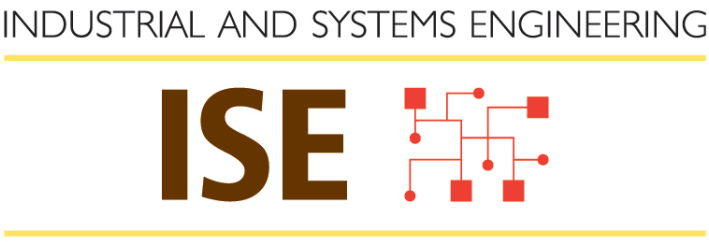


# Quantum Interior Point Methods for Conic Linear Optimization

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### Article Outline



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### Introduction

This article studies quantum interior point methods (QIPMs) for solving conic linear optimization (CLO) problems [3]. Let $\mathcal{J} \subset \mathbb{R}^n$ be an $n$-dimensional vector space over the real numbers and $\mathcal{K} \subset \mathcal{J}$ be a closed, pointed convex cone in $\mathbb{R}^n$ with a non-empty interior. The dual cone of $\mathcal{K}$ is denoted by $\mathcal{K}^*$, which is given as

$$\mathcal{K}^* \equiv \left\{ s \in \mathbb{R}^n : \langle s, x \rangle \geq_{\mathcal{K}} 0 \text{ for every } x \in \mathcal{K} \right\},$$

where $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{J}$ and the notation $x \geq_{\mathcal{K}} \bar{x}$ means $x - \bar{x} \in \mathcal{K}$. Letting $\text{int}(\mathcal{K})$ denote the interior of $\mathcal{K}$, $x >_{\mathcal{K}} \bar{x}$ means $x - \bar{x} \in \text{int}(\mathcal{K})$.

Letting $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$ be given, the primal and dual pair of CLO problems can be defined in standard form as follows:

$$z_P = \min_x \{ \langle c, x \rangle : Ax = b, x \geq_{\mathcal{K}} 0 \}, \tag{1}$$

$$z_D = \max_{(y,s)} \left\{ \langle b, y \rangle : A^\top y + s = c, s \geq_{\mathcal{K}^*} 0 \right\}. \tag{2}$$

By appropriately choosing cone $\mathcal{K}$, any convex optimization problem can be formulated in the form given in (1)–(2) [3]. Further, whenever there exists a primal feasible $x \in \text{int}(\mathcal{K})$ and a dual


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feasible point $(y, s)$ with $s \in \text{int}(\mathcal{K}^*)$, then the so-called interior point condition (IPC) is said to hold. Under the IPC, *strong duality* holds and an optimal solution $(x, y, s)$ with zero duality gap exists, i.e., the primal and dual optimal values coincide, i.e., $z_P = z_D$.

A cone $\mathcal{K}$ is symmetric if it is self-dual, i.e., $\mathcal{K} = \mathcal{K}^*$, and homogeneous, i.e., for any $(x, \bar{x}) \in \text{int}(\mathcal{K})$, there exists a linear transformation $M$ such that $M(x) = \bar{x}$ and $M(\mathcal{K}) = \mathcal{K}$. Symmetric cones are well-studied in optimization and include the nonnegative orthant $\mathbb{R}^n_+$, the *Lorentz* or second-order cone $\mathcal{L}^n$, as well as the cone of $n \times n$ symmetric positive semidefinite matrices $\mathcal{S}^n_+$. Accordingly, the primal and dual symmetric conic optimization (SCO) problems (1)–(2) include linear optimization (LO) problems upon choosing $\mathcal{K} = \mathbb{R}^n_+$; Second-order conic optimization (SOCO) problems when $\mathcal{K} = \otimes^r_{i=1}\mathcal{L}^{n_i}$, and when $\mathcal{K} = \mathcal{S}^n_+$, the resulting problem is a semidefinite optimization (SDO) problem. Accordingly, SCO problems allow one to model a wide variety of fundamental optimization problems with applications in finance [20] and control [6], as well as computing approximations to NP-hard problems [13, 19], just to give a few examples.

In the classical setting, due to their strong theoretical guarantees and practical efficiency, interior point methods (IPMs) are the prevailing methodology for solving optimization problems over symmetric cones [34]. The development of Karmarkar's IPM for LO was closely followed by Nesterov and Nemirovskii [28], who introduced the paradigm of self-concordant barrier functions. This seminal work inspired a great deal of interest in IPMs, which were subsequently applied to broad classes of optimization problems, such as SDO [10, 22, 25–27, 38] and SOCO [35, 36].

Nesterov and Todd were the first to study primal-dual IPMs for SCO [29, 30]. Implicitly, the work of Faybusovich [12] and Tsuchiya [35, 36] was instrumental in the development and analysis of IPMs applied to SCO. They showed that the complexity analysis of Monteiro and Zhang for SDO [27] can be extended to analyze IPMs for SOCO using Jordan algebras. Shortly thereafter, Schmieta and Alizadeh [32, 33] further generalized Monteiro's analysis to SCO. Specifically, upon demonstrating the symmetrization that takes place in the context of SDO to a similar scaling on associatively induced symmetric cones, Monteiro's proof extends to all symmetric cones. Affine scaling algorithms for SCO have also been proposed; see [21].

The bottleneck operation at every iteration of IPMs is the solution of the linear approximation of a perturbed set of Karush-Kuhn-Tucker (KKT) optimality conditions known as the *Newton linear system*. Given that quantum linear systems algorithms (QLSAs) [8, 9, 14] can prepare a quantum state encoding the solution to a linear equation system at cost that is polylogarithmic in the dimension, researchers have investigated whether substituting the classical solution of the Newton system with a QLSA would provide a faster interior point algorithm for solving convex optimization problems. Kerenidis and Prakash [16] were the first to propose a QIPM for solving SDO and LO problems in this manner. However, the errors resulting from the use of a QLSA to solve the Newton system and quantum state tomography to classically estimate the resulting state encoding the solution are not properly handled by their framework, and thus convergence is not guaranteed.

Augustino et al. [4] overcame the issues arising from quantum errors by presenting two schemes. The first directly quantizes a classical inexact-infeasible IPM [39], in which residuals are added to the right-hand side of the Newton linear system to capture infeasibility and inexactness of the search directions. However, the authors determined that the resulting II-QIPM does not provide an advantage in any parameter with respect to overall complexity when compared to the best classical feasible IPMs. The second scheme they present is a novel framework which they call an inexact-feasible QIPM (IF-QIPM). In the IF-QIPM, the primal and dual search directions are defined as linear combinations of the nullspace and row space of the constraint matrices, resulting in what is referred to as an orthogonal subspace system formulation of the Newton system. As a consequence, primal and dual feasibility are guaranteed to be satisfied *exactly*, in spite

of the errors from QLSA and quantum state tomography. These ideas were also applied to LO by Mohammadisiahroudi et al. [23]. Since the analysis presented in [4] extends Monteiro's proof in [25] to account for inexactness in the complementarity condition, convergence of an IF-QIPM for SCO problems can be obtained by generalizing their proof to symmetric cones in the same way as Schmieta and Alizadeh [32] proceeded with [25].

The rest of this chapter is structured as follows: section "Preliminaries" reviews the theory of Euclidean Jordan algebras, which serves as the foundation for analysis of IPMs applied to symmetric cones. Section "Classical IPMs for Optimization over Symmetric Cones" discusses the theory of the central path and classical IPMs, as well as how to formulate the Newton linearsystem for use in a QIPM. Section "Technical Results" details technical results which are necessary to prove convergence of an inexact-feasible (Q)IPM for SCO, and Section "A QIPM for Optimization over Symmetric Cones" presents the IF-QIPM for symmetric cones.

## Preliminaries

In this section, several foundational components of the analysis are summarized. The theory of Euclidean Jordan algebras is reviewed at a high level, providing only the necessary concepts for this paper. For a detailed discussion, the reader is referred to [11, 32].

### "Big O" Notation

In what follows, $\mathcal{O}(\cdot)$ is defined as

$$f(x) = \mathcal{O}(g(x)) \iff \exists \ell \in \mathbb{R}, \vartheta \in \mathbb{R}_+, \text{ such that } f(x) \leq \vartheta g(x) \quad \forall x > \ell.$$

When the function depends polylogarithmically on other quantities, e.g., $\kappa$ and $\rho$, the following notation is used:

$$\widetilde{\mathcal{O}}_{\kappa,\rho}(f(x)) = \mathcal{O}(f(x) \cdot \text{polylog}(\kappa, \rho, f(x))).$$

### Basic Definitions

Suppose that $\mathcal{A}$ is an $n$-dimensional vector space endowed with a bilinear mapping $(x, s) \mapsto xs$. Whenever $x(uv) = (xu)v$ for any $x, u, v \in \mathcal{A}$ (i.e., this multiplication is associative for all $x$, $u$, and $v$), then $\mathcal{A}$ is an *associative algebra*. An element $e \in \mathcal{A}$ is the *identity element* of $\mathcal{A}$ if $xe = ex = x$ for every $x$. An element $x'$ is the *adjoint* of $x$ if there is a (linear) one-to-one *conjugation* "$'$" on $\mathcal{A}$ such that $(x')' = x$ and $(xs)' = s'x'$. Hence, $x$ is *self-adjoint* whenever $x = x'$, and the set of self-adjoint elements of $\mathcal{A}$ form a subspace of $\mathcal{A}$.

