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Beyond hypergraph acyclicity: limits of tractability for pseudo-Boolean optimization

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Beyond hypergraph acyclicity: limits of tractability for pseudo-Boolean optimization

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Abstract

In this paper, we study the problem of minimizing a polynomial function with literals over all binary points, often referred to as *pseudo-Boolean optimization.* We investigate the fundamental limits of computation for this problem by providing new necessary conditions and sufficient conditions for tractability. On one hand, we obtain the first intractability results for pseudo-Boolean optimization problems on signed hypergraphs with bounded rank, in terms of the treewidth of the intersection graph. On the other hand, we introduce the nest-set gap, a new hypergraph-theoretic notion that enables us to move beyond hypergraph acyclicity, and obtain a polynomial-size extended formulation for the pseudo-Boolean polytope of a class of signed hypergraphs whose underlying hypergraphs contain β-cycles.

Key words: *binary polynomial optimization; pseudo-Boolean optimization; treewidth; pseudo-Boolean polytope; polynomial-size extended formulation.*

1 Introduction

Binary polynomial optimization, i.e., the problem of maximizing a multivariate polynomial function over the set of binary points, is a fundamental NP-hard problem in discrete optimization. In this paper, we consider a formulation that encodes the objective function using pseudo-Boolean functions, often referred to as pseudo-Boolean optimization. To formally define the problem, we make use of signed hypergraphs, a representation scheme that was recently introduced in [19]. Recall that a *hypergraph* G is a pair (V, E), where V is a finite set of nodes and E is a set of subsets of V of cardinality at least two, called the edges of G. A *signed hypergraph* H is a pair (V, S), where V is a finite set of nodes and S is a set of signed edges. A *signed edge* $s \in S$ is a pair (e, η_s), where e is a subset of V of cardinality at least two, and η_s is a map that assigns to each $v \in e$ a *sign* $\eta_s(v) \in \{-1, +1\}$. The *underlying edge* of a signed edge $s = (e, \eta_s)$ is e. Two signed edges $s = (e, \eta_s)$, $s' = (e', \eta_{s'}) \in S$ are said to be *parallel* if $e = e'$, and they are said to be *identical* if $e = e'$ and $\eta_s = \eta_{s'}$. Throughout this paper, we consider signed hypergraphs with no identical signed edges. However, our signed hypergraphs often contain parallel signed edges. With any signed hypergraph $H = (V, S)$, and cost vector $c \in \mathbb{R}^{V \cup S}$, we associate the *pseudo-Boolean*

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optimization problem:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{s \in S} c_s \prod_{v \in s} \sigma_s(z_v) \\ \text{s.t.} \quad & z \in \{0,1\}^V, \end{aligned} \tag{PBO}$$

where

$$\sigma_s(z_v) := \begin{cases} z_v & \text{if } \eta_s(v) = +1 \\ 1 - z_v & \text{if } \eta_s(v) = -1, \end{cases}$$

and without loss of generality we assume $c_s \neq 0$ for all $s \in S$. Several applications such as maximum satisfiability problems [26], and graphical models [28] can naturally be formulated as pseudo-Boolean optimization problems. See [4] for a review of existing results on pseudo-Boolean optimization. We then linearize the objective function of Problem PBO by introducing a variable z_s, for each signed edge $s \in S$, to obtain an equivalent formulation in a lifted space:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{s \in S} c_s z_s \\ \text{s.t.} \quad & z_s = \prod_{v \in s} \sigma_s(z_v) \qquad \forall s \in S \\ & z \in \{0,1\}^{V \cup S}. \end{aligned} \tag{L-PBO}$$

In [19], the authors introduced the *pseudo-Boolean set* of the signed hypergraph $H = (V, S)$, as the feasible region of Problem L-PBO:

$$\text{PBS}(H) := \Big\{ z \in \{0,1\}^{V \cup S} : z_s = \prod_{v \in s} \sigma_s(z_v), \ \forall s \in S \Big\},$$

and they refer to its convex hull as the *pseudo-Boolean polytope* and denote it by $\text{PBP}(H)$. With each signed hypergraph $H = (V, S)$, we associate two key hypergraphs: (i) the *underlying hypergraph* of H, which is the hypergraph obtained from H by ignoring the signs and dropping parallel edges, and (ii) the *multilinear hypergraph* of H, which is the hypergraph $\text{mh}(H) = (V, E)$, where E is constructed as follows: For each $s \in S$, and every $t \subseteq s$ with $\eta_s(v) = -1$ for all $v \in t$, the set E contains $\{v \in s : \eta_s(v) = +1\} \cup t$, if this set has cardinality at least two.

Let us consider an important special case of Problem PBO obtained by letting $\eta_s(v) = +1$ for every $s \in S$ and $v \in s$. In this case, a signed hypergraph $H = (V, S)$ essentially coincides with its underlying hypergraph $G = (V, E)$, and hence Problem PBO can be equivalently written as the following *binary multilinear optimization problem*:

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v z_v + \sum_{e \in E} c_e \prod_{v \in e} z_v \\ \text{s.t.} \quad & z \in \{0,1\}^V, \end{aligned} \tag{BMO}$$

where again without loss of generality we assume $c_e \neq 0$ for all $e \in E$. In [15], the authors defined the *multilinear set* as the feasible region of a linearized multilinear optimization problem:

$$\mathcal{S}(G) := \Big\{ z \in \{0,1\}^{V+E} : z_e = \prod_{v \in e} z_v, \ \forall e \in E \Big\},$$

and referred to its convex hull as the *multilinear polytope* $\text{MP}(G)$. Notice that by expanding the objective function of Problem PBO over a signed hypergraph H, this problem can be reformulated in the form of problem BMO over the multilinear hypergraph of H; i.e., $\text{mh}(H)$.

In the special case where $|e| = 2$ for all $e \in E$, Problem BMO simplifies to a *binary quadratic optimization (BQO)* problem and the multilinear polytope coincides with the well-known Boolean quadric polytope introduced by Padberg [34] and later studied by others.

1.1 Polynomial-time solvable classes

It is well known that if the objective function of Problem PBO is super-modular, then the pseudo-Boolean optimization problem can be solved in strongly polynomial time [37]. However, the recognition of supermodularity of pseudo-Boolean functions of degree four (or higher) is NP-complete [25]. Another celebrated line of research relates the complexity of combinatorial optimization problems to the treewidth of a corresponding graph. In [11] the authors proved that if the intersection graph of the underlying hypergraph of Problem PBO has a bounded treewidth, then this problem can be solved in linear time using dynamic programming. Recall that given a hypergraph $G = (V, E)$, the *intersection graph of* G is the graph with node set V, and where two nodes $v, v' \in V$ are adjacent if $v, v' \in e$ for some $e \in E$. In [6], authors present a strongly polynomial-time algorithm for Problem PBO, assuming that the incidence graph of the underlying hypergraph has a bounded treewidth. Recall that given a hypergraph $G = (V, E)$, the *incidence graph of* G is a bipartite graph whose vertex set is $V \cup E$ and the edge set is $\{\{v, e\} : v \in V, e \in E, v \in e\}$. For any hypergraph, the treewidth of the incidence graph is upper bounded by the treewidth of the intersection graph. Moreover, while the problem of computing the treewidth of a graph is NP-hard in general, it is fixed-parameter tractable when parameterized by the treewidth [3].

Next, we review the existing results on polynomial-size extended formulations for the multilinear polytope and the pseudo-Boolean polytope. Note that if a polynomial-size extended formulation of PBP(H) (resp. MP(G)) is readily available, then Problem PBO (resp. Problem BMO) can be solved in polynomial time. However, the converse may not hold, in general. In [34], Padberg proved that if $G = (V, E)$ is an acyclic graph, then the Boolean quadric polytope BQP(G) is defined by $4|E|$ inequalities in the original space. The notion of graph acyclicity has been extended to several different notions of hypergraph acyclicity; in increasing order of generality, one can name Berge-acyclicity, γ-acyclicity, β-acyclicity, and α-acyclicity. We should remark that polynomial-time algorithms for determining acyclicity level of hypergraphs are available [24]. In [16, 5, 14, 18], the authors obtained a complete characterization of acyclic hypergraphs whose multilinear polytopes admit polynomial-size extended formulations. Henceforth, we denote by r the *rank* of the hypergraph $G = (V, E)$, defined as the maximum cardinality of an edge in E. In [16, 5], the authors proved that if G is Berge-acyclic, then MP(G) is defined by $|V| + (r+2)|E|$ inequalities in the original space. Moreover, in [16], the authors proved that if G is γ-acyclic, then MP(G) has a polynomial-size extended formulation with at most $|V| + 2|E|$ variables and at most $|V| + (r+2)|E|$ inequalities. In [18], the authors present a polynomial-size extended formulation for the multilinear polytope of β-acyclic hypergraphs with at most $(r-1)|V| + |E|$ variables and at most $(3r-4)|V| + 4|E|$ inequalities. In [19], the authors introduced the pseudo-Boolean polytope, and subsequently generalized the earlier results by proving that if the underlying hypergraph of a signed hypergraph $H = (V, S)$ is β-acyclic, then PBP(H) has a polynomial-size extended formulation with $O(r|V||S|)$ variables and inequalities. Note that this result is more general than the previous ones because it only requires the β-acyclicity of the underlying hypergraph of H. Indeed, the multilinear hypergraph of H may contain many β-cycles. In this paper, we obtain a significant generalization of this result.

