676 *testing environment with safe threads*, technical report, Rutherford Appleton Laboratory,  
677 Chilton, England, 2013, <https://doi.org/10.1007/s10589-014-9687-3>, [http://dx.doi.org/10.](http://dx.doi.org/10.1007/s10589-014-9687-3)  
678 [1007/s10589-014-9687-3](http://dx.doi.org/10.1007/s10589-014-9687-3).
- 679 [19] N. I. M. GOULD AND D. P. ROBINSON, *A second derivative SQP method: Global convergence*,  
680 SIAM J. Optim., 20 (2010), pp. 2023–2048.
- 681 [20] N. I. M. GOULD AND D. P. ROBINSON, *A second derivative SQP method: Local convergence*  
682 *and practical issues*, SIAM J. Optim., 20 (2010), pp. 2049–2079.
- 683 [21] GUROBI OPTIMIZATION, LLC, *Gurobi Optimizer Reference Manual*, 2023, [https://www.gurobi.](https://www.gurobi.com)  
684 [com](https://www.gurobi.com).
- 685 [22] C. B. GURWITZ AND M. L. OVERTON, *Sequential quadratic programming methods based on*  
686 *approximating a projected hessian matrix*, SIAM J. Sci. and Statist. Comput., 10 (1989),  
687 pp. 631–653.
- 688 [23] S.-P. HAN, *A globally convergent method for nonlinear programming*, Journal of optimization  
689 theory and applications, 22 (1977), pp. 297–309.
- 690 [24] S. P. HAN AND O. L. MANGASARIAN, *Exact penalty functions in nonlinear programming*, Math-  
691 ematical programming, 17 (1979), pp. 251–269.
- 692 [25] H. KARIMI, J. NUTINI, AND M. SCHMIDT, *Linear convergence of gradient and proximal-gradient*  
693 *methods under the Polyak-Lojasiewicz condition*, in Joint European Conference on Machine  
694 Learning and Knowledge Discovery in Databases, Springer, 2016, pp. 795–811.
- 695 [26] C.-P. LEE AND S. J. WRIGHT, *Inexact successive quadratic approximation for regularized opti-*  
696 *mization*, Computational Optimization and Applications, 72 (2019), pp. 641–674.
- 697 [27] Y.-F. LIU, S. MA, Y.-H. DAI, AND S. ZHANG, *A smoothing sqp framework for a class of compos-*  
698 *ite l q minimization over polyhedron*, Mathematical Programming, 158 (2016), pp. 467–500.
- 699 [28] D. Q. MAYNE AND N. MARATOS, *A first-order, exact penalty function algorithm for equality*  
700 *constrained optimization problems*, Mathematical Programming, 16 (1979), pp. 303–324.
- 701 [29] J. L. MORALES, J. NOCEDAL, AND Y. WU, *A sequential quadratic programming algorithm with*  
702 *an additional equality constrained phase*, IMA J. Numer. Anal., 32 (2012), pp. 553–579.
- 703 [30] J. NOCEDAL AND S. WRIGHT, *Numerical optimization*, Springer Science & Business Media,  
704 2006.
- 705 [31] M. J. POWELL, *A fast algorithm for nonlinearly constrained optimization calculations*, in Numerical  
706 Analysis: Proceedings of the Biennial Conference Held at Dundee, June 28–July  
707 1, 1977, Springer, 2006, pp. 144–157.
- 708 [32] D. P. ROBINSON, *Primal-dual active-set methods for large-scale optimization*, Submitted to  
709 Journal of Optimization Theory and Applications, (2013).
- 710 [33] A. WÄCHTER AND L. T. BIEGLER, *On the implementation of an interior-point filter line-search*  
711 *algorithm for large-scale nonlinear programming*, Mathematical programming, 106 (2006),  
712 pp. 25–57.
- 713 [34] N. XIAO, X. LIU, AND Y.-X. YUAN, *A penalty-free infeasible approach for a class of nonsmooth*  
714 *optimization problems over the stiefel manifold*, arXiv preprint arXiv:2103.03514, (2021).
- 715 [35] G. YUAN AND B. GHANEM, *\ell\_0 tv: A sparse optimization method for impulse noise image*  
716 *restoration*, IEEE transactions on pattern analysis and machine intelligence, 41 (2017),  
717 pp. 352–364.
- 718 [36] V. M. ZAVALA AND M. ANITESCU, *Scalable nonlinear programming via exact differentiable*  
719 *penalty functions and trust-region newton methods*, SIAM Journal on Optimization, 24  
720 (2014), pp. 528–558.
- 721 [37] D. ZHU, L. ZHAO, AND S. ZHANG, *A first-order primal-dual method for nonconvex constrained*  
722 *optimization based on the augmented lagrangian*, Mathematics of Operations Research,  
723 (2023).