### Euclidean Jordan Algebras

Suppose that $\mathcal{J}$ is an $n$-dimensional vector space endowed with a bilinear mapping $\circ : (x, s) \mapsto x \circ s$. It is said that $(\mathcal{J}, \circ)$ is a *Jordan algebra* if

$$x \circ s = s \circ x, \tag{3a}$$

$$x \circ (x^2 \circ s) = x^2 \circ (x \circ s), \tag{3b}$$

with $x^2 = x \circ x$. A Jordan algebra $\mathcal{J}$ is *Euclidean* whenever $\langle x, x \rangle > 0$ for all $x \neq 0$, where $\langle \cdot, \cdot \rangle$ is an inner product defined over $\mathcal{J}$.

Observe that while Jordan algebras are commutative, they are not associative in general. Yet, Euclidean Jordan algebras are power associative; any $x \in \mathcal{J}$ satisfies $x^{p+q} := x^p \circ x^q$. These algebras lend naturally to many properties of symmetric matrices, such as the concept of positive semidefiniteness and possessing real eigenvalues and orthogonal eigenvectors. The *degree* of $x \in \mathcal{J}$ to be the smallest integer $r$ such that $\{e, x, x^2, \ldots, x^r\}$ is linearly dependent, where $e$ is the unit element satisfying $x \circ e = e \circ x = x$, writing $\deg(x) = r$. The *rank* of $\mathcal{J}$ is defined as the maximum degree across all elements of $\mathcal{J}$, i.e., $\text{rank}(\mathcal{J}) = \max_{x \in \mathcal{J}} \deg(x)$, and it is assumed that $\max \deg(x)$ is finite. Whenever the degree of an element $x \in \mathcal{J}$ equals the rank of the algebra, it is said that $x$ is *regular*. Moving forward, it is also assumed that $\text{rank}(\mathcal{J}) = r$.

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For all $x, s \in \mathcal{J}$, there exists a symmetric matrix $L(x)$ which characterizes the bilinear map $\circ : (x, s) \mapsto x \circ s$:

$$L(x)s = x \circ s.$$

Observe that $L(x)e = x$ and $L(x)x = x^2$. It can also be observed that due to (3b) $L(x)$ and $L(x^2)$ commute. The *quadratic representation* of $x$ is defined as

$$Q_x = 2L^2(x) - L(x^2). \tag{4}$$

Recall that an *idempotent* is a nonzero element $v \in \mathcal{J}$ such that $v^2 = v$ and a *primitive* is an idempotent that cannot be defined as the sum of two other idempotents. A *Jordan frame* is defined as a complete set of primitive idempotents, i.e., a set $\{v_1, \ldots, v_r\}$ satisfying

$$v_i \circ v_j = \mathbf{0}\ \forall i \neq j, \text{ and } \sum_{i=1}^{r} v_i = e.$$

**Theorem 1 (Theorem III.1.2 in [11])** *Let $(\mathcal{J}, \circ)$ be a rank $r$ Euclidean Jordan algebra. Then, every $x \in \mathcal{J}$ can be represented as*

$$x = \lambda_1 v_1 + \cdots + \lambda_r v_r, \tag{5}$$

*where $\{v_1, \ldots, v_r\}$ denotes a Jordan frame and $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of $x$, such that (5) is the spectral decomposition of $x$.*

Using the eigenvalues of elements of Jordan algebras, these definitions extend to any real-valued function $f(\cdot)$. That is, $f(x) \equiv f(\lambda_1)v_1 + \cdots + f(\lambda_r)v_r$. This work will make use of both the square root and inverse, respectively, defined to be

$$x^{1/2} \equiv \lambda_1^{1/2} v_1 + \cdots + \lambda_r^{1/2} v_r,$$

and

$$x^{-1} \equiv \lambda_1^{-1} v_1 + \cdots + \lambda_r^{-1} v_r.$$

The square root $x^{1/2}$ is defined whenever $\lambda_i \geq 0$ for all $i = 1, \ldots, r$, and undefined otherwise. The inverse $x^{-1}$ is defined whenever $\lambda_i \neq 0$ for all $i = 1, \ldots, r$, and undefined otherwise. Whenever each of the eigenvalues $\lambda_i$ of $x$ is nonnegative (positive), $x$ is said to be positive semidefinite (positive definite), writing $x \succeq 0$ ($x \succ 0$) to indicate when this is the case.

Norms and inner products on $\mathcal{J}$ can also be defined as functions of the eigenvalues of $x$. Namely,

$$\operatorname{tr}(x) = \sum_{i=1}^{r} \lambda_i, \quad \langle x, s \rangle = \operatorname{tr}(x \circ s), \quad \|x\|_F \equiv \left( \sum_{i=1}^{r} \lambda_i^2 \right)^{1/2} = \sqrt{\operatorname{tr}(x^2)}, \quad \|x\|_2 = \max_i |\lambda_i|.$$

It should be clear from these definitions that $\operatorname{tr}(e) = r$ and $\|e\|_F = \sqrt{r}$. It also follows from the associativity of the trace that $L(x)$ is a symmetric operator with respect to the inner product that our space is equipped with.

**Lemma 1 (Lemma 1 in [32])** *Let $x \in \mathcal{J}$ with spectral decomposition*

$$x = \lambda_1 v_1 + \cdots + \lambda_r v_r.$$

*Then, the eigenvalues of the symmetric linear operator $L(x)$ are all of the form*

$$\frac{\lambda_i + \lambda_j}{2}, \quad i, j = 1, \ldots, r.$$

The following definition motivates our use of Euclidean Jordan algebras in SCO.

**Definition 1 (Definition 2 in [32])** If $\mathcal{J}$ is a Euclidean Jordan algebra, then its cone of squares is the set

$$\mathcal{K}(\mathcal{J}) \equiv \{x^2 : x \in \mathcal{J}\}.$$

It follows that $\mathcal{K}(\mathcal{J})$ is a closed, pointed, convex cone with a non-empty interior.

The following theorem from [32] establishes the role of Euclidean Jordan algebras in the context of optimization over symmetric cones.

**Theorem 2 (Theorem 2 in [32])** *A cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra.*

This section concludes with the definition of *simple* elements and an important result for our analysis that uses such elements.

**Definition 2 (Definition 5 in [32])** Let $\mathcal{A}$ be an associative algebra. An element $x \in \mathcal{A}$ is simple if $xx' \in \mathcal{J}$.

**Lemma 2 (Lemma 5 in [32])** *Let $x, y, u, v \in \mathcal{J}$ with $u, v$ invertible, $a$ be a simple element of an associative algebra $\mathcal{A}$ and $\alpha \in \mathbb{R}$. Then, $xy, xa, u^{-1}xyu, uv + \alpha u^{-1}v^{-1}$ and $xy + u$ are also simple.*

### Solving the Newton Linear System with a Quantum Computer

A detailed discussion on how quantum computers can be used to solve linear systems of equations is provided in the chapter on quantum linear algebra. These concepts are only summarized at a high level here, assuming a basic familiarity with the framework of block-encoded matrices and quantum random access memory (QRAM).

Let $U$ be an $(\alpha, a, \delta)$-block-encoding of a matrix $M \in \mathbb{R}^{n\times n}$ that can be implemented in time $T_U$, and suppose that $M$ has condition number $\kappa_M$. Then, $U$ can be used to implement a block-encoding $V$ of the inverse of $M$:

$$V = \begin{pmatrix} \frac{M^{-1}}{2\kappa_M} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

and amplitude amplification is subsequently applied to

$$V |u\rangle |0\rangle^{\otimes a} = \frac{1}{2\kappa_M} M^{-1} |u\rangle |0\rangle^{\otimes a} + |\cdot\rangle .$$

This idea is formalized in the next result from [8].