On the other hand, in [14], the authors prove that Problem BMO is strongly NP-hard over α-acyclic hypergraphs. This result implies that, unless $\mathcal{P} = \mathcal{NP}$, one cannot construct a polynomial-size extended formulation for the multilinear polytope of α-acyclic hypergraphs. However, in [17], the authors showed that if the rank r of an α-acyclic hypergraph is upper bounded by the logarithm of a polynomial in the size of the hypergraph, then the multilinear polytope has a polynomial-size extended formulation with $O(2^r|V|)$ variables and inequalities. Interestingly, this sufficient condition is equivalent to assuming a bounded treewidth for the intersection graph of Problem BMO [39, 33, 2] (see Section 4 of [17] for the proof of equivalence). For further results regarding polyhedral relaxations of multilinear sets of degree at least three, see [12, 20, 17, 13, 29, 9, 21, 31].

1.2 Our contributions

In this paper, we investigate the fundamental limits of computation for pseudo-Boolean optimization, by providing new necessary conditions and sufficient conditions for tractability. Henceforth, for brevity, whenever we say the treewidth of a hypergraph (resp. a signed hypergraph), we refer to the treewidth of the intersection graph of the hypergraph (resp. underlying hypergraph of the signed hypergraph). The main contributions of this paper are twofold:

Intractability results. In Section 2, we obtain the first intractability results for pseudo-Boolean optimization. Namely, under some mild assumptions, we show that for every sequence of hypergraphs $\{G_k\}_{k=1}^{\infty}$ indexed by the treewidth k and with bounded rank, the complexity of solving Problem PBO on a signed hypergraph whose underlying hypergraph is G_k grows super-polynomially in k (see Theorem 4). To prove this result, we first obtain an intractability result for BQO problems which is based on a complexity result for inference in graphical models [7] (see Theorem 3). Subsequently, using the inflation operation introduced in [19], we present a polynomial-time reduction of intractable BQO instances to pseudo-Boolean optimization instances. Notice that the bounded rank assumption is key here, as for example Problem PBO over signed hypergraphs whose underlying hypergraphs are β-acyclic can be solved in polynomial time and a β-acyclic hypergraph with unbounded rank has an unbounded treewidth as well.

Tractability results. In Section 3, we move beyond hypergraph acyclicity and obtain new polynomial-size extended formulations for the pseudo-Boolean polytope of signed hypergraphs whose underlying hypergraphs contain β-cycles. To this end, we make use of the hypergraph-theoretic notion of nest-sets recently introduced in [32], which is a natural generalization of nest-points. We then introduce the notion of nest-set gap for a hypergraph, a quantity equal to zero if and only if the hypergraph is β-acylic. We then prove that if the nest-set gap of the underlying hypergraph is bounded, then the pseudo-Boolean polytope admits a polynomial-size extended formulation (see Theorem 10). The complexity of checking whether the nest-set gap of a hypergraph is bounded is unknown. However, checking the boundedness of a related quantity; namely, the nest-set width of a hypergraph can be solved in polynomial time. The nest-set width of a hypergraph equals one if and only if the hypergraph is β-acyclic. Moreover, the nest-set width of a hypergraph is lower bounded by its nest-set gap; this, in turn, implies that the pseudo-Boolean polytope of a signed hypergraph with bounded nest-set width has a polynomial-size extended formulation as well.

Figure 1 summarizes various types of hypergraphs (resp. underlying hypergraphs) for which tractability or intractability of Problem BMO (resp. Problem PBO) is known.

2 Intractability results for pseudo-Boolean optimization

In this section, we discuss the limits of tractability of Problems BMO and PBO. For a graph G, we denote by $\mathrm{tw}(G)$ the treewidth of G. For a hypergraph G, we denote by $\mathrm{tw}(G)$ the treewidth

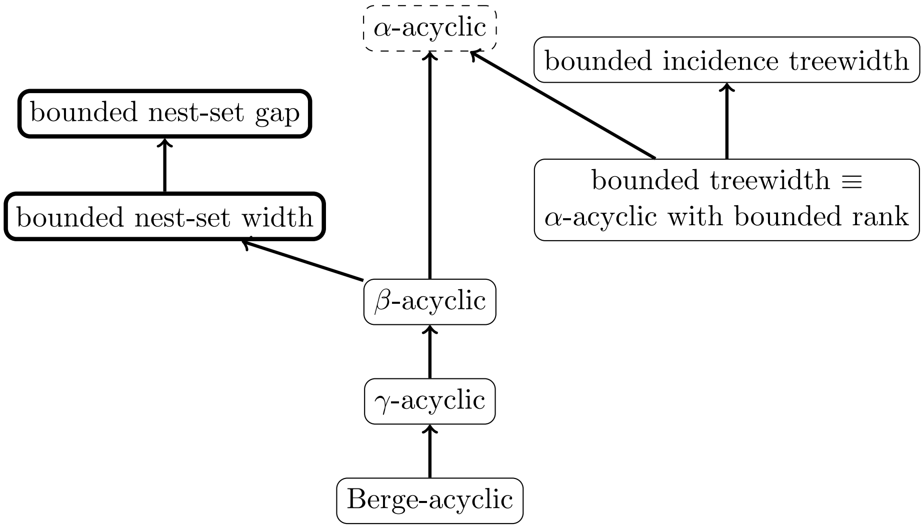

Figure 1: Hypergraph classes (resp. underlying hypergraph classes) for which tractability or intractability of Problem BMO (resp. Problem PBO) is known. The NP-hard class is depicted in dashed lines, while polynomial-time solvable classes are depicted in solid lines. Our contributions in this paper are depicted in thick solid lines. Arcs are directed from less general to more general. Properties with no directed connection are incomparable.

of the intersection graph of G. For a signed hypergraph H, we denote by $\text{tw}(H)$ the treewidth of the underlying hypergraph of H. Henceforth, we denote by $\text{poly}(n)$, a polynomial function in n. In the combinatorial optimization literature, it is well-understood that treewidth is a crucial metric in measuring the difficulty of many graph problems. In our context, if the treewidth of the intersection graph of the hypergraph is bounded, then the multilinear polytope admits a polynomial size extended formulation [39, 33, 2]:

Theorem 1. *Let $G = (V, E)$ be hypergraph with $\text{tw}(G) = k$. Then $\text{MP}(G)$ has an extended formulation with $O(2^k|V|)$ variables and inequalities. Moreover, if $k \in O(\log \text{poly}(|V|, |E|))$, then $\text{MP}(G)$ has a polynomial-size extended formulation.*

Theorem 1 implies that if the hypergraph has a bounded treewidth, then Problem BMO can be solved in polynomial-time. A similar result can be stated for the pseudo-Boolean polytope $\text{PBP}(H)$ in terms of $\text{tw}(H)$ (see theorem 6 in [19]). In this section, we obtain necessary conditions in terms of treewidth of G (resp. H) for polynomial-time solvability of Problem BMO (resp. Problem PBO). To this end, we first obtain an intractability result for the special case where the objective function is quadratic. Building upon this result, we then obtain intractability results for higher degree binary polynomial optimization problems.

Given an instance Λ of Problem BMO or Problem PBO, in this section, we refer to the *size* of Λ (also known as *bit size*, or *length*), denoted by $\|\Lambda\|$, as the number of bits required to encode it, which is the standard definition in the mathematical programming literature (see for example [36, 10]). Recall that the Bounded-error Probabilistic Polynomial time ($\mathcal{BPP}$) is the class of decision problems solvable by a probabilistic Turing machine in polynomial time with an error probability bounded by 1/3 for all instances. Our tractability results in this section are under the assumption that $\mathcal{NP} \not\subseteq \mathcal{BPP}$. It is widely believed that $\mathcal{P} = \mathcal{BPP}$, which implies that $\mathcal{NP} \not\subseteq \mathcal{BPP}$ is equivalent to $\mathcal{P} \neq \mathcal{NP}$. For a precise definition of $\mathcal{BPP}$ and the commonly believed $\mathcal{NP} \not\subseteq \mathcal{BPP}$ hypothesis, we refer the reader to [1].