724 **Appendix A. Detailed Results for the Test Problems.**

725 In this appendix we provide the detailed output from our Algorithm 2.1 and  
 726 Bazinga for the test problems in Table A.1 and Table A.2. See Section 4 for details  
 727 on the problem formulation, the test functions used, and the implementation details.

728 The columns of Table A.1 and Table A.2 have the following meanings.

- 729 • **Problem.** The name of the test problem. Specifically, the value in this col-  
 730 umn gives the name of the CUTEst test problem used to obtain the objective  
 731 function  $f$  and constraint function  $c$  in the test problem formulation (4.1).
- 732 • **Method.** The name of the method used.
- 733 • **Obj.** The value of the objective function in problem (4.1) at the final iterate  
 734 returned by the solver.
- 735 • **RE.** The relative error between the objective function value returned by the  
 736 algorithm and the optimal objective function value. In particular, if we let  
 737  $(f+r)$  denote the objective function value returned by a solver on a particular  
 738 problem and let  $(f+r)_{\text{opt}}$  denote the optimal objective value for that same  
 739 problem (as determined by the CUTEst documentation), then we define the  
 740 relative error for that method on that problem as

$$741 \quad \mathbf{RE} = \frac{|(f+r) - (f+r)_{\text{opt}}|}{\max(1, |(f+r)_{\text{opt}}|)}.$$

- 742 •  $\|c(x) + a\|_2$ . The value of  $\|c(x) + a\|_2$  at the point returned by the solver.
- 743 •  $\|a\|_\infty$ . The value of  $\|a\|_\infty$  at the point returned by the solver.
- 744 • **Status.** A three letter string that indicates the outcome when the given  
 745 method was used to solve the given test problem. In particular, the value  
 746 “Opt” means that the method returned a final iterate that was an **approx-**  
 747 **imate KKT point** as described in Section 4.2. The value “Max” indicates  
 748 that the method reached its maximum allowed number of iterations as de-  
 749 scribed under **Maximum Iterations** in Section 4.2. The value “Err” only  
 750 occurred for Algorithm 2.1 and indicates that a Gurobi error occurred as de-  
 751 scribed under **Gurobi error** in Section 4.2. Finally, the value “NaN” only  
 752 occurred for Bazinga and indicates that the data type not-a-number occurred.
- 753 • **Time.** The computational time measured in seconds.