**Theorem 3 (Theorem 30 in [8])** *(Solution of linear system) Let $\zeta \in (0, 1)$, $\kappa \geq 2$ and $H$ be a Hermitian matrix such that its nonzero eigenvalues lie in $[-1, -1/\kappa]\cup[1/\kappa, 1]$. Suppose that*

$$\delta = o\left(\frac{\zeta}{\kappa^2 \log^3 \frac{\kappa^2}{\zeta}}\right)$$

*and $U$ is an $(\alpha, a, \delta)$-block-encoding of $H$ that can be implemented in time $T_U$. Suppose further that a state $|v\rangle$ that is in the image of $H$ can be prepared in time $T_v$. Then, for any $\zeta$, we can output a state that is $\zeta$-close to $H^{-1}|v\rangle / \|H^{-1}v\|$ in time*

$$\mathcal{O}\left(\kappa\left(\alpha(a + T_U)\log^2\left(\frac{\kappa}{\zeta}\right) + T_v\right)\log\kappa\right).$$

In the context of QIPMs, QLSAs prepare a *quantum state* encoding the solution to the Newton linear system, and no classical information on this state can be accessed directly. Therefore, a procedure for preparing a classical estimate of the solution is required. At the time of writing, it is not clear how to prepare the Newton system for the subsequent iteration without a classical description of the solution. For this task, the quantum state tomography algorithm of van Apeldoorn et al. [37] is employed.

**Theorem 4 ([37])** *Let $\Re(v)$ denote the real part of $v$. Let $|\psi\rangle = \sum_{j=0}^{d-1} v_j |j\rangle$ be a quantum state, $v \in \mathbb{C}^d$ the vector with elements $v_j$, and $U|0\rangle = |\psi\rangle$. There is a quantum algorithm that, with probability at least $1 - \delta$, outputs $\tilde{v} \in \mathbb{R}^d$ such that $\|\Re(v) - \tilde{v}\|_2 \leq \varepsilon$ using $\mathcal{O}(\frac{d}{\varepsilon}\log\frac{d}{\delta})$ applications of $U$, $\widetilde{\mathcal{O}}_{d,\frac{1}{\delta},\frac{1}{\varepsilon}}(\frac{d}{\varepsilon})$ indexed-SWAP gates acting on $d$ qubits, and $\widetilde{\mathcal{O}}_{d,\frac{1}{\delta},\frac{1}{\epsilon}}(\frac{d}{\varepsilon})$ additional gates. With access to a classical-write, quantum-read QRAM of size $\widetilde{\mathcal{O}}_{d,\frac{1}{\delta},\frac{1}{\varepsilon}}(2^{d/\varepsilon})$, the indexed-SWAP gates are not required.*

To summarize, the tomography subroutine can obtain a classical description in time that is linear in the dimension and inverse precision, which is an $\mathcal{O}(1/\epsilon)$ improvement over all other tomography algorithms in the literature. Note that the tomography algorithm in [37] applies to any state preparation unitary. Mohammadisiahroudi et al. [24] showed that for the special case in

which the state preparation unitary is a QLSA, the dependence on precision can be exponentially improved by employing an iterative refinement scheme. At a high level, their idea reduces to solving a polylogarithmic number of linear systems rather than a single one and using the sequence of solutions obtained by the refinement scheme to construct a highly precise solution to the original system of interest.

**Theorem 5 (Theorem 7 in [24])** *Assume classical access to an $s$-sparse matrix $M \in \mathbb{R}^{d\times d}$ and a vector $z \in \mathbb{R}^d$. Let $\zeta \in (0,1)$. Then, a quantum computer with access to a classical-write, quantum-read QRAM outputs a $\zeta$-precise solution (in the $\ell_2$-norm) to the linear system $Mx = z$ using at most*

$$\mathcal{O}\left(d\kappa_M \cdot \text{polylog}\left(d, \kappa_M, \frac{1}{\zeta}\right)\right),$$

*accesses to the QRAM, and $\mathcal{O}\left(ds \cdot \text{polylog}\left(d, \kappa_M, \frac{1}{\zeta}\right)\right)$ arithmetic operations.*

## Classical IPMs for Optimization over Symmetric Cones

This section reviews the theory of IPMs as it is applied to SCO. It is then shown how to formulate the Newton linear system to ensure primal and dual feasibility exactly holds while using an inexact linear system subroutine such as a QLSA.

### The Central Path in SCO

Suppose that $\mathcal{J} \subset \mathbb{R}^n$ is a Euclidean Jordan algebra with rank $r$, and let $\mathcal{K}$ be the cone of squares of $\mathcal{J}$. Let $c, a_1, \ldots, a_m \in \mathcal{J}$ and $b \in \mathbb{R}^m$. Defining

$$Ax = \begin{pmatrix} \langle a_1, x\rangle \\ \vdots \\ \langle a_m, x\rangle \end{pmatrix},$$

where $Ax$ is assumed to be a linear surjective map, the primal-dual pair (1)-(2) can be compactly written as follows:

$$z_P = \min_x \{\langle c, x\rangle : Ax = b,\ x \in \mathcal{K}\} \tag{P}$$

$$z_D = \max_{(y,s)} \left\{b^\top y : A^\top y + s = c,\ s \in \mathcal{K}\right\}. \tag{D}$$

The set of *interior feasible solutions* of (P) and (D) are defined to be

$$\mathcal{P}^0 = \{x \in \text{int}(\mathcal{K}) : Ax = b\}$$

$$\mathcal{D}^0 = \left\{(y,s) \in \mathbb{R}^m \times \text{int}(\mathcal{K}) : A^\top y + s = c\right\}.$$

When $\mathcal{P}^0$ and $\mathcal{D}^0$ are nonempty, the IPC is satisfied, then the *primal and dual optimal sets*

$$\mathcal{P}^* = \{x \in \mathcal{K} : Ax = b,\ \langle c, x\rangle = z_P\}$$

$$\mathcal{D}^* = \left\{(y,s) \in \mathbb{R}^m \times \mathcal{K} : A^\top y + s = c,\ b^\top y = z_D\right\}$$

are non-empty and bounded, and strong duality holds with $z_P = z_D$.

Whenever the IPC holds, determining an optimal solution to (P)–(D) is equivalent to solving the system

$$\begin{aligned} Ax &= b, \\ A^\top y + s &= c, \\ x \circ s &= \mathbf{0}, \\ x, s &\in \mathcal{K}. \end{aligned}$$

In the framework of primal-dual IPMs, the complementarity condition $x \circ s = \mathbf{0}$ is perturbed to $x \circ s = \mu e$, where $e$ is the identity element. Proceeding in this way gives rise to the *central path equation system*

$$\begin{aligned} Ax &= b, \\ A^\top y + s &= c, \\ x \circ s &= \mu e, \\ x, s &\in \mathcal{K}, \end{aligned}$$

where $\mu$ is the central path parameter. Clearly the limit of the central path exists if $\mu \to 0$, and the

limit point provides an optimal solution to (P)–(D). Having $x, s \in \text{int}(\mathcal{K})$, let us define $\mu = \frac{\langle x,s\rangle}{r}$ and reduce $\mu$ to $\sigma\mu$ for $\sigma \in (0,1)$. Applying Newton's method to the resulting system yields the following system of equations:

$$\begin{aligned} A\Delta x &= 0, \\ A^\top \Delta y + \Delta s &= 0, \\ \Delta x \circ s + x \circ \Delta s &= \sigma\mu e - x \circ s, \end{aligned} \tag{6}$$

where $(\Delta x, \Delta y, \Delta s) \in \mathcal{J} \times \mathbb{R}^m \times \mathcal{J}$ denotes the Newton step.

However, the system (6) is not solvable; $x$ and $s$ may not operator commute as $L(x)L(s) \neq L(s)L(x)$ in general. When applying primal-dual IPMs to symmetric cones, a scaling is therefore necessary to ensure convergence of our algorithm. This is discussed next.

### Similarity Scaling

This section details a group of scaling automorphisms that map the cone $\mathcal{K}$ onto itself. A detailed discussion here describes how scaling gives rise to different search directions and, as a consequence, solutions to our primal and dual CLO problems which possess attractive features such as maximal complementarity.