2.1 Intractability of binary quadratic optimization

Our intractability result, which is closely related to theorem 5.1 in [7] and theorem 7 in [23], implies that there exists no class of graphs with unbounded treewidth for which every BQO problem on these graphs can be solved in time polynomial in the treewidth. More precisely, for every sequence of graphs $\{G_k\}_{k=1}^{\infty}$ indexed by treewidth k, there exists a choice of objective function coefficients c such that the runtime of any algorithm that solves the BQO problem is super-polynomial in treewidth k. In theorem 5.1 of [7], the authors prove a similar result for the inference problem on binary pair-wise graphical models. It is known that this inference problem can be formulated as a BQO problem. However, we state an independent proof since first our proof is direct; i.e., it is concerned with a BQO problem and not inference, and second theorem 5.1 in [7] is proved under the stronger assumption $\mathcal{NP} \not\subseteq \mathcal{P}/\operatorname{poly}$, where $\mathcal{P}/\operatorname{poly}$ denotes the class of decision problems that can be solved by a polynomial-time Turing machine with *advice* strings of length polynomial in the input size. Unlike other polynomial-time classes such as $\mathcal{P}$ or $\mathcal{BPP}$, the class $\mathcal{P}/\operatorname{poly}$, is considered impractical for computing. In theorem 7 of [23], the authors prove an intractability result similar to ours, for quadratically constrained quadratic optimization problems and their reduction arguments relies on the fact that the optimization problem has linear and quadratic constraints. In spite of these technical differences, we should mention that our proof technique follows the general scheme first proposed in [7] and later refined in [23].

Our proof relies on the next theorem, which is essentially a consequence of the celebrated graph minor theorem [8]. The proof of this theorem is stated inside the proof of theorem 7 in [23]. We include it here for completeness.

Theorem 2. *Any planar graph $\bar{G}$ with n nodes is a minor of any graph G with treewidth at least $\kappa(n) \in O(n^{98} \operatorname{poly}\log(n))$. Furthermore, there is a randomized algorithm that, given G, outputs the sequence of minor operations transforming G into $\bar{G}$ in time $O(\operatorname{poly}(|V(G)| \cdot \kappa(n)))$, with high probability.*

Proof. Let $\bar{G}$ be a planar graph with n nodes, and let G be a graph with treewidth at least $\kappa(n) \in O(n^{98} \operatorname{poly}\log(n))$. Since $\bar{G}$ is planar, it is a minor of the $n/c \times n/c$ grid, for some constant c [35], and the sequence of minor operations can be found in linear time [38]. It follows from [8] that the $n/c \times n/c$ grid is a minor of G, and that we can find the corresponding sequence of minor operations (with high probability) in $O(\operatorname{poly}(|V(G)| \cdot \kappa(n)))$ time. □

Henceforth, we say that a countable family of graphs $\{G_k\}_{k=1}^{\infty}$ is *polynomial-time enumerable* if a description of G_k is computable in $\operatorname{poly}(k)$ time. This in turn implies that an encoding of G_k of size polynomial in k exists. The next theorem provides a necessary condition for polynomial-time solvability of BQO problems.

Theorem 3. *Let $\{G_k\}_{k=1}^{\infty}$ be a polynomial-time enumerable family of graphs with $\operatorname{tw}(G_k) = k$, for all k. Let f be an algorithm that solves any instance Λ_k of BQO on graph G_k in time $T(k) \cdot \operatorname{poly}(\|\Lambda_k\|)$. Then, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k.*

Proof. The problem *Max 2-SAT* on a planar graph, also known as *planar Max-2SAT,* is NP-complete [27]. It can be checked that an instance of Max-2SAT on a planar graph $\bar{G} = (\bar{V}, \bar{E})$ can be formulated as a BQO problem on the same graph $\bar{G}$, and with objective coefficients $\bar{c}_v \in \{\pm 1, 0\}$ for all $v \in \bar{V}$ and $\bar{c}_e \in \{\pm 2, \pm 1, 0\}$ for all $e \in \bar{E}$. Let $n := |\bar{V}|$. By Theorem 2, $\bar{G}$ is a minor of $G_{\kappa(n)} = (V_{\kappa(n)}, E_{\kappa(n)})$, where $\kappa(n)$ is as defined in the statement of the theorem. Since by the enumerability assumption $V_{\kappa(n)}$ is bounded by a polynomial in $\kappa(n)$, by Theorem 2, there is a polynomial-time randomized algorithm that, given $G_{\kappa(n)}$, outputs the sequence of minor operations transforming $G_{\kappa(n)}$ into $\bar{G}$, with high probability.

We now show that, given an instance of a BQO problem on a planar graph $\bar{G} = (\bar{V}, \bar{E})$ with $\bar{c}_v \in \{\pm 1, 0\}$ for all $v \in \bar{V}$ and $\bar{c}_e \in \{\pm 2, \pm 1, 0\}$ for all $e \in \bar{E}$, we can construct in polynomial time an equivalent instance of a BQO problem on $G_{\kappa(n)}$. First, we observe that $G_{\kappa(n)}$ can be constructed in time bounded by a polynomial in $\kappa(n)$, due to our enumerability assumption. It suffices to show the equivalence for a single minor operation; The result then follows from a repeated application of this technique. Let $H = (V_H, E_H)$ be graph and denote by $H' = (V_{H'}, E_{H'})$ a graph obtained from H after a single minor operation. Consider an instance Λ' of a BQO problem on H' with objective function coefficients c'. We show how to solve this instance Λ' by solving an instance Λ of a BQO problem on H with objective function coefficients c. Recall that there are three graph minor operations (see for example [7]):

1. Node deletion: Let $u \in V_H$ and denote by E_u all edges of H containing u. Then the minor of H obtained by deleting node u is $H' = (V_H \setminus \{u\}, E_H \setminus E_u)$. In this case, the objective function coefficients c of Λ can defined as follows: $c_u = 0$, $c_e = 0$ for all $e \in E_u$ and $c_p = c'_p$ for all $p \in V_H \setminus \{u\}$ and $p \in E_H \setminus E_u$.

2. Edge deletion: Let $f \in E_H$. Then the minor of H obtained by deleting edge f is $H' = (V_H, E_H \setminus \{f\})$. In this case, the objective function coefficients c of Λ can defined as follows: $c_f = 0$ and $c_p = c'_p$ for all $p \in V_H$ and $p \in E_H \setminus \{f\}$.

3. Edge contraction: Let $\{u, v\} \in E_H$ and denote by E_u all edges of H containing u. Let E'_u be set obtained by replacing u with v in every element of E_u and subsequently dropping $\{v, v\}$ from it. Then the minor of H obtained by contracting edge $\{u, v\}$ to node v is $H' = (V_H \setminus \{u\}, E_H \cup E'_u \setminus E_u)$. Define $M := \sum_{p \in V_{H'} \cup E_{H'}} |c'_p|$. In this case, the objective function coefficients c of Λ can defined as follows: $c_u = -M$, $c_v = c'_v - M$, $c_{\{u,v\}} = 2M$, $c_p = c'_p$ for all $p \in V_H \setminus \{u, v\}$ and $p \in E_H \setminus E_u$, and $c_e = c'_{e \cup \{v\} \setminus \{u\}}$ for all $e \in E_u \setminus \{u, v\}$. To see that Λ' is equivalent to Λ, first note that Λ' is given by

$$\begin{aligned} \max \sum_{w \in V_{H'}} c'_w x_w &+ \sum_{\{w,q\} \in E_{H'}} c'_{w,q} x_w x_q \\ \text{s.t.} \quad x_w &\in \{0, 1\}, w \in V_{H'}. \end{aligned}$$

Then Λ' can be equivalently solved by solving the following optimization problem on H:

$$\begin{aligned} \max \sum_{w \in V_H} c''_w x_w &+ \sum_{\{w,q\} \in E_H} c''_{w,q} x_w x_q \\ \text{s.t.} \quad x_u &= x_v \\ x_w &\in \{0, 1\}, w \in V_H, \end{aligned} \tag{1}$$

where $c''_u = 0$, $c''_{\{u,v\}} = 0$, $c''_p = c'_p$ for all $p \in V_H \setminus \{u\}$ and $p \in E_H \setminus E_u$, and $c''_e = c'_{e \cup \{v\} \setminus \{u\}}$ for all $e \in E_u \setminus \{u, v\}$. In order to reformulate Problem (1) as a BQO problem on H, it suffices to remove the constraint $x_u = x_v$ and instead subtract the penalty term $M(x_u - x_v)^2$ from the objective function. The equivalence then follows.