Problem	Method	Obj	RE	$\ c(x) + a\ _2$	$\ a\ _\infty$	Status	Time
BT11	Alg. 2.1	8.25E-01	1.58E-07	1.01E-13	0.00E+00	Opt	1.71E+00
	Bazinga	2.22E+01	2.79E+01	8.23E-07	6.62E-01	Max	3.50E+02
BT12	Alg. 2.1	6.19E+00	3.15E-10	4.26E-10	0.00E+00	Opt	2.99E-01
	Bazinga	3.75E+02	5.95E+01	1.02E-07	1.24E+01	Max	3.33E+02
BT1	Alg. 2.1	-9.00E+01	8.90E+01	4.77E-13	1.00E+00	Opt	3.35E-01
	Bazinga	-5.54E+01	5.44E+01	3.68E-08	6.08E-01	Opt	9.37E+00
BT2	Alg. 2.1	1.02E+05	3.14E+06	2.24E-05	1.02E+04	Max	5.17E+00
	Bazinga	3.26E-02	2.83E-07	8.22E-07	0.00E+00	Opt	1.24E+01
BT3	Alg. 2.1	4.09E+00	3.10E-06	4.71E-15	0.00E+00	Opt	2.58E-01
	Bazinga	3.41E+01	7.33E+00	3.91E-09	5.92E-01	Max	3.39E+02
BT4	Alg. 2.1	-2.67E+31	5.87E+29	2.63E+21	1.84E+10	Err	3.24E-02
	Bazinga	4.00E+01	1.88E+00	1.22E-07	2.69E+00	Max	3.26E+02
BT5	Alg. 2.1	9.62E+02	7.32E-11	4.05E-10	0.00E+00	Opt	1.31E-01
	Bazinga	1.03E+03	7.60E-02	7.06E-08	3.11E+00	Max	3.26E+02
BT6	Alg. 2.1	2.77E-01	4.89E-07	1.82E-12	0.00E+00	Opt	6.57E-01
	Bazinga	2.65E+01	9.45E+01	1.32E-07	1.19E+00	Max	3.42E+02
BT7	Alg. 2.1	3.96E+01	8.71E-01	2.44E-13	4.54E-01	Opt	4.60E+00
	Bazinga	9.26E+02	2.02E+00	9.74E-07	2.50E-01	Opt	1.27E+02
BT8	Alg. 2.1	1.00E+00	2.61E-06	8.67E-13	2.37E-07	Max	7.83E+00
	Bazinga	1.00E+00	9.86E-10	1.31E-09	0.00E+00	Opt	9.57E+00
BT9	Alg. 2.1	-1.00E+00	1.05E-11	1.63E-11	0.00E+00	Opt	1.46E-01
	Bazinga	2.60E+01	2.70E+01	5.87E-07	1.24E+00	Max	3.23E+02
BYRDSPHR	Alg. 2.1	-4.68E+00	7.61E-08	6.55E-09	0.00E+00	Opt	7.89E-02
	Bazinga	6.27E+00	2.34E+00	2.23E-08	4.92E-01	Opt	2.06E+01
DIXCHLNG	Alg. 2.1	1.59E+02	1.59E+02	1.98E-07	9.70E-01	Max	1.55E+01
	Bazinga	NaN	NaN	NaN	NaN	NaN	1.35E+01
ELEC	Alg. 2.1	1.46E+04	2.08E-01	1.12E-06	2.97E+00	Max	2.24E+04
	Bazinga	1.58E+04	1.42E-01	8.22E-08	6.45E-01	Max	3.34E+04
EXTROSNBNE	Alg. 2.1	-2.00E+00	2.00E+00	1.86E-06	0.00E+00	Max	1.38E+05
	Bazinga	4.11E+03	4.11E+03	3.32E-07	4.12E-01	Max	3.65E+04
GENHS28	Alg. 2.1	9.27E-01	9.27E-01	2.47E-15	0.00E+00	Opt	2.07E+00
	Bazinga	9.27E-01	9.27E-01	1.01E-10	0.00E+00	Opt	9.35E+00
HS100LNP	Alg. 2.1	6.81E+02	1.09E-10	2.27E-13	0.00E+00	Opt	6.93E-01
	Bazinga	7.26E+02	1.16E+02	1.17E-07	1.46E+00	Max	3.36E+02
HS111LNP	Alg. 2.1	-4.78E+01	1.10E-03	4.99E-10	0.00E+00	Max	1.15E+01
	Bazinga	-5.22E+01	9.52E-02	7.12E-08	2.60E+00	Max	4.58E+02
HS26	Alg. 2.1	1.09E+00	1.09E+00	5.12E-08	0.00E+00	Max	5.27E+00
	Bazinga	1.05E-13	1.05E-13	2.93E-10	0.00E+00	Opt	9.14E+00
HS27	Alg. 2.1	4.00E-02	5.63E-10	9.48E-15	0.00E+00	Opt	3.48E-01
	Bazinga	4.00E-02	1.46E-07	1.46E-07	0.00E+00	Opt	9.12E+00
HS28	Alg. 2.1	5.05E-11	5.05E-11	8.88E-16	0.00E+00	Opt	2.26E-01
	Bazinga	7.95E-17	7.95E-17	8.01E-10	0.00E+00	Opt	9.05E+00
HS39	Alg. 2.1	-1.00E+00	1.05E-11	1.63E-11	0.00E+00	Opt	1.37E-01
	Bazinga	2.60E+01	2.70E+01	5.87E-07	1.24E+00	Max	3.25E+02

TABLE A.1  
Results for test problems BT11-HS39.