In what follows, suppose that $x, s \in \text{int}(\mathcal{K})$ and $p >_{\mathcal{K}} 0$ (hence, $p$ is invertible). Consider a bilinear operator on $x$ and $s$

$$H_p(x,s) \equiv (Q_p x) \circ (Q_{p^{-1}} s),$$

where $Q_p$ is defined according to (4), and for any $u \in \text{int}(\mathcal{K})$, define

$$\tilde{u} = Q_p u, \quad \underset{\sim}{u} = Q_{p^{-1}} u.$$

It follows (see, e.g., Section 3.2 in [32]) that $\tilde{\cdot}$ and $\underset{\sim}{\cdot}$ are inverses of each other, due to the fact that $Q_p Q_{p^{-1}} = I$, where $I$ is the identity matrix. Further, whenever $u$ is positive semidefinite, so are $\tilde{u}$ and $\underset{\sim}{u}$. Observe that the cone $\mathcal{K}$ of squares of $\mathcal{J}$ remains invariant under the transformations $\tilde{\cdot}$ and $\underset{\sim}{\cdot}$. Consequently, a solution $(x, y, s)$ is feasible for (P)–(D) if and only if $(\tilde{x}, y, \underset{\sim}{s})$ is feasible for the scaled primal-dual pair:

$$z_P = \min_{\tilde{x}} \{\langle \underset{\sim}{c}, \tilde{x}\rangle : \langle \underset{\sim}{a}_i, \tilde{x}\rangle = b_i \ \forall i \in [m], \tilde{x} \in \mathcal{K}\} \tag{$\tilde{\text{P}}$}$$

$$z_D = \max_{(y,s)} \left\{ b^\top y : \sum_{i=1}^m y_i \underset{\sim}{a}_i + \underset{\sim}{s} = \underset{\sim}{c}, \underset{\sim}{s} \in \mathcal{K} \right\}. \tag{$\tilde{\text{D}}$}$$

The following two crucial lemmas from [32] establish the intrinsic relationship between (P)–(D) and the scaled problems ($\tilde{\text{P}}$)–($\tilde{\text{D}}$).

**Lemma 3 (Lemma 17 in [32])** *It follows*

1. $\langle x, s\rangle = \langle \tilde{x}, \underset{\sim}{s}\rangle$;
2. $Au = \mathbf{0} \iff \underset{\sim}{A}\tilde{u} = \mathbf{0}$ *and* $A^\top v + w = \mathbf{0} \iff \underset{\sim}{A}^\top v + \underset{\sim}{w} = \mathbf{0}$;
3. $H_p(x,s) = H_e(\tilde{x}, \underset{\sim}{s}) = \tilde{x} \circ \underset{\sim}{s}$;
4. $d(x,s) = d(\tilde{x}, \underset{\sim}{s})$.

**Lemma 4 (Lemma 18 in [32])** $(\widetilde{\Delta x}, \Delta y, \underset{\sim}{\Delta s})$ *is a solution to the system*

$$\begin{aligned} \underset{\sim}{A}\widetilde{\Delta x} &= b - \underset{\sim}{A}\tilde{x}, \\ \underset{\sim}{A}^\top \Delta y + \underset{\sim}{\Delta s} &= \underset{\sim}{c} - \underset{\sim}{A}^\top y - \underset{\sim}{s}, \\ \widetilde{\Delta x} \circ \underset{\sim}{s} + \tilde{x} \circ \underset{\sim}{\Delta s} &= \sigma\mu e - \tilde{x} \circ \underset{\sim}{s}, \end{aligned}$$

*if an only if* $(\Delta x, \Delta y, \Delta s)$ *is a solution to the system*

$$\begin{aligned} A\Delta x &= b - Ax, \\ A^\top \Delta y + \Delta s &= c - s - A^\top y, \\ H_p(\Delta x, s) + H_p(x, \Delta s) &= \sigma\mu e - H_p(x,s). \end{aligned} \tag{7}$$

There are many possible choices for $p$, and our results extend to all of these directions, the simplest being the Alizadeh-Haeberly-Overton (AHO) direction [2] with $p = e$. Upon setting $p = s^{1/2}$, the so-called HKM direction is obtained, which was independently discovered in [15, 18, 25]. The Nesterov-Todd [30] direction corresponds to choosing $p = w^{-1/2}$ where

$$w = Q_{x^{1/2}} \left(Q_{x^{1/2}} s\right)^{-1/2} = Q_{s^{-1/2}} \left(Q_{s^{1/2}} x\right)^{1/2}.$$

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While more computationally demanding to compute than the AHO and HKM directions, the Nesterov-Todd direction guarantees the optimal solution is maximally complementary.

For any $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$, the centrality measure is defined to be

$$d(x,s) = \left\| Q_{x^{1/2}} s - \mu e \right\|_F = \left\| x^{1/2} s x^{1/2} - \mu e \right\|_F = \sqrt{\frac{1}{k} \sum_{i=1}^{n} (\lambda_i(xs) - \mu)^2}, \tag{8}$$

where $Q_{x^{1/2}}$ denotes the quadratic representation at $x^{1/2}$; see section "Euclidean Jordan Algebras." This work considers a small neighborhood of the central path, which for a constant $\gamma \in (0, 1)$ is defined to be

$$\mathcal{N}_F(\gamma) = \left\{ (x, y, s) \in \mathcal{P}^0 \times \mathcal{D}^0 : d(x, s) \leq \gamma \mu \right\}. \tag{9}$$

The next result from [32] establishes that the system (7) has a unique solution whenever the point $(x, y, s)$ is close to the central path.

**Lemma 5** *If* $(x, y, s) \in \text{int}(\mathcal{K}) \times \mathbb{R}^m \times \text{int}(\mathcal{K})$ *such that*

$$d(x,s) := \left\| Q_{x^{1/2}} s - \mu e \right\|_F \leq \frac{\mu}{2},$$

*then system* (7) *has a unique solution.*

### The Classical IPM for SCO

A classical IPM for SCO is outlined in Algorithm 1. That algorithm commences from an interior feasible solution $(x^{(0)}, y^{(0)}, s^{(0)}) \in \mathcal{N}_F(\gamma)$ with duality gap $\mu^{(0)} = \langle x^{(0)}, s^{(0)} \rangle / r$. At every iteration, the Newton linear system (7) is solved to obtain $(\Delta x, \Delta y, \Delta s)$, which is used to update the solution for the following iterate using the rule $(x, y, s) := (x + \Delta x, y + \Delta y, s + \Delta s)$, and subsequently calculate the value of the central path parameter $\mu = \langle x, s \rangle / r$ at this solution. If $\mu \leq \epsilon$, the algorithm terminates as an $\epsilon$-optimal solution to the primal-dual SCO pair (P)-(D) has been reached; otherwise the algorithm proceeds to the next iteration.

**Algorithm 1** Classical interior point method for SCO problems

Choose constants $\gamma \in (0, 1/3)$, $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$;
Set $\sigma = 1 - \delta/\sqrt{r}$;
Choose initial point $(x^{(0)}, y^{(0)}, s^{(0)}) \in \mathcal{N}_F(\gamma)$
Set $\mu = \langle x^{(0)}, s^{(0)} \rangle / r$
**while** $\mu > \epsilon$:

1. Solve the Newton linear system (7) to obtain $(\Delta x, \Delta y, \Delta x)$
2. Update solution

$$x \leftarrow x + \Delta x \quad s \leftarrow s + \Delta s \quad y \leftarrow y + \Delta y$$

3. $\mu = \langle x, s \rangle / r$

**end**

In order ensure convergence in the quantum setting, additional precautions need to be taken with regard to how quantum noise impacts to the solution to the Newton system.

### The Newton Linear System and Quantum Noise

Observe that, from the definition of primal and dual feasibility, any exact solution $(\Delta x, \Delta y, \Delta s)$ to (7) must satisfy $\langle \Delta x, \Delta s \rangle = 0$. However, by directly applying a QLSA to solve (7), and subsequently apply a quantum state tomography to obtain a classical estimate of the state obtained by our QLSA, the noise introduced by these subroutines means the classical representation only satisfies

$$A\Delta x = \xi_p,$$
$$A^\top \Delta y + \Delta s = \xi_d,$$
$$H_p(\Delta x, s) + H_p(x, \Delta s) = \sigma\mu e - H_p(x, s) + \xi_c, \tag{10}$$

where $\xi_p$, $\xi_d$, and $\xi_c$ are nonzero error terms.

As a consequence, the primal and dual search directions are not primal-dual feasible, and accordingly no longer guaranteed to be members of orthogonal subspaces. One possible way to reconcile this issue would be to quantize an inexact-infeasible IPM, but this approach has been shown to exhibit worse overall complexity in all parameters compared to the best performing classical feasible IPMs (see, e.g., [4,23]). Rather, it is preferred to design QIPMs that can maintain feasibility.

In order to guarantee that primal and dual feasibility are satisfied exactly, the approach in [4] is adopted here. Let $\text{Null}(A)$ and $\mathcal{R}(A)$ denote the nullspace and rangespace of $A$, respectively. Letting $\mathcal{B}_{\text{Null}(A)}$ be a basis for $\text{Null}(A)$ and $\mathcal{B}_{\mathcal{R}}$ be a basis for $\mathcal{R}(A)$, the primal and dual search directions can be defined as linear combinations of these basis elements, respectively. Note that for the rangespace, it suffices to choose $\mathcal{B}_{\mathcal{R}} = A^\top$ and, hence, define

$$\Delta x = \mathcal{B}_{\text{Null}(A)} \Delta z \tag{11a}$$
$$\Delta s = -A^\top \Delta y. \tag{11b}$$

If the input data is sparse, sparse matrix multiplication [31] can be used to compute $\mathcal{B}_{\text{Null}(A)}$ efficiently. Moreover, if the primal-dual pair was provided in canonical form, the nullspace can be obtained directly from the coefficient matrix. While the nullspace representation of the search directions is novel in its application to IPMs, choosing a basis is in general a common pre-processing step used by many optimization solvers in theory and in practice. The next result is a direct consequence of definition of the search directions in (11).