We have explained how to reduce (with high probability) a BQO problem on $\bar{G} = (\bar{V}, \bar{E})$ to a BQO problem on $G_{\kappa(n)} = (V_{\kappa(n)}, E_{\kappa(n)})$. The number of arithmetic operations performed in the reduction is clearly polynomially bounded, therefore it suffices to show that the size of the objective function coefficients of the constructed instance is polynomially bounded by the size of the original BQO instance. Recall that the objective coefficients of the original instance are $\bar{c}_v \in \{\pm 1, 0\}$ for all $v \in \bar{V}$, and $\bar{c}_e \in \{\pm 2, \pm 1, 0\}$ for all $e \in \bar{E}$. Let us now consider the objective function coefficients of the constructed instance of BQO problem , denoted by c, and observe that c has integer components. Let M be the sum of the absolute values of all coefficients in the original instance, i.e., $M := \sum_{v\in\bar{V}} |\bar{c}_v| + \sum_{e\in\bar{E}} |\bar{c}_e|$ and observe that $M \leq |\bar{V}| + 2|\bar{E}|$. Our enumerability assumption, together with $|\bar{V}| \leq |V_{\kappa(n)}|$, $|\bar{E}| \leq |E_{\kappa(n)}|$, imply that M is polynomial in $\kappa(n)$ as well. Now consider the three types of minor operations defined above. First note that the first two minor operations do not change the sum of the absolute values of all coefficients. Let us consider the last minor operation; i.e., the edge contraction. By construction, after each edge contraction, the sum of the absolute values of all coefficients doubles. Since there are at most $|E_{\kappa(n)}|$ contractions, we conclude that in the constructed instance, the sum of the absolute values of all coefficients is at most $2^{|E_{\kappa(n)}|}M$, whose size, by our enumerability assumption, is polynomial in $\kappa(n)$.

Now let us use Algorithm f to solve the resulting BQO problem over $G_{\kappa(n)}$ in time $T(\kappa(n)) \cdot \text{poly}(\|\Lambda_{\kappa(n)}\|)$ and hence equivalently solving the initial BQO problem over the graph $\bar{G}$ with n nodes. We already showed that $\|\Lambda_{\kappa(n)}\|$ is upper bounded by a polynomial in $\kappa(n)$, which in turn is bounded by a polynomial in n. Assume, for a contradiction, that $T(k)$ is bounded by a polynomial in k. This implies that $T(\kappa(n))$ is bounded by a polynomial in $\kappa(n)$, and therefore by a polynomial in n. It then follows that planar MAX-2SAT$\in \mathcal{BPP}$, which contradicts the assumption that $\mathcal{NP} \not\subseteq \mathcal{BPP}$. Therefore, $T(k)$ grows super-polynomially in k. □

The following result is a direct consequence of Theorem 3.

Corollary 1. *For every positive k, let $\mathcal{G}_k$ be the set of all graphs with treewidth k. Let f be an algorithm that solves any instance Λ_k of a BQO problem on a graph in $\mathcal{G}_k$ in time $T(k)\cdot\text{poly}(\|\Lambda_k\|)$. Then, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k.*

Proof. For every positive k, let $\mathcal{G}_k$ be the set of all graphs with treewidth k. Let f be an algorithm that solves any instance Λ_k of a BQO problem on a graph in $\mathcal{G}_k$ in time $T(k)\cdot\text{poly}(\|\Lambda_k\|)$. Consider a polynomial-time enumerable family of graphs $\{G_k\}_{k=1}^{\infty}$ as in the statement of Theorem 3, and note that $G_k \in \mathcal{G}_k$ for every positive k. Then, f satisfies the assumption of Theorem 3. Hence, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k. □

Together with Theorem 1, Corollary 1 suggests that a bounded treewidth is necessary and sufficient for polynomial-time solvability of BQO problems. However, notice that there is a gap between these two results. For every positive integer k, let $G_k = (V_k, E_k)$ be a graph with treewidth k and with $n_k := |V_k| \in \Theta(2^{\sqrt{k}})$, hence $k \in \Theta(\log^2 n_k)$. Clearly, $\{G_k\}_{k=1}^{\infty}$ does not satisfy the assumption of Theorem 1 because $k \notin O(\log \text{poly}(n_k))$. On the other hand, $\{G_k\}_{k=1}^{\infty}$ is not a polynomial-time enumerable family because n_k is not bounded by a polynomial in k and hence Theorem 3 is not applicable. To the best of our knowledge, at the time of this writing, no tractability or intractability result is known for the regime $k \in \omega(\log n_k)$ and $k \in o(n_k^{1/c})$ for any constant integer c.

2.2 Intractability of Problem BMO and Problem PBO

Our next goal is to obtain necessary conditions for polynomial-time solvability of higher-degree binary polynomial optimization problems. Henceforth, we say that a countable family of hypergraphs $\{G_k\}_{k=1}^{\infty}$ is *polynomial-time enumerable* if a description of G_k is computable in poly(k) time. A fairly straightforward application of Theorem 3 gives the following intractability result for Problem BMO.

Corollary 2. *Let $\{G_k\}_{k=1}^{\infty}$ be a polynomial-time enumerable family of hypergraphs with $\text{tw}(G_k) = k$. Suppose that for every two nodes u, v contained in an edge of G_k, $\{u, v\}$ also is an edge of G_k for all k. Let f be an algorithm that solves any instance Λ_k of Problem BMO on hypergraph G_k in time at most $T(k)\cdot\text{poly}(\|\Lambda_k\|)$. Then, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k.*

Proof. For every positive k, denote by G'_k the intersection graph of G_k. Note that, by assumption, each edge of G'_k is also an edge of G_k. Therefore, the family of intersection graphs $\{G'_k\}_{k=1}^{\infty}$ is polynomial-time enumerable. The proof then follows from Theorem 3, by setting to zero the costs of all the edges of G_k of cardinality greater than two. □

We observe that the inference problem on higher-order graphical models can be formulated as Problem BMO satisfying the assumption of Corollary 2. Namely, the corresponding hypergraph G_k has a fixed rank and therefore has polynomially many edges in $|V_k|$ (and hence in k). Moreover, for every edge of the hypergraph, all possible subsets of cardinality at least two are also in the edge set (see [30] for a detailed derivation). Nonetheless, Corollary 2 is rather restrictive, as it does not consider general *sparse* hypergraphs. Interestingly, as we show next, for Problem PBO on signed hypergraphs with a bounded rank, no such restrictive assumption is required.

To prove our next intractability result, we make use of the so-called *inflation operation* that was recently introduced in [19] Let $H = (V, S)$ be a signed hypergraph, let $s \in S$, and let $e \subseteq V$ such that $s \subset e$. Denote by $I(s, e)$ the set of all possible signed edges s' parallel to e such that $\eta_s(v) = \eta_{s'}(v)$ for every $v \in s$. Define $S' := S \cup I(s, e) \setminus \{s\}$. We say that $H' = (V, S')$ is obtained from H by *inflating* s to e.

In the following, we say that a function $r(k)$ is log-poly if $2^{r(k)}$ is a polynomial function. Recall that a hypergraph $G' = (V', E')$ is said to be a *partial hypergraph* of hypergraph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We are now ready to state our intractability result for pseudo-Boolean optimization:

Theorem 4. *Let $\{G_k\}_{k=1}^{\infty}$ be a polynomial-time enumerable family of hypergraphs with $\mathrm{tw}(G_k) = k$ for all k, and with rank $r(k) \geq 2$ that is upper bounded by a log-poly function in k. Let f be an algorithm that solves any instance Λ_k of Problem PBO on a signed hypergraph with the underlying hypergraph G_k, in time $T(k) \cdot \mathrm{poly}(\|\Lambda_k\|)$. Then, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k.*

Proof. For every positive k, let $G_k = (V_k, E_k)$, and let $G'_k = (V_k, E'_k)$ be the intersection graph of G_k, which has the treewidth k. Note that $|E'_k| \leq |V_k|^2$, thus $|E'_k|$ is bounded by a polynomial in k. It follows that $\{G'_k\}_{k=1}^{\infty}$ is a polynomial-time enumerable family of graphs with $\mathrm{tw}(G'_k) = k$ for all k. In the remainder of the proof, we show that every instance Λ'_k of problem BQO on graph G'_k can be polynomially reduced to an instance Λ_k of Problem PBO on a signed hypergraph H_k, whose underlying hypergraph is a partial hypergraph of G_k. Since each obtained Problem PBO can be stated as a Problem PBO on a signed hypergraph with underlying hypergraph G_k, by setting the costs of additional $\mathrm{poly}(k)$ signed edges to zero, the result then follows from Theorem 3. In the remainder of the proof, we consider one fixed k.