Problem	Method	Obj	RE	$\ c(x) + a\ _2$	$\ a\ _\infty$	Status	Time (s)
HS40	Alg. 2.1	-2.50E-01	1.39E-10	5.91E-11	0.00E+00	Opt	1.05E-01
	Bazinga	1.43E+01	5.81E+01	9.97E-10	4.78E-01	Opt	1.25E+02
HS42	Alg. 2.1	1.39E+01	2.68E-10	3.06E-14	0.00E+00	Opt	2.68E-01
	Bazinga	2.76E+01	9.90E-01	5.91E-10	1.13E+00	Opt	9.38E+01
HS46	Alg. 2.1	5.73E-10	5.73E-10	4.09E-12	0.00E+00	Opt	1.84E+00
	Bazinga	NaN	NaN	NaN	NaN	NaN	3.09E+02
HS47	Alg. 2.1	1.33E-05	1.33E-05	4.70E-10	0.00E+00	Max	8.48E+00
	Bazinga	4.21E+01	4.21E+01	1.27E-10	1.36E+00	Opt	3.84E+01
HS48	Alg. 2.1	2.69E-11	2.69E-11	4.44E-16	0.00E+00	Opt	1.84E-01
	Bazinga	2.00E-13	2.00E-13	1.59E-10	0.00E+00	Opt	9.06E+00
HS49	Alg. 2.1	2.20E-04	2.20E-04	1.81E-13	0.00E+00	Max	7.22E+00
	Bazinga	2.90E+01	2.90E+01	2.02E-07	1.43E+00	Max	3.34E+02
HS50	Alg. 2.1	2.86E-11	2.86E-11	1.72E-14	0.00E+00	Opt	6.58E+00
	Bazinga	4.53E+00	4.53E+00	1.66E-09	7.65E-02	Max	3.33E+02
HS51	Alg. 2.1	1.48E-11	1.48E-11	6.28E-16	0.00E+00	Opt	1.71E-01
	Bazinga	1.61E-15	1.61E-15	3.17E-09	0.00E+00	Opt	9.05E+00
HS52	Alg. 2.1	5.33E+00	4.31E-12	7.69E-16	0.00E+00	Opt	1.37E+00
	Bazinga	5.33E+00	6.50E-07	2.98E-07	0.00E+00	Opt	9.15E+00
HS56	Alg. 2.1	-5.81E+83	1.68E+83	4.07E+28	4.94E+27	Err	6.57E-02
	Bazinga	NaN	NaN	NaN	NaN	NaN	7.44E+01
HS61	Alg. 2.1	-1.44E+02	1.55E-11	2.03E-13	0.00E+00	Opt	1.75E-01
	Bazinga	-1.44E+02	4.29E-09	4.48E-07	0.00E+00	Opt	9.10E+00
HS6	Alg. 2.1	1.16E-10	1.16E-10	2.38E-11	0.00E+00	Opt	4.11E-01
	Bazinga	4.46E-14	4.46E-14	2.49E-09	0.00E+00	Opt	1.20E+01
HS77	Alg. 2.1	2.42E-01	6.83E-09	3.16E-12	0.00E+00	Opt	5.32E-01
	Bazinga	NaN	NaN	NaN	NaN	NaN	1.12E+01
HS78	Alg. 2.1	1.10E+01	4.77E+00	4.73E-10	1.00E+00	Max	8.44E+00
	Bazinga	5.44E+01	1.96E+01	3.93E-09	1.74E+00	Max	3.44E+02
HS79	Alg. 2.1	7.88E-02	7.11E-10	6.50E-14	0.00E+00	Opt	3.51E+00
	Bazinga	2.82E+01	3.57E+02	3.74E-08	9.33E-01	Max	3.48E+02
HS7	Alg. 2.1	-1.73E+00	2.44E-10	8.18E-11	0.00E+00	Opt	1.81E+00
	Bazinga	-1.73E+00	2.12E-08	1.29E-07	0.00E+00	Opt	9.11E+00
HS9	Alg. 2.1	-5.00E-01	1.32E-09	1.78E-15	0.00E+00	Opt	7.98E-02
	Bazinga	-5.00E-01	9.02E-10	1.37E-08	0.00E+00	Opt	9.10E+00
LCH	Alg. 2.1	-1.23E+02	2.77E+01	3.42E-06	9.47E+00	Max	1.84E+03
	Bazinga	1.11E+00	1.26E+00	7.89E-09	3.50E-01	Max	1.07E+03
MARATOS	Alg. 2.1	-1.00E+00	2.00E+00	4.71E-10	0.00E+00	Opt	6.19E-02
	Bazinga	5.48E+00	4.48E+00	7.69E-11	6.10E-01	Opt	9.56E+00
MWRIGHT	Alg. 2.1	2.50E+01	2.40E-01	1.09E-13	0.00E+00	Opt	6.92E-01
	Bazinga	2.67E+02	7.11E+00	3.41E-07	3.02E+00	Max	3.46E+02
ORTHREGB	Alg. 2.1	2.09E-12	2.09E-12	9.39E-13	0.00E+00	Opt	5.19E-01
	Bazinga	3.90E+02	3.90E+02	6.22E-08	6.49E+00	Max	5.21E+02
S316-322	Alg. 2.1	3.34E+02	1.29E-06	3.31E-08	0.00E+00	Opt	6.14E-02
	Bazinga	9.12E+02	1.73E+00	3.26E-07	9.80E-01	Opt	1.59E+01
SPIN2OP	Alg. 2.1	1.23E-09	1.23E-09	1.22E-09	0.00E+00	Opt	6.55E+02
	Bazinga	2.09E+03	2.09E+03	2.02E-08	2.09E+00	Max	4.45E+03
STREGNE	Alg. 2.1	4.93E-12	4.93E-12	1.11E-15	0.00E+00	Opt	1.99E-01
	Bazinga	NaN	NaN	NaN	NaN	NaN	1.82E+01

TABLE A.2  
Results for test problems HS40-STREGNE.