**Lemma 6** *For any $\Delta x$ and $\Delta s$ defined according to (11), it follows*

$$\langle \Delta x, \Delta s \rangle = 0. \tag{12}$$

***Proof*** The result directly follows from the fact that $\Delta x$ and $\Delta s$ are defined to be the linear combinations of basis elements of subspaces that are orthogonal to one another. □

Define $\Delta x$ and $\Delta s$ according to (11), and substitute these expressions in the third equation of (7). Defining the quantities

$$\mathcal{E} = L(Q_{p^{-1}} s) Q_p,$$
$$\mathcal{F} = L(Q_p x) Q_{p^{-1}},$$

the complementarity equation that arises in (7) can be equivalently expressed as

$$\mathcal{E}\Delta x + \mathcal{F}\Delta s = \sigma\mu e - H_p(x, s).$$

Hence, substituting the orthogonal subspace definitions of the primal and dual search directions in (11), the *quantum Newton linear system* is given by

$$\left[\mathcal{E}\mathcal{B}_{\text{Null}(A)} \; -\mathcal{F}A\right] \left|(\Delta z, \Delta y)^\top\right\rangle = \left|\sigma\mu e - H_p(x, s)\right\rangle. \tag{13}$$

Lemma 5 asserts that system (13) has a unique solution, provided that the solution $(x, y, s)$ is sufficiently close to the central path.

Additionally, even for inexact solutions $\Delta z$ and $\Delta y$ to (13), primal and dual feasibility will be satisfied exactly regardless of quantum noise. To see this, suppose that a quantum computer is used to prepare a state $|(\Delta z, \Delta y)\rangle$ encoding the solution to the linear system. Letting $(\overline{\Delta z}, \overline{\Delta y})^\top$ be a classical estimate of $\left|(\Delta z, \Delta y)^\top\right\rangle$ obtained by quantum state tomography, it follows

$$\overline{\Delta z} = \Delta z + \xi_z, \quad \overline{\Delta y} = \Delta y + \xi_y,$$

where $\xi_z$ and $\xi_y$ are terms that characterize the noise present in our classical estimate. However, for $\overline{\Delta x} = \mathcal{B}_{\mathrm{Null}(A)}\overline{\Delta z}$ and $\overline{\Delta s} = -A^\top \overline{\Delta y}$, note that for any $(x, y, s) \in \mathcal{P} \times \mathcal{D}$, it follows $A\overline{\Delta x} = 0$ and $A^\top \overline{\Delta y} + \overline{\Delta s} = 0$. Thus primal and dual feasibility is preserved as long as $x + \overline{\Delta x} \in \mathrm{int}(\mathcal{K})$ and $s + \overline{\Delta s} \in \mathrm{int}(\mathcal{K})$.

On the other hand, the symmetrized central path equation will only be satisfied inexactly. Indeed,

$$\overline{\Delta x} \circ s + x \circ \overline{\Delta s} = \sigma\mu e - H_p(x, s) + \xi$$

where $\xi \in \mathbb{R}^n$ is a residual term coming from our use of QLSAs and quantum state tomography. Note that in this context $\xi$ is equivalent to the complementarity error term $\xi_c$ that appears in (10); the subscript $c$ has been dropped because using the nullspace representation of the search directions will ensure that the primal and dual residuals $\xi_p$ and $\xi_d$ are always zero. In other words, the only nontrivial error term is with respect to complementarity.

In order to guarantee convergence of our algorithm, $\xi$ is required to be chosen such that

$$\|\xi\|_F \le \beta \|\sigma\mu e - x^{\frac{1}{2}} s x^{\frac{1}{2}}\|_F \qquad \text{(AR1)}$$

for a properly chosen $\beta \in (0, 1)$. This is a standard assumption for inexact IPMs and ensures that residuals are driven toward 0 as the optimal solution is approached; the norm present in the right-hand side is our distance metric $d(x, s)$. It is also assumed that this relationship holds upon multiplying by $x^{-1/2}$ from the left and $x^{1/2}$ from the right:

$$\|x^{-1/2}\xi x^{1/2}\|_F \le \beta \left\| x^{-1/2}\left[\sigma\mu e - x^{\frac{1}{2}} s x^{\frac{1}{2}}\right] x^{1/2}\right\|_F. \qquad \text{(AR2)}$$

## Technical Results

In this section, the work of Schmieta and Alizadeh [32] is extended to the inexact setting. The IF-QIPM is presented in section "A QIPM for Optimization over Symmetric Cones." The analysis in [32] can be viewed as a generalization of the results of Monteiro [25], using the framework of Euclidean Jordan algebras. Conveniently, in proving convergence of their IF-QIPM for SDO, Augustino et al. [4] extended the analysis in [25] to account for inexactness of the solution/Newton step with respect to the linearized complementarity condition. Thus, the results here can be viewed as a generalization of those found in [4].

Although the notation $\tilde{\cdot}$ and $\underset{\sim}{\cdot}$ is dropped for ease of presentation, the reader should note that the analysis performed in this section is with respect to the scaled problems ($\tilde{\mathrm{P}}$)–($\tilde{\mathrm{D}}$) and the associated system

$$\begin{aligned} \underset{\sim}{A}\widetilde{\Delta x} &= b - \underset{\sim}{A}\tilde{x}, \\ \underset{\sim}{A}^\top \Delta y + \underset{\sim}{\Delta s} &= \underset{\sim}{c} - \underset{\sim}{A}^\top y - \underset{\sim}{s}, \\ \widetilde{\Delta x} \circ \underset{\sim}{s} + \tilde{x} \circ \underset{\sim}{\Delta s} &= \sigma\mu e - \tilde{x} \circ \underset{\sim}{s}, \end{aligned}$$

having dropped the accents $\tilde{\cdot}$ and $\underset{\sim}{\cdot}$.

What remains will make use of the following quantities:

$$x(\alpha) = x + \alpha\Delta x, \qquad s(\alpha) = s + \alpha\Delta s, \quad y(\alpha) = y + \alpha\Delta y, \tag{14a}$$

$$w_x = x^{-1/2}\left[\Delta x s + x\Delta s + xs - \sigma\nu e - \xi\right] x^{1/2}, \tag{14b}$$

$$\mu = \frac{\langle x, s\rangle}{r}, \qquad \mu(\alpha) = \frac{\langle x(\alpha), s(\alpha)\rangle}{r}, \tag{14c}$$

$$\Phi_\theta(u, v) = v^{1/2}u^{1/2} - \theta\frac{\langle u, v\rangle}{r} v^{-1/2}u^{-1/2}, \qquad \phi_\theta(u, v) = \|\Phi_\theta(u, v)\|_F, \tag{14d}$$

$$\Delta_x = x^{-1/2}\Delta x s^{1/2}, \quad \delta_x = \|\Delta_x\|_F, \qquad \Delta_s = s^{-1/2}\Delta s x^{1/2}, \quad \delta_s = \|\Delta_s\|_F. \tag{14e}$$

The main goal of this section is to establish a bound on the quantity

$$\left\| x^{-1/2} \left[ \Delta x s + x \Delta s + x s - \sigma \nu e - \xi \right] x^{1/2} \right\|_F,$$

which majorizes the centrality measure $d(x(\alpha), s(\alpha))$. The following result comes from [32] and originally appeared in its application to SDO in [25]. It establishes a useful feature of the distance to the central path known as scale invariance.

**Lemma 7 (Lemma 9 in [32])** *Suppose that $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$ and $p$ is invertible. Then,*

*(a)* $d(pxp, p^{-1}sp^{-1}) = d(x, s)$,
*(b)* $d(x, s) \leq \left\| \frac{1}{2}(pxsp^{-1} + p^{-1}sxp) - \mu e \right\|_F$, *with equality holding if* $pxsp^{-1} \in \mathcal{J}$.

The next lemma can be viewed as the inexact analogue of Lemma 11 in [32] and a generalization of Lemma 7 in [4].