First, we define the signed hypergraph H_k. By the definition of G'_k, for each edge $e \in E'_k$, there exists an edge $g \in E_k$ such that $e \subseteq g$; we denote by $g(e)$ one such edge of E_k. We then define the signed hypergraph $H_k = (V_k, S_k)$ obtained from G'_k by inflating e to $I(e, g(e))$. Note that, technically, the inflation operation is defined only for signed hypergraphs, while here G'_k is non-signed; however, we can think of G'_k as signed, with all signs equal to $+1$. Since the rank of G_k is upper bounded by $r(k)$, we have $|g(e)| \leq r(k)$, therefore the number of signed edges in S_k parallel to $g(e)$ that we just introduced is at most $2^{r(k)}$. As a result, $|S_k| \leq 2^{r(k)} \cdot |E'_k|$, thus $|S_k|$ is bounded by a polynomial in k. By definition of H_k, we then have $I(e, g(e)) \subseteq S_k$, where we recall that $I(e, g(e))$ denotes the set of all possible signed edges e' parallel to $g(e)$ such that $\eta_e(v) = \eta_{e'}(v)$ for every $v \in e$.

Denote by c'_v for $v \in V_k$, and c'_e for $e \in E'_k$, the objective function coefficients of the instance Λ'_k. Next, we define the cost coefficients of the instance Λ_k, which we denote by c_v for $v \in V_k$, and

c_s for $s \in S_k$. We initialize $c_v := c'_v$ for every $v \in V_k$, and $c_s := 0$ for every $s \in S_k$. Next, for each edge $e \in E'_k$, we update the cost coefficients of Λ_k recursively as follows: For every $s \in I(e, g(e))$, we update $c_s := c_s + c'_e$.

To complete the proof, we observe that the instances Λ'_k and Λ_k are equivalent, in the sense that, for every feasible point $z_v \in \{0, 1\}$, for $v \in V$, its objective value in Λ'_k is equal to its objective value in Λ_k. This claim follows because, for every $e \in E'_k$ the following equation holds:

$$c'_e \prod_{v \in e} z_v = c'_e \prod_{v \in e} z_v \prod_{v \in g(e) \setminus e} (z_v + (1 - z_v)) = c'_e \sum_{s \in I(e, g(e))} \prod_{v \in s} \sigma_s(z_v).$$

□

Note that Theorem 4 also holds if $r(k)$ is a constant greater than or equal to two. We then obtain the following corollary.

Corollary 3. *For every positive k, let $\mathcal{G}_k$ be the set of all hypergraphs with treewidth k and with rank equal to a constant $r \geq 2$. Let f be an algorithm that solves any instance Λ_k of Problem PBO on a signed hypergraph with underlying hypergraph in $\mathcal{G}_k$, in time $T(k) \cdot \text{poly}(\|\Lambda_k\|)$. Then, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k.*

Proof. Consider a polynomial-time enumerable family of hypergraphs $\{G_k\}_{k=1}^{\infty}$ as in the statement of Theorem 4 with $r(k)$ equal to a constant $r \geq 2$. For every positive k, let $\mathcal{G}_k$ be the set of all hypergraphs with rank r and with treewidth k. Then, $G_k \in \mathcal{G}_k$ for every positive k. Let f be an algorithm that solves any instance Λ_k of Problem PBO on a signed hypergraph with underlying hypergraph in $\mathcal{G}_k$, in time $T(k) \cdot \text{poly}(\|\Lambda_k\|)$. Theorem 4 implies that, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k. □

Together with Theorem 1, Theorem 4 suggests that for pseudo-Boolean optimization on signed hypergraphs of log-poly rank, a bounded treewidth is a necessary and sufficient condition for tractability. It is important to note that the log-poly rank assumption is the key here, as otherwise such a conclusion is not valid, see for example Theorem 5. Similar to our discussion following Corollary 1, there is a subtle gap between these necessary and sufficient conditions; that is, to the best of our knowledge, for treewidth k satisfying $k \in \omega(\log n_k)$ and $k \in o(n_k^{1/c})$ for any constant integer c, no tractability or intractability result is known.

2.3 Two open questions

We conclude this section by posing two open questions; below we give two statements which we do not know if they are true or false. We say that a countable family of signed hypergraphs $\{H_k\}_{k=1}^{\infty}$ is *polynomial-time enumerable* if a description of H_k is computable in $\text{poly}(k)$ time. The first statement is again for Problem PBO.

Statement 1. *Let $\{H_k\}_{k=1}^{\infty}$ be a polynomial-time enumerable family of signed hypergraphs with $\text{tw}(H_k) = k$ for all k, and with rank $r(k) \geq 2$ that is upper bounded by a log-poly function in k. Let f be an algorithm that solves any instance Λ_k of Problem PBO on signed hypergraph H_k, in time $T(k) \cdot \text{poly}(\|\Lambda_k\|)$. Then, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k.*

If true, Statement 1 would constitute as a stronger version of Theorem 4. In fact, while the thesis is the same, the assumptions of Statement 1 are much weaker than those in Theorem 4. Namely, for each k, in Theorem 4, we assume that f solves Problem PBO over instances on all signed hypergraphs with underlying hypergraph G_k. On the other hand, in Statement 1, we assume that f solves Problem PBO over instances on just one signed hypergraph H_k. The idea used in the proof of Theorem 4 does not seem to be applicable to Statement 1, since in the reduction we need to introduce a very specific set of signed edges.

Our second statement is similar to Theorem 4, but is concerned with Problem BMO. It is important to note that the proof technique employed in Theorem 4 is not applicable to prove hardness for Problem PBO, as it uses the inflation operation, which involves the introduction of signed edges.

Statement 2. *Let $\{G_k\}_{k=1}^{\infty}$ be a polynomial-time enumerable family of hypergraphs with $\mathrm{tw}(G_k) = k$ for all k, and with rank $r(k) \geq 2$ that is upper bounded by a log-poly function in k. Let f be an algorithm that solves any instance Λ_k of Problem BMO on hypergraph G_k, in time $T(k) \cdot \mathrm{poly}(\|\Lambda_k\|)$. Then, assuming $\mathcal{NP} \not\subseteq \mathcal{BPP}$, $T(k)$ grows super-polynomially in k.*

We remark that, to the best of our knowledge, even if instead of a log-poly function in k, we assume that the rank r is a constant, Statements 1 and 2 remain open.

3 Tractability beyond hypergraph acyclicity for pseudo-Boolean optimization

In this section, we move beyond acyclic hypergraphs and identify more general classes of binary polynomial optimization problems that are solvable in polynomial time. In a recent work [19], the authors introduce the pseudo-Boolean polytope, a generalization of the multilinear polytope that enables them to unify and extend all prior results on the existence of polynomial-size extended formulations for the convex hull of the feasible region of binary polynomial optimization problems of degree at least three. In particular, given a signed hypergraph $H = (V, S)$, the authors prove that if the underlying hypergraph of H is β-acyclic, then $\mathrm{PBP}(H)$ has an extended formulation of size polynomial in $|V|, |S|$. Recall that a *β-cycle* of length ℓ for some $\ell \geq 3$ in a hypergraph G is a sequence $v_1, e_1, v_2, e_2, \ldots, v_\ell, e_\ell, v_1$ such that $v_1, v_2, \ldots, v_\ell$ are distinct nodes, $e_1, e_2, \ldots, e_\ell$ are distinct edges, and v_i belongs to e_{i-1}, e_i and no other e_j for all $i = 1, \ldots, \ell$, where $e_0 = e_\ell$. A hypergraph is *β-acyclic* if it does not contain any β-cycles.

Theorem 5. *[19] Let $H = (V, S)$ be a signed hypergraph of rank r whose underlying hypergraph is β-acyclic. Then the pseudo-Boolean polytope $PBP(H)$ has a polynomial-size extended formulation with at most $O(r|S||V|)$ variables and inequalities. Moreover, all coefficients and right-hand side constants in the extended formulation are $0, \pm 1$.*

Theorem 5 is a generalization of the earlier result in [18] stating that the multilinear polytope of a β-acyclic hypergraph has a polynomial-size extended formulation. Namely, Theorem 5 only requires the β-acyclicity of the underlying hypergraph of H. Indeed, the multilinear hypergraph of H, i.e., $\mathrm{mh}(H)$ may contain many β-cycles. In this section, we present a significant generalization of Theorem 5 that gives polynomial-size extended formulations for the pseudo-Boolean polytope of a class of signed hypergraphs whose underlying hypergraph contains β-cycles. To define this class of hypergraphs, we make use of the notion of "gap" introduced in [19]. In this paper, we use this concept in a slightly more general form, which we define next. Consider a hypergraph $G = (V, E)$, and let $V' \subseteq V$; we define the *gap* of G induced by V' as

$$\mathrm{gap}(G, V') := \max \Big\{ |V'| - |e \cap V'| : e \in E, \; e \cap V' \neq \emptyset \Big\}. \qquad (2)$$

Moreover, for a signed hypergraph $H = (V, S)$, and $V' \subseteq V$, the gap of H induced by V', denoted by $\mathrm{gap}(H, V')$ is defined as the gap of the underlying hypergraph of H induced by V'. In [19] the authors prove that thanks to the inflation operation, if $\text{gap}(H, V) = O(\log \text{poly}(|V|, |S|))$, the pseudo-Boolean polytope $\text{PBP}(H)$ has a polynomial-size extended formulation.

3.1 Nest-set, nest-set width, and nest-set gap

In the following, we introduce some new hypergraph theoretic notions that enable us to obtain new polynomial-size extended formulations for the pseudo-Boolean polytope. The proof of Theorem 5 relies on the key concept of nest points. Let $G = (V, E)$ be a hypergraph. A node $v \in V$ is a *nest point* of G if the set of the edges of G containing v is totally ordered with respect to inclusion. It is simple to see that nest points can be found in polynomial time. Let $N \subset V$. We define the hypergraph obtained from G by *deleting* N as the hypergraph $G - N$ with set of nodes $V \setminus N$ and set of edges $\{e \setminus N : e \in E,\ |e \setminus N| \geq 2\}$. When $N = \{v\}$, we write $G - v$ instead of $G - N$. A *nest point elimination order* is an ordering $v_1, \ldots, v_n$ of the nodes of G, such that v_1 is a nest point of G, v_2 is a nest point of $G - v_1$, and so on, until v_n is a nest point of $G - v_1 - \cdots - v_{n-1}$. The following result provides a characterization of β-acyclic hypergraphs, in terms of nest points:

Theorem 6 ([22])**.** *A hypergraph G is β-acyclic if and only if it has a nest point elimination order.*

Recently, in the context of satisfiability problems, Lanzinger [32] generalizes the concept of nest points to *nest-sets*, which we define next. Let $G = (V, E)$ be a hypergraph and let $N \subseteq V$. We say that N is a *nest-set* of G if the set

$$\{e \setminus N : e \in E,\ e \cap N \neq \emptyset\}, \tag{3}$$

is totally ordered with respect to inclusion. Clearly if $|N| = 1$, then N consists of a nest point of G. Let $N_1, N_2, \ldots, N_t$ be pairwise disjoint subsets of V such that $\cup_{i \in [t]} N_i = V$. We say that $\mathcal{N} = N_1, \cdots, N_t$ is a *nest-set elimination order* of G, if N_1 is a nest-set of G, N_2 is a nest-set of $G - N_1$, and so on, until N_t is nest-set of $G - N_1 - \cdots - N_{t-1}$.

Given a nest-set elimination order $\mathcal{N}$ of G, we define *nest-set width* of $\mathcal{N}$, denoted by $\text{nsw}_{\mathcal{N}}(G)$, as the maximum cardinality of any element in $\mathcal{N}$. We then define *nest-set gap* of $\mathcal{N}$ as

$$\text{nsg}_{\mathcal{N}}(G) := \max\Big\{\text{gap}(G - N_1 - \cdots - N_{i-1}, N_i) : i \in [t]\Big\},$$

where $\text{gap}(\cdot)$ is defined by (2). From these definitions, it follows that

$$\text{nsg}_{\mathcal{N}}(G) \leq \text{nsw}_{\mathcal{N}}(G) - 1. \tag{4}$$

In fact, $\text{nsg}_{\mathcal{N}}(G)$ can be much smaller than $\text{nsw}_{\mathcal{N}}(G)$. The following example demonstrates this fact.

Example 1. *Consider a hypergraph $G = (V, E)$ whose edge set E consists of all subsets of V of cardinality $|V| - 1$. Letting $\mathcal{N} = V \setminus \{\bar{v}\}, \{\bar{v}\}$ for some $\bar{v} \in V$, it follows that* $\text{nsw}_{\mathcal{N}}(G) = |V| - 1$, *while* $\text{nsg}_{\mathcal{N}}(G) = 1$.

We define the nest-set width of G, denoted by $\text{nsw}(G)$, as the minimum value of $\text{nsw}_{\mathcal{N}}(G)$ over all nest-set elimination orders $\mathcal{N}$ of G. Similarly, we define the nest-set gap of G, denoted by $\text{nsg}(G)$, as the minimum value of $\text{nsg}_{\mathcal{N}}(G)$ over all nest-set elimination orders $\mathcal{N}$ of G. Consider the hypergraph of Example 1. It can be checked that $\text{nsw}(G) = |V| - 1$, while $\text{nsg}(G) = 1$. We define the nest-set gap of a signed hypergraph as the nest-set gap of its underlying hypergraph. Similarly, we define the notions of nest point, nest-set, and nest-set width, also for signed hypergraphs. See Figures 2 to 4 for an illustration of nest-sets, nest-set widths, and nest-set gaps.

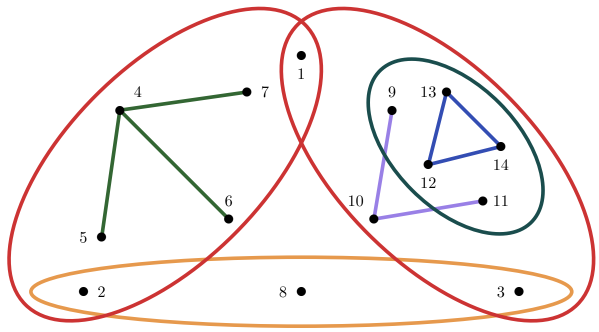

Figure 2: A hypergraph G with $\text{nsw}(G) = 2$ and $\text{nsg}(G) = 1$. A nest-set elimination order of G is given by $\mathcal{N} = \{8\}, \{7\}, \{6\}, \{5\}, \{4\}, \{12, 13\}, \{14\}, \{9, 10\}, \{11\}, \{2, 3\}, \{1\}$. G contains β-cycles of length three.

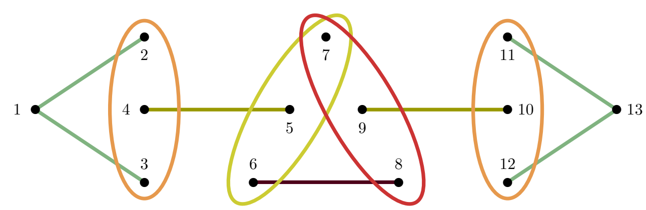

Figure 3: A hypergraph G with $\text{nsw}(G) = 2$ and $\text{nsg}(G) = 1$. A nest-set elimination order of G is given by $\mathcal{N} = \{1, 2\}, \{3\}, \{4\}, \{5\}, \{6, 7\}, \{8\}, \{9\}, \{10\}, \{11, 12\}, \{13\}$. G contains β-cycles of length three.

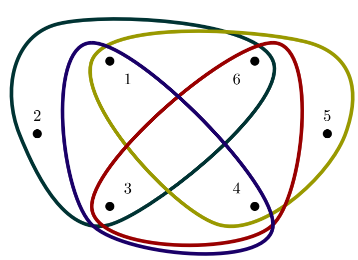

Figure 4: A hypergraph G with $\text{nsw}(G) = 3$ and $\text{nsg}(G) = 1$. A nest-set elimination order of G is given by $\mathcal{N} = \{2\}, \{5\}, \{1, 3, 4\}, \{6\}$. G contains β-cycles of length three.

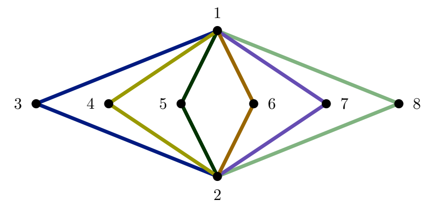

Figure 5: An illustration of the graph of Example 2 with $n = 8$. This graph contains cycles of length four only, however, its nest-set width is seven.

From Theorem 6 it follows that $\text{nsw}(G) = 1$, if and only if G is β-acyclic; that is, hypergraphs with $\text{nsw}(G) \geq 2$ contain β-cycles. Hence, a natural question is whether $\text{nsw}(G)$ is in general related to the length of β-cycles of G. In lemma 6 in [32], the author proves that if G has a β-cycle of length ℓ, then $\text{nsw}(G) \geq \ell - 1$. Next, we strengthen this lower bound on the nest-set width:

Proposition 1. *Let $C = v_1, e_1, v_2, e_2, \ldots, v_\ell, e_\ell, v_1$ be a β-cycle of length ℓ in a hypergraph G. Let $U_1 := (e_\ell \cap e_1) \setminus \cup_{j=2}^{\ell-1} e_j$. Similarly, for $i = 2, \ldots, \ell$, let $U_i := (e_{i-1} \cap e_i) \setminus \cup_{j \in [\ell] \setminus \{i-1,i\}} e_j$. For every nest-set s of G we have that $s \cap (\cup_{j=1}^{\ell} U_j)$ is either the empty set, or contains at least $\ell - 1$ sets among U_j, for $j \in [\ell]$. Furthermore, $\text{nsw}(G) \geq \min_{k \in [j]} \sum_{j \in [\ell] \setminus k} |U_j|$.*

Proof. Let $U := \cup_{j=1}^{\ell} U_j$. Suppose that $s \cap U$ is nonempty. That is, at least one node in U is in s. Since we can rotate the indices of a cycle arbitrarily, we assume, without loss of generality that this is a node $u_1 \in U_1$. Then e_1 and e_ℓ are both in $I(s)$, where we denote by $I(s)$ the set of edges of G that contain at least one node in s. By the definition of U_2, we have $U_2 \subseteq e_1 \setminus e_\ell$. Similarly, $U_\ell \subseteq e_\ell \setminus e_1$. Thus, e_1 and e_ℓ can only be comparable to $\subseteq_s$ if U_2 or U_ℓ is contained in s. Suppose, without loss of generality, $U_2 \subseteq s$.