**Lemma 8** *It follows*

$$\mu(\alpha) = (1 - \alpha - \sigma\alpha)\mu + \alpha \frac{\langle e, \xi \rangle}{r}, \tag{15a}$$

$$x^{-1/2} \left[ x(\alpha) s(\alpha) - \mu(\alpha) e \right] x^{1/2} = (1 - \alpha) \left( x^{1/2} s x^{1/2} - \mu e \right) + \alpha w_x + \alpha^2 \Delta_x \Delta_s + \alpha x^{-1/2} \left[ \xi - \frac{\langle e, \xi \rangle}{r} \right] x^{1/2}. \tag{15b}$$

***Proof*** The proof directly follows the proof of Lemma 7 in [4], upon accountingfor $\langle e, e \rangle = r$. First note that $\langle \Delta x, \Delta s \rangle = 0$ by Lemma 6, and hence

$$\begin{aligned} \langle x(\alpha), s(\alpha) \rangle = \langle x + \alpha \Delta x, s + \alpha \Delta s \rangle &= \langle x, s \rangle + \alpha \left( \langle x, \Delta s \rangle + \langle s, \Delta x \rangle \right) + \alpha^2 \langle \Delta x, \Delta s \rangle \\ &= (1 - \alpha) \langle x, s \rangle + \alpha \langle x \circ \Delta s + s \circ \Delta x + x \circ s + \xi, e \rangle \\ &= (1 - \alpha) \langle x, s \rangle + \alpha \langle \sigma \mu e, e \rangle + \alpha \langle \xi, e \rangle \\ &= r \left[ (1 - \alpha)\mu + \alpha \sigma \mu \right] + \alpha \langle \xi, e \rangle. \end{aligned}$$

Dividing through by $r$, the result in (15a) is obtained. Next, observe that

$$x(\alpha) s(\alpha) = (x + \alpha \Delta x)(s + \alpha \Delta s) = xs + \alpha(x \Delta s + \Delta x s) + \alpha^2 \Delta x \Delta s.$$

Therefore, it follows from (15a) that

$$
\begin{aligned}
x(\alpha)s(\alpha) - \mu(\alpha)e &= (x + \alpha\Delta x)(s + \alpha\Delta s) - \mu(\alpha)e \\
&= xs + \alpha(x\Delta s + \Delta x s) + \alpha^2\Delta x\Delta s - \mu(\alpha)e \\
&= (1-\alpha)xs + \alpha(x\Delta s + \Delta x s + xs) + \alpha^2\Delta x\Delta s - \left[(1-\alpha-\sigma\alpha)\mu + \alpha\frac{\langle e,\xi\rangle}{r}\right] \\
&\quad e + \underbrace{\alpha\,(\xi - \xi)}_{=0} \\
&= (1-\alpha)(xs-\mu e) + \alpha(x\Delta s + \Delta x s + xs - \sigma\mu e - \xi) + \alpha^2\Delta x\Delta s \\
&\quad + \alpha\left[\xi - \frac{\langle e,\xi\rangle}{r}\right].
\end{aligned}
$$

The result in (15b) follows from multiplying by $x^{-1/2}$ from the left and from the right by $x^{1/2}$. The proof is complete. □

The next result from [32] plays an important role in the analysis, which generalizes Lemma 3.5 of [25] to representable symmetric elements.

**Lemma 9 (Lemma 12 in [32])** *Let $w \in \mathcal{J}$ be such that $qwq^{-1} + (qwq^{-1})' = \mathbf{0}$ for some invertible $q \in \mathcal{J}$. Then*

$$\left\|\frac{w+w'}{2}\right\|_F \le \frac{1}{2}\left\|w - w'\right\|_F, \tag{16a}$$

$$\|w\|_F \le \frac{\sqrt{2}}{2}\left\|w - w'\right\|_F. \tag{16b}$$

*In particular, if $w = u_1 + u_2$ for some $u_1 \in \mathcal{J}$, and arbitrary $u_2 \in \mathcal{A}$, then*

$$\|w\|_F \le \sqrt{2}\|u_2\|_F. \tag{17}$$

The following result extends Lemma 13 in [32] to account for inexactness in the complementarity equation.

**Lemma 10** *For every $\theta \in \mathbb{R}$, it follows*

$$\|w\|_F \le \sqrt{2}\left(\delta_x\phi_\theta(x,s) + \|x^{-1/2}\xi x^{1/2}\|_F\right), \tag{18}$$

*and*

$$
\begin{aligned}
\left\|x^{-1/2}\left[x(\alpha)s(\alpha) - \mu(\alpha)e\right]x^{1/2}\right\|_F &\le (1-\alpha)d(x,s) + \alpha^2\delta_x\delta_s + \alpha\sqrt{2}\delta_x\phi_\theta(x,s) \\
&\quad + \alpha\left\|x^{-1/2}\left[\xi - \frac{\langle e,\xi\rangle}{r}\right]x^{1/2}\right\|_F
\end{aligned} \tag{19}
$$

*for all $\alpha \in \mathbb{R}$.*

***Proof*** Our proof combines the proof of Lemma 13 in [32], with the proof of Lemma 9 in [4] (both of which follow from the proof of Lemma 3.6 in [25]).

Note that upon choosing $q = x^{1/2}$, $w$ satisfies the conditions of Lemma 9. Defining

$$u_1 = x^{1/2}\Delta s x^{1/2} + \theta\mu x^{-1/2}\Delta x x^{-1/2} + x^{1/2}s x^{1/2} - \sigma\mu e, \tag{20a}$$

$$u_2 = x^{-1/2}\Delta x s^{1/2}\Phi_\theta(x,s) - x^{-1/2}\xi x^{1/2}, \tag{20b}$$

and noting that $w = u_1 + u_2$, with $u_1 \in \mathcal{J}$ and $u_2$ being simple, a combined application of Lemmas 2 and 9 yields

$$\|w_x\|_F \le \sqrt{2}\|u_2\|_F \le \sqrt{2}\left(\delta_x\phi_\theta(x,s) + \|x^{-1/2}\xi x^{1/2}\|_F\right).$$

In other words, (18) holds.

Next, observe that (15b) and a triangle inequality to obtain

$$\begin{aligned}
&\left\|x^{-1/2}\left[x(\alpha)s(\alpha) - \mu(\alpha)e\right]x^{1/2}\right\|_F \\
&\quad = \left\|(1-\alpha)\left(x^{1/2}sx^{1/2} - \mu e\right) + \alpha w_x + \alpha^2\Delta_x\Delta_s + \alpha x^{-1/2}\left[\xi - \frac{\langle e,\xi\rangle}{r}\right]x^{1/2}\right\|_F \\
&\quad \le (1-\alpha)d(x,s) + \alpha^2\delta_x\delta_s + \alpha\sqrt{2}\delta_x\phi_\theta(x,s) + \alpha\left\|x^{-1/2}\left[\xi - \frac{\langle e,\xi\rangle}{r}\right]x^{1/2}\right\|_F.
\end{aligned}$$

That is, (10) holds and the proof is complete. □

This section concludes with four more results, each of which is a generalization of a result from [4]. For brevity, proofs of these results are omitted here as each argument exactly follows its counterpart in [4], upon replacing $n$ with $r$. The next two results provide upper bounds on norms that appear in (18). The first is adapted from Lemma 11 in [4] and provides bounds on norms involving the residual term, while the second comes from [32] (which itself was adapted from Lemma 3.7 in [25]).

**Lemma 11** *Suppose that the error tolerances are chosen such that the assumptions* (AR1) *and* (AR2) *hold. For* $\beta \in (0,1)$ *it follows*

$$\|\xi\|_F \le \beta\gamma\sigma\mu, \quad (21a)$$

$$\left\|x^{-1/2}\xi x^{1/2}\right\|_F \le \beta\gamma\sigma\mu, \quad (21b)$$

$$\left\|x^{-1/2}\left[\xi - \frac{\langle\xi,e\rangle}{r}I\right]x^{1/2}\right\|_F \le 2\beta\gamma\sigma\mu. \quad (21c)$$

**Lemma 12 (Lemma 14 in [32])** *If* $d(x,s) \le \gamma\mu$ *for some* $\gamma \in (0,1)$*, then*

$$\left\|x^{-1/2}s^{-1/2}\right\|_2^2 \le \frac{1}{(1-\gamma)\mu}, \quad (22a)$$

$$\phi_\theta(x,s)^2 \le \frac{\gamma + (1-\theta)^2 r}{1-\gamma}. \quad (22b)$$

The next result can be viewed as an extension of Lemma 12 in [4] from SDO to SCO.

**Lemma 13** *If* $(x,y,s) \in \mathcal{N}_F(\gamma)$ *for some* $\gamma > 0$ *satisfying*

$$2\sqrt{2}\frac{\gamma}{1-\gamma}, \quad (23)$$

*and* $\beta \in (0,1)$ *such that*

$$\beta\sigma \le \sqrt{\frac{\gamma^2 + (1-\sigma)^2 r}{1-\gamma}},$$

*then*

$$\max\{\delta_x,\delta_s\} \le 2\left(\phi_\sigma(x,s) + \sqrt{\frac{\gamma^2 + (1-\sigma)^2 r}{1-\gamma}}\mu\right).$$

The final lemma of this section follows from the above lemmas and is the direct extension of Lemma 13 in [4].

**Lemma 14** *If* $(x,y,s) \in \mathcal{N}_F(\gamma)$ *for some* $\gamma > 0$ *satisfying* (23)*, then, for every* $\alpha \in [0,1]$ *and* $\beta \in (0,1)$*, it follows*

$$\left\| x^{-1/2} \left[ x(\alpha)s(\alpha) - \mu(\alpha)e \right] x^{1/2} \right\|_F$$
$$\leq \left( (1-\alpha)\gamma + 4\sqrt{2}\alpha \frac{\gamma \left[ \gamma^2 + (1-\sigma)^2 r \right]^{1/2}}{1-\gamma} + 16\alpha^2 \frac{\gamma^2 + (1-\sigma)^2 r}{1-\gamma} + \alpha(2+\sqrt{2})\beta\sigma\gamma \right) \mu.$$

## A QIPM for Optimization over Symmetric Cones

An IF-QIPM for SCO is outlined in Algorithm 2. That algorithm commences from an interior feasible solution $(x^{(0)}, y^{(0)}, s^{(0)}) \in \mathcal{N}_F(\gamma)$ which is stored in QRAM and exhibits a duality gap of $\mu^{(0)} = \langle x^{(0)}, s^{(0)} \rangle / r$. In every iteration, the combined use of a QLSA with quantum state tomography is employed to solve the orthogonal subspace system to obtain $(\Delta z, \Delta y)$, which is used to update the solution for the following iterate using the rule (see, e.g., (11)):

$$(x, y, s) = (x + \mathcal{B}_{\mathrm{Null}(A)} \Delta z, y + \Delta y, s - A^\top \Delta y).$$

From here, the updated solution is stored in QRAM, and compute the value of the central path parameter at this point $\mu = \langle x, s \rangle / r$. If $\mu \leq \epsilon$, the algorithm terminates, reporting the current solution as $\epsilon$-optimal for the primal-dual SCO pair (P)-(D); otherwise the algorithm proceeds to the next iteration.

**Algorithm 2** Inexact-feasible quantum interior point method for SCO problems

Choose constants $\gamma \in (0, 1/3)$ and $\delta \in [0, \sqrt{r})$; set optimality gap to $\epsilon \in (0, 1)$
Set $\sigma = 1 - \delta/\sqrt{r}$;
Choose initial point $(x^{(0)}, y^{(0)}, s^{(0)}) \in \mathcal{N}_F(\gamma)$, store in QRAM
Set $\mu^{(0)} = \langle x^{(0)}, s^{(0)} \rangle / r$
**while** $\mu > \epsilon$:

1. Prepare and solve the Newton linear system (13) to obtain the quantum state $|(\Delta z, \Delta y)\rangle$
2. Perform quantum state tomography to estimate $(\Delta z, \Delta y)$
3. Classically update solution
$$x \leftarrow x + \mathcal{B}_{\mathrm{Null}(A)} \Delta z \quad s \leftarrow s - A^\top \Delta y \; y \leftarrow y + \Delta y$$
4. Store $(x, y, s)$ in QRAM
5. $\mu = \dfrac{\langle x, s \rangle}{r}$

### Convergence

Our use of inexact quantum subroutines requires us to appropriately choose the tolerated amount of error for each of these steps. Recalling Assumption (AR1), bounds for the error must be set for the tomography step at iteration $k$ such that it satisfies

$$\zeta^{(k)} \leq \frac{\beta}{\varrho} \| \mu e - H_p(x, s) \|,$$

where $\varrho$ is the upper bound on norm of the solution to the Newton linear system and $\beta \in (0, 1)$. This is necessary as the tomography error bound is relative, assuming the vector that will be estimated is a unit vector. Noting that $\epsilon$-optimality implies

$$\| \mu e - H_p(x, s) \| \leq \epsilon,$$

and so the precision of the tomography step is set according to

$$\zeta^{(k)} = \frac{\beta}{\varrho} \cdot \max \left\{ \| \xi^{(k)} \|_F, \frac{1}{25} \epsilon \right\}. \tag{24}$$

The main result of this section analyzes one iteration of an inexact-feasible (Q)IPM applied to SCO.

**Theorem 6** *Let $\gamma, \beta \in (0, 1)$ and $\delta \in [0, \sqrt{r})$ be constants satisfying*

$$\frac{2\sqrt{2}\gamma}{1-\gamma} \leq 1, \quad \beta\sigma \leq \sqrt{\frac{\gamma^2 + (1-\sigma)^2 r}{1-\gamma}},$$
$$\beta \leq 1 - \frac{\gamma}{\sqrt{r}} - \frac{21.7(\gamma^2 + \delta^2)}{(2+\sqrt{2})\left(1 - \frac{\delta}{\sqrt{r}}\right)\gamma(1-\gamma)}. \tag{25}$$

*Suppose that $(x, y, s) \in \mathcal{N}_F(\gamma)$ and let $(\Delta x, \Delta y, \Delta s)$ denote the solution obtained from*

*solving system (13), where* $\sigma = 1 - \delta/\sqrt{r}$, $\mu = \langle x, s\rangle/r$. *Then,*

*(a)* $(\hat{x}, \hat{y}, \hat{s}) = (x + \Delta x, y + \Delta y, s + \Delta s) \in \mathcal{N}_F(\gamma)$;
*(b)* $\langle \hat{x}, \hat{s}\rangle = \left(1 - \frac{\delta}{\sqrt{r}}\right)\langle x, s\rangle$.

***Proof*** The proof is exactly the same as the proof of Lemma 13 given in [4], and the result follows upon applying Lemma 10 with $\theta = 1$, in combination with Lemmas 12–14. □

**Corollary 1** *Let* $\gamma, \beta \in (0, 1)$ *and* $\delta \in [0, \sqrt{r})$ *be constants satisfying*

$$\frac{2\sqrt{2}\gamma}{1-\gamma} \leq 1, \quad \beta\sigma \leq \sqrt{\frac{\gamma^2 + (1-\sigma)^2 r}{1-\gamma}},$$

$$\beta \leq 1 - \frac{\gamma}{\sqrt{r}} - \frac{21.7(\gamma^2 + \delta^2)}{(2+\sqrt{2})\left(1 - \frac{\delta}{\sqrt{r}}\right)\gamma(1-\gamma)}.$$

*Then, the IF-QIPM in Algorithm 2 converges to an* $\epsilon$*-optimal solution to the SCO primal-dual pair (P)-(D) in at most* $\mathcal{O}(\sqrt{r}\log(r/\epsilon))$ *iterations.*

Recall that Lemmas 3 and 4 assert that the analysis performed for the unscaled problem applies directly to the scaled problem as well. More specifically, the analysis of one iteration of our IPM applies to all directions obtained from replacing $x \circ s = \mu e$ with $H_p(x, s) = \mu e$. The following lemma establishes the validity of such a replacement.

**Lemma 15 (Lemma 19 in [32])** *If* $(x, s) \in \text{int}(\mathcal{K}) \times \text{int}(\mathcal{K})$, *then*

$$x \circ s = \mu e \iff \tilde{x} \circ \underline{s} = \mu e.$$

Having demonstrated convergence, the cost of preparing and solving the Newton linear system (13) using a quantum computer is analyzed next.

### Implementing the Orthogonal Subspace System

In order to fully leverage the benefits of the block-encoding framework, the scaling matrix $Q_p$ and its inverse $Q_{p^{-1}}$ are classically computed and subsequently stored in QRAM. This step requires $\mathcal{O}(n^\omega)$ arithmetic operations, where $\omega \in [2, 2.38)$ is the matrix multiplication constant. Then, a factorization of the Newton system is utilized to construct the entire system in time that is polylogarithmic in the dimension $n$.

The following two results establish the factors for the quantum Newton linear system (13).