Then we have e_2 and e_ℓ in $I(s)$ and the same argument can be applied again, as long as the two edges are not adjacent in the cycle, and we find that U_3 or U_ℓ is contained in s, without loss of generality, $U_3 \subseteq s$. We then obtain $U_2, U_3, \ldots, U_{\ell-1} \subseteq s$.

Next, we consider the edges e_1 and $e_{\ell-1}$, which are both in $I(s)$. We have $U_1 \subseteq e_1 \setminus e_{\ell-1}$ and $U_\ell \subseteq e_{\ell-1} \setminus e_1$. Thus, e_1 and $e_{\ell-1}$ can only be comparable to $\subseteq_s$ if U_1 or U_ℓ is contained in s. In either case, s contains at least $\ell - 1$ sets among U_j, for $j \in [\ell]$.

Since any node in the β-cycle C has to be removed at some point in any nest-set elimination order, we obtain $\text{nsw}(G) \geq \min_{k \in [j]} \sum_{j \in [\ell] \setminus k} |U_j|$. □

Notice that if the edge set of hypergraph G of Proposition 1 is $E = \{e_1, \cdots, e_\ell\}$, then the bound given in this lemma is sharp; i.e., $\text{nsw}(G) = \min_{k \in [j]} \sum_{j \in [\ell] \setminus k} |U_j|$. The next example demonstrates that the lower bound given in Proposition 1 is not sharp, in general.

Example 2. *Consider a graph $G = (V, E)$ with $V = \{v_1, \cdots, v_n\}$ for some $n \geq 5$ and $E = \{\{v_1, v_i\}, \{v_2, v_i\}, \forall i \in \{3, \cdots, n\}\}$. See Figure 5 for an illustration. It can be checked that G contains cycles of length four only hence the lower bound on $\text{nsw}(G)$ given by Proposition 1 is three. However, $\text{nsw}(G) = n - 1 > 3$ since by assumption $n \geq 5$. To see this, first note that either v_1 or v_2 should be in a nest set, as otherwise set (3) will contain $\{v_1\}$ and $\{v_2\}$ and hence will not be totally ordered. Without loss of generality, suppose that v_1 is in the nest set. Second, note that v_i, for all $i \in \{3, \cdots, n\}$ should also be in the nest set as otherwise set (3) will contain $\{v_2\}$ and $\{v_j\}$ for some $j \in \{3, \cdots, n\}$ and hence will not be totally ordered. Therefore, $\text{nsw}(G) = n - 1$.*

A future direction of research is to investigate whether it is possible to give a characterization of nest-set width in terms of the "size" of the β-cycles in the hypergraph.

As we detail in the next section, if the nest-set gap of the underlying hypergraph is bounded, then the pseudo-Boolean polytope has a polynomial-size extended formulation. Hence, we would like to understand the complexity of checking whether the nest-set gap of a hypergraph is bounded. Notice that by (4), if the nest-set width is bounded, so is the nest-set gap. The complexity of checking the nest-set width of a hypergraph was settled in [32]. Lanzinger proves that deciding whether $\text{nsw}(G) \leq k$ for any integer k is NP-complete (see Theorem 10 in [32]). However, when parameterized by k, this problem is fixed-parameter tractable:

Proposition 2 (Theorem 15 in [32])**.** *There exists a $2^{O(k^2)}\,\text{poly}(|V|, |E|)$ time algorithm that takes as input hypergraph $G = (V, E)$ and integer $k \geq 1$ and returns a nest-set elimination order $\mathcal{N}$ with $\text{nsw}_{\mathcal{N}}(G) = k$ if one exists, or rejects otherwise.*

As Example 1 indicates, there exist hypergraphs $G = (V, E)$ with large nest-set width (e.g. $\text{nsw}(G) = \theta(|V|)$) but with fixed nest-set gap. At the time of this writing, the complexity of checking whether the nest-set gap of a hypergraph is bounded is an open question.

3.2 The pseudo-Boolean polytope of signed hypergraphs with a bounded nest-set gap

In this section, we present a polynomial-size extended formulation for the pseudo-Boolean polytope of signed hypergraphs with a bounded nest-set gap. The result we present in this section serves as a significant generalization of Theorem 5, which was proved in [19]. To obtain this extended formulation, we make use of three tools developed in [19]. The first result is concerned with *decomposability* of the pseudo-Boolean polytope $\text{PBP}(H)$. Consider a signed hypergraph $H = (V, S)$, let $s = (e, \eta_s) \in S$, and let $U \subseteq V$. In the following, when we write $s \in U$, we mean $e \in U$. If $v \in s$, we denote by $s - v$ the signed edge $s' = (e', \eta_{s'})$, where $e' := e \setminus \{v\}$, and $\eta_{s'}$ is the restriction of η_s that assigns to each $v \in e'$ the sign $\eta_{s'}(v) = \eta_s(v)$.

Let $H = (V, S)$ be a signed hypergraph and let $V_1, V_2 \subseteq V$ such that $V = V_1 \cup V_2$, let $S_1 \subseteq \{s \in S : s \subseteq V_1\}$, $S_2 \subseteq \{s \in S : s \subseteq V_2\}$ such that $S = S_1 \cup S_2$. Let $H_1 := (V_1, S_1)$ and $H_2 := (V_2, S_2)$. We say that $\text{PBP}(H)$ is decomposable into $\text{PBP}(H_1)$ and $\text{PBP}(H_2)$, if the system comprised of a description of $\text{PBP}(H_1)$ and a description of $\text{PBP}(H_2)$, is a description of $\text{PBP}(H)$. The next theorem provides a sufficient condition for decomposability of the pseudo-Boolean polytope.

Theorem 7 (Theorem 1 in [19])**.** *Let $H = (V, S)$ be a signed hypergraph, and assume that it has a nest point v. Let $s_1 \subseteq s_2 \subseteq \cdots \subseteq s_k$ be the signed edges of H containing v, and assume that S contains the signed edges $s_i - v$ such that $|s_i - v| \geq 2$, for every $i \in [k]$. Then $PBP(H)$ is decomposable into $PBP(H_1)$ and $PBP(H_2)$, where H_1 and H_2 are defined as follows. $H_1 := (V_1, S_v \cup P_v)$, where V_1 is the underlying edge of s_k, $S_v := \{s_1, \ldots, s_k\}$, $P_v := \{s_i - v : |s_i - v| \geq 2,\ i \in [k]\}$, and $H_2 := H - v$.*

The next theorem provides a polynomial-size extended formulation for the pseudo-Boolean polytope of a special type of signed hypergraphs, which we refer to as "pointed." Consider a signed hypergraph $H = (V, S)$ and let $v \in V$ be a nest point of H. Denote by S_v the set of all signed edges in S containing v. Define $P_v := \{s - v : s \in S_v, |s| \geq 3\}$. We say that H is *pointed* at v (or is a *pointed signed hypergraph*) if V coincides with the underlying edge of the signed edge of maximum cardinality in S_v and $S = S_v \cup P_v$.

Theorem 8 (Theorem 2 in [19])**.** *Let $H = (V, S)$ be a pointed signed hypergraph. Then the pseudo-Boolean polytope $PBP(H)$ has a polynomial-size extended formulation with at most $2|V|(|S| + 1)$ variables and at most $4(|S|(|V| - 2) + |V|)$ inequalities. Moreover, all coefficients and right-hand side constants in the system defining $PBP(H)$ are $0, \pm 1$.*

The final result is concerned with the inflation operation. Let $H = (V, S)$ be a signed hypergraph, let $s \in S$, and let $e \subseteq V$ such that $s \subset e$. Let $H' = (V, S')$ be obtained from H by inflating s to e. The following theorem indicates that if an extended formulation for PBP(H') is available, one can obtain an extended formulation for PBP(H) as well.