**Proposition 1** *Let* $\mathcal{E}$ *and* $\mathcal{B}_{\text{Null}(A)}$ *be stored in QRAM, and define*

$$M_1 = \begin{bmatrix} \mathcal{E}\mathcal{B}_{\text{Null}(A)} & 0 \end{bmatrix}.$$

*Then, a* $(\|M_1\|_F, \mathcal{O}(\log n), \zeta_{M_1})$*-block-encoding of* $M_1$ *can be constructed using* $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\zeta}}(1)$ *accesses to the QRAM.*

***Proof*** Noting that $\mathcal{E}$ is stored in QRAM, a $(\|\mathcal{E}\|_F, \mathcal{O}(\log n), \zeta_{\mathcal{E}})$-block-encoding of $\mathcal{E}$ using $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\zeta_E}}(1)$ can be constructed using accesses to the QRAM. Likewise a $(\|\mathcal{B}_{\text{Null}(A)}\|_F, \mathcal{O}(\log n), \zeta_{\mathcal{B}_{\text{Null}(A)}})$-block-encoding of $\mathcal{B}_{\text{Null}(A)}$ is implemented using $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\zeta_{\mathcal{B}_{\text{Null}(A)}}}}(1)$ accesses to the QRAM. Taking the product of our block-encodings, and choosing

$$\zeta_{\mathcal{E}} = \frac{\zeta_{M_1}}{2\|\mathcal{B}_{\text{Null}(A)}\|_F} \text{ and } \xi_{\mathcal{B}_{\text{Null}(A)}} = \frac{\zeta_{M_1}}{2\|\mathcal{E}\|_F},$$

yields a $(\|M_1\|_F, \mathcal{O}(\log n), \zeta_{M_1})$-block-encoding of $M_1$ which can be constructed with at most $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\zeta}}(1)$ accesses to the QRAM, as desired. □

**Proposition 2** *Let* $\mathcal{F}$ *and* $A$ *be stored in QRAM, and define*

$$M_2 = \begin{bmatrix} 0 & -\mathcal{F}A^\top \end{bmatrix}.$$

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*Then, a* $(\|M_2\|_F, \mathcal{O}(\log n), \zeta_{M_2})$*-block-encoding of* $M_2$ *can be constructed using at most* $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\zeta}}(1)$ *accesses to the QRAM.*

***Proof*** The proof follows the exact same steps as the proof of Proposition 1, repeating the argument with $\mathcal{F}$ and $A^\top$. □

**Proposition 3** *Let* $\mathcal{E}, \mathcal{F}, A,$ *and* $\mathcal{B}_{\text{Null}(A)}$ *be stored in QRAM, and consider the quantum Newton linear system coefficient matrix in* (13) *as*

$$M_{NT} = \left[\mathcal{E}\mathcal{B}_{\text{Null}(A)} \; -\mathcal{F}A\right]. \tag{26}$$

*Then, a* $(\|M_{NT}\|_F, \mathcal{O}(\log n), \zeta/(\kappa^2 \log^2 \frac{\kappa}{\zeta}))$*-block-encoding of* $M_{NT}$ *can be constructed using at most* $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\zeta}}(1)$ *accesses to the QRAM.*

***Proof*** Carrying out the calculations shows that (26) corresponds to the Newton linear system. Next, the two following block-encodings are constructed:

$$M_1 = \left[\mathcal{E}\mathcal{B}_{\text{Null}(A)} \; 0\right] \quad M_2 = \left[0 \; -\mathcal{F}A^\top\right],$$

using Propositions 1–2, choosing precision of this step in order to obtain

$$(\|M_1\|_F, \mathcal{O}(\log n), \zeta/(\|M_1\|_F \kappa^2 \log^2 \frac{\kappa}{\zeta}))$$

and $(\|M_2\|_F, \mathcal{O}(\log n), \zeta/(\|M_2\|_F \kappa^2 \log^2 \frac{\kappa}{\zeta}))$-block-encodings, respectively, where $\kappa$ refers to the condition number of (26), here and below. Upon adding these two block-encodings, a

$$(\max\{\|M_1\|_F, \|M_2\|_F\}, \mathcal{O}(\log n), \zeta/(\kappa^2 \log^2 \frac{\kappa}{\zeta}))$$

-block-encoding of (26) is obtained, having used $\widetilde{\mathcal{O}}_{n,\kappa,\frac{1}{\zeta}}(1)$ accesses to the QRAM. □

The following theorem is an immediate consequence of the preceding results.

**Theorem 7** *There is a quantum algorithm that, given*

$$\left|\sigma\mu e - H_p(x,s)\right\rangle$$

*and access to QRAM data structures encoding* $\mathcal{E}, \mathcal{F}, A,$ *and* $\mathcal{B}_{\text{Null}(A)}$*, outputs a state* $\zeta$*-close to* $\left|(\Delta z, \Delta y)^\top\right\rangle$ *in time* $\widetilde{\mathcal{O}}_{d,\kappa,\frac{1}{\xi}}(\kappa)$*, using the Nesterov-Todd direction.*

***Proof*** The result follows from Proposition 3 and Theorem 3. □

### Running Time Analysis

The following result summarizes the overall complexity of the IF-QIPM applied to SCO problems.

**Theorem 8** *Let the desired duality gap be* $\epsilon \in (0, 1)$*. Each iteration* $k$ *of the IF-QIPM in Algorithm 2 requires*

$$\mathcal{O}\left((m+n)\kappa \cdot \text{polylog}\left(m, n, \kappa, \frac{1}{\epsilon}\right)\right)$$

*accesses to a QRAM data structure and* $\mathcal{O}(mn + (m+n)^\omega)$ *classical arithmetic operations. Here,* $\kappa$ *is an upper bound for the condition number of the Newton linear systems that arise over the course of the algorithm, and* $\omega \in [2, 2.38)$ *is the matrix multiplication exponent.*

***Proof*** In every iteration, the NT Newton linear system (13) is prepared and solved, before obtaining classical estimate of the quantum state encoding its solution via a state tomography algorithm. Applying Theorem 5 with precision $\zeta^{(k)}$ chosen according to (24), a classical solution to the (13) can be obtained using

$$T_{NT} = \widetilde{\mathcal{O}}_{m,n,\kappa,\frac{1}{\epsilon}}((m+n)\kappa)$$

accesses to the QRAM and $\widetilde{\mathcal{O}}_{m,n,\kappa,\frac{1}{\epsilon}}(mn)$ arithmetic operations.

The primal and dual search directions $\Delta x$ and $\Delta s$ are classically from the nullspace representation and used to update the solution $(x, y, s)$ to the SCO to prepare the next iteration. The scaling element $p$ and its inverse are also computed classically. Computing $\Delta x$ and $\Delta s$ classically can be accomplished via $\mathcal{O}(mn + n^2)$ arithmetic operations. Likewise, updating the solution is element-wise addition, which requires $\mathcal{O}(m+n)$

arithmetic operations. Lastly, classically computing $p$ can be accomplished using $\mathcal{O}((m+n)^\omega)$ arithmetic operations. In total, this implies

$$\mathcal{O}(mn + n^2 + (m+n)^\omega) = \mathcal{O}(mn + (m+n)^\omega),$$

classical arithmetic operations take place in each of the iterates.

Summarizing, each iteration uses $\widetilde{\mathcal{O}}_{m,n,\kappa,\frac{1}{\epsilon}}((m+n)\kappa)$ accesses to the QRAM and $\mathcal{O}(mn + (m+n)^\omega)$ classical arithmetic operations, which yields the stated per-iteration cost. The proof is complete. □

**Corollary 2** *The IF-QIPM for SCO problems provided in Algorithm 2 requires*

$$\mathcal{O}\left(\sqrt{r}(m+n)\kappa \cdot \text{polylog}\left(n, \kappa, \frac{1}{\epsilon}\right)\right)$$

*accesses to a QRAM data structure and* $\mathcal{O}\left(\sqrt{r}\,(mn + (m+n)^\omega) \cdot \text{polylog}\left(n, \kappa, \frac{1}{\epsilon}\right)\right)$ *classical operations. Here,* $\kappa$ *is an upper bound for the condition number of the Newton linear systems that arise over the course of the algorithm, and* $\omega \in [2, 2.38)$ *is the matrix multiplication exponent.*

***Proof*** The result follows directly from combining Corollary 1 and Theorem 8. □

## Conclusion

This chapter studied and analyzed an inexact-feasible quantum interior point method for SCO. The resulting overall complexity depends on a condition number bound for the Newton linear systems that arise during the algorithm. It is well-known, however, that the condition number of the Newton system coefficient matrices tends toward infinity as optimality is reached, and thus it is of interest to determine approaches that mitigate this dependence for QIPMs.

Alternatively, it may be worthwhile to develop QIPMs that eschew the use of QLSAs entirely. Recent works [1,5] have drawn on an equivalence between simulated annealing [17] and IPMs that make use of the entropic barrier function. Under this framework, the key mechanism is the use of a Markov chain Monte Carlo random walk known as hit-and-run sampling to approximate the gradient and Hessian of the barrier function in every iteration. Simulated annealing and hit-and-run walks have been quantized in application to estimate the volume of convex bodies [7], and the quantum hit-and-run walk achieves a superlinear speedup in $n$ over its classical counterpart.

## See Also

- Quantum Annealing
- Quantum IPMs for Linear Optimization
- Quantum Linear Algebra
- The Quantum Approximate Optimization Algorithm (QAOA) and Quantum Walks.

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