Theorem 9 (Theorem 3 in [19]). *Let $H = (V, S)$ be a signed hypergraph, let $s \in S$, and let $e \subseteq V$ such that $s \subset e$. Let $H' = (V, S')$ be obtained from H by inflating s to e. Then an extended formulation of PBP(H) can be obtained by juxtaposing an extended formulation of PBP(H') and the equality constraint*

$$z_s = \sum_{s' \in I(s,e)} z_{s'}. \tag{5}$$

Moreover, if PBP(H') has a polynomial-size extended formulation and $|e| - |s| = O(\log \text{poly}(|V|, |S|))$, then PBP$(H)$ has a polynomial-size extended formulation as well.

We are now ready to state the main result of this section.

Theorem 10. *Let $H = (V, S)$ be a signed hypergraph of rank r whose underlying hypergraph $G = (V, E)$ satisfies $\text{nsg}(G) \leq k$. Then the pseudo-Boolean polytope PBP(H) has an extended formulation with $O(r2^k|V||S|)$ variables and inequalities. In particular, if $k \in O(\log \text{poly}(|V|, |E|))$, then PBP$(H)$ has a polynomial-size extended formulation. Moreover, all coefficients and right-hand side constants in the system defining PBP(H) are $0, \pm 1$.*

Proof. Denote by $\mathcal{N} = N_1, \cdots, N_t$ a nest-set elimination order of G with $\text{nsg}_{\mathcal{N}}(G) \leq k$. Consider the nest-set N_1.

Assume $N_1 = \{v\}$ for some $v \in V$. Then, v is a nest point of G. Denote by S_v the set of all signed edges of H containing v. By definition, the set S_v is totally ordered. Define the signed hypergraph $H'_1 := (V, S_v \cup P_v)$, where $P_v := \{s - v : s \in S_v, |s| \geq 3\}$. Clearly, an extended formulation for PBP(H'_1) serves as an extended formulation for PBP(H) as well. Now define the pointed signed hypergraph $H_v := (V_1, S_v \cup P_v)$, where V_1 denotes the underlying edge of a signed edge of maximum cardinality in S_v. We then have $H'_1 = H_v \cup (H - v)$, where we used the fact that $H'_1 - v = H - v$. Hence by Theorem 7, the pseudo-Boolean polytope PBP(H'_1) is decomposable into pseudo-Boolean polytopes PBP(H_v) and PBP$(H - v)$, and a polynomial-size extended formulation of PBP(H_v) exists by Theorem 8.

Now assume $|N_1| \geq 2$; let $S_1 \subseteq S$ denote the set of signed edges containing some $v \in N_1$. Consider $s \in S_1$, denote by e the underlying edge of s, and define $\bar{e} := e \cup N_1$. Inflate s to $\bar{e}$; repeat a similar inflation operation for every $s \in S_1$ to obtain a new signed hypergraph $\bar{H} = (V, \bar{S})$. It then follows that $\bar{H}$ satisfies two key properties:

(i) $|\bar{S}| \leq 2^k|S|$, since by assumption $\text{nsg}_{\mathcal{N}}(G) \leq k$, implying that $\text{gap}(G, N_1) \leq k$,

(ii) all nodes in N_1 are nest points of the underlying hypergraph of $\bar{H}$. To see this, denote by $\bar{G}$ the underlying hypergraph of $\bar{H}$ and denote by $\bar{F}$ the set of all edges in $\bar{G}$ containing some node in N_1. By definition of a nest-set, $\{f \setminus N_1 : f \in \bar{F}\}$ is totally ordered. Moreover, by the inflation operation defined above, we have $f \cap N_1 = N_1$ for all $f \in \bar{F}$. It then follows that $\bar{F}$ is totally ordered as well; hence, all nodes in N_1 are nest points of $\bar{G}$.

By Theorem 9 and property (i) above, by obtaining a polynomial-size extended formulation for PBP$(\bar{H})$, we obtain a polynomial-size extended formulation for PBP(H) as well. Now consider a node $v_1 \in N_1$. Since v_1 is a nest point of $\bar{G}$, we can use the technique described above to decompose PBP$(\bar{H})$ into PBP$(\bar{H}_{v_1})$ and PBP$(\bar{H} - v_1)$, where as before $\bar{H}_{v_1}$ is a signed hypergraph pointed at

v_1. Moreover, denoting by $\bar{S}_{v_1}$ the number of edges in $\bar{H}_{v_1}$ containing v_1 we have $\bar{S}_{v_1} \leq 2^k|S|$. Next consider the signed hypergraph $\bar{H} - v_1$; it follows that any node $v \in N_1 \setminus \{v_1\}$ is a nest point of the underlying hypergraph of $\bar{H} - v_1$. Hence we can apply our decomposition technique recursively to all nest points in N_1 to decompose PBP($\bar{H}$) into PBP($\bar{H} - N_1$) and PBP($\bar{H}_v$) for all $v \in N_1$, where PBP($\bar{H}_v$) is a pointed signed hypergraph at v with at most $2^{k+1}|S|$ edges for all $v \in N_1$. Since the nodes in N_1 are contained in the same signed edges of $\bar{H}$ obtained by inflating any edge in H containing some node in N_1 to their union, it follows that the number of signed edges of $\bar{H} - N_1$ is upper bounded by $|S|$. Next, we apply the above inflation and decomposition technique to the signed hypergraph $\bar{H} - N_1$ together with the nest-set N_2. Repeating this argument t times for all $N_i \in \mathcal{N}$, we conclude that an extended formulation for PBP(H) is obtained by juxtaposing extended formulations of pointed signed hypergraphs PBP($\bar{H}_{v_i}$) for all $v_i \in V$, together with t equalities of the form (5).

Now consider the pointed signed hypergraph $\bar{H}_{v_i}$ for some $v_i \in V$. Denote by V_i the node set of $\bar{H}_{v_i}$, denote by $\bar{S}_{v_i}$ the set of edges containing v_i and let $\bar{P}_{v_i}$ denote the remaining edges of $\bar{H}_{v_i}$. By Theorem 8 the polytope PBP($\bar{H}_{v_i}$) has an extended formulation consisting of at most $2|V_i|(|S_{v_i}|+|P_{v_i}|+1) \leq 2r(2^k|S|+2^k|S|+1)$ variables and at most $4(|S_{v_i}|+|P_{v_i}|)(|V_i|-2)+4|V_i| \leq (r-2)2^{k+3}|S| + 4|V|$ inequalities, where the inequalities follow since $|V_i| \leq r$ and $|\bar{P}_{v_i}| \leq |\bar{S}_{v_i}| \leq 2^k|S|$. Therefore, PBP($H$) has an extended formulation with $O(r2^k|V||S|)$ variables and inequalities. Moreover, by Theorem 8 and Theorem 9 all coefficients and right-hand side constants in the system defining this extended formulation are $0, \pm1$. □

Notice that Theorem 5 is a special case of Theorem 10 obtained by letting $k = 1$. Using inequality (4), we obtain the following result regarding the pseudo-Boolean polytope of signed hypergraphs with bounded nest-set width:

Corollary 4. *Let $H = (V, S)$ be a signed hypergraph of rank r whose underlying hypergraph $G = (V, E)$ satisfies $\mathrm{nsw}(G) \leq k$. Then the pseudo-Boolean polytope PBP(H) has an extended formulation with $O(r2^k|V||S|)$ variables and inequalities. In particular, if $k \in O(\log \mathrm{poly}(|V|, |E|))$, then PBP($H$) has a polynomial-size extended formulation. Moreover, all coefficients and right-hand side constants in the system defining PBP(H) are $0, \pm1$.*

Recall that by Proposition 2, if $k \in O(\sqrt{\log \mathrm{poly}(|V|, |E|)})$, then deciding whether $\mathrm{nsw}(G) \leq k$ can be performed in polynomial-time. Together with Corollary 4 this settles the complexity of recognition and solution of Problem PBO over signed hypergraphs with $\mathrm{nsw}(G) \in O(\sqrt{\log \mathrm{poly}(|V|, |E|)})$. However, for the regime $k \in \omega(\sqrt{\log \mathrm{poly}(|V|, |E|)})$ and $k \in O(\log \mathrm{poly}(|V|, |E|))$, while by Corollary 4 we can solve Problem PBO in polynomial-time, the complexity of checking whether $\mathrm{nsw}(G) \leq k$ remains an open question.

Finally, let us examine the connections between Theorem 1, Theorem 4, and Corollary 4. From Lemma 9 of [32] it follows that for a hypergraph G of rank r we have

$$\mathrm{tw}(G) \leq \mathrm{nsw}(G)(r - 1).$$

Hence, if the rank r is fixed, then Corollary 4 is implied by Theorem 1 as in this case, bounded nest-set width implies bounded treewidth. However, if r is not fixed, then one can consider hypergraphs with bounded nest-set width and unbounded treewidth in which case Theorem 1 is not applicable, while Corollary 4 gives polynomial-size extended formulations for the pseudo-Boolean polytope.

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