



---

# A Quantum Central Path Algorithm for Linear Optimization

BRANDON AUGUSTINO<sup>1</sup>, JIAQI LENG<sup>2,6,8</sup>, GIACOMO NANNICINI<sup>3</sup>, TAMÁS  
TERLAKY<sup>4,5</sup>, AND XIAODI WU<sup>7,8</sup>

<sup>1</sup>Sloan School of Management, Massachusetts Institute of Technology USA

<sup>2</sup>Simons Institute for the Theory of Computing and Department of Mathematics, University of  
California, Berkeley USA

<sup>3</sup>Department of Industrial and Systems Engineering, University of Southern California USA

<sup>4</sup>Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA, 18015  
USA

<sup>5</sup>Quantum Computing and Optimization Lab, Lehigh University, Bethlehem, PA, 18015 USA

<sup>6</sup>Department of Mathematics, University of Maryland, College Park, MD, USA

<sup>7</sup>Department of Computer Science, University of Maryland, College Park, MD, USA

<sup>8</sup>Joint Center for Quantum Information and Computer Science, University of Maryland, College  
Park, MD, USA

ISE Technical Report 24T-022



# A quantum central path algorithm for linear optimization

Brandon Augustino<sup>\*1</sup>, Jiaqi Leng<sup>2,6,8</sup>, Giacomo Nannicini<sup>3</sup>, Tamás Terlaky<sup>4,5</sup>, and Xiaodi Wu<sup>7,8</sup>

<sup>1</sup>*Sloan School of Management, Massachusetts Institute of Technology*

<sup>2</sup>*Simons Institute for the Theory of Computing and Department of Mathematics, University of California, Berkeley*

<sup>3</sup>*Department of Industrial and Systems Engineering, University of Southern California*

<sup>4</sup>*Department of Industrial and Systems Engineering, Lehigh University*

<sup>5</sup>*Quantum Computing and Optimization Lab, Lehigh University*

<sup>6</sup>*Department of Mathematics, University of Maryland, College Park*

<sup>7</sup>*Department of Computer Science, University of Maryland, College Park*

<sup>8</sup>*Joint Center for Quantum Information and Computer Science, University of Maryland*

## Abstract

We propose a novel quantum algorithm for solving linear optimization problems by quantum-mechanical simulation of the central path. While interior point methods follow the central path with an iterative algorithm that works with successive linearizations of the perturbed KKT conditions, we perform a single simulation working directly with the nonlinear complementarity equations. This approach yields an algorithm for solving linear optimization problems involving  $m$  constraints and  $n$  variables to  $\varepsilon$ -optimality using  $\mathcal{O}\left(\sqrt{m+n} \cdot \frac{R_1}{\varepsilon}\right)$  queries to an oracle that evaluates a potential function, where  $R_1$  is an  $\ell_1$ -norm upper bound on the size of the optimal solution. In the standard gate model (i.e., without access to quantum RAM) our algorithm can obtain highly-precise solutions to LO problems using at most

$$\mathcal{O}\left(\sqrt{m+n} \cdot \text{nnz}(A) \frac{R_1}{\varepsilon}\right)$$

elementary gates, where  $\text{nnz}(A)$  is the total number of non-zero elements found in the constraint matrix.

---

\*Corresponding Author: [baug@mit.edu](mailto:baug@mit.edu)

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Related work . . . . .	5
1.2	Contributions . . . . .	6
1.3	Technical summary and roadmap . . . . .	8
<b>2</b>	<b>Preliminaries</b>	<b>11</b>
2.1	Interior Point Methods for Linear Optimization . . . . .	12
2.1.1	Primal and dual LO problems . . . . .	12
2.1.2	The self-dual embedding model . . . . .	13
2.1.3	The central path and its neighborhood . . . . .	14
2.2	Quantum adiabatic theorem for unbounded Hamiltonians . . . . .	15
2.3	Quantum algorithms for Schrödinger equations . . . . .	16
<b>3</b>	<b>A quantum algorithm that traces the central path</b>	<b>17</b>
3.1	Quantum representation of the central path . . . . .	17
3.2	Quantum simulation of the central path . . . . .	18
3.3	Quantum central path algorithm . . . . .	19
3.4	Constructing an evaluation oracle for $f_\mu$ . . . . .	22
3.5	Comparison to existing LO solvers . . . . .	25
<b>4</b>	<b>Conclusion and outlook</b>	<b>26</b>
<b>A</b>	<b>Technical lemmas</b>	<b>34</b>
<b>B</b>	<b>Proof of Proposition 1</b>	<b>35</b>
<b>C</b>	<b>Additional Proofs</b>	<b>36</b>

# 1 Introduction

Given a matrix  $A \in \mathbb{R}^{m \times n}$  and vectors  $b \in \mathbb{R}^m$  and  $c \in \mathbb{R}^n$ , we are interested in solving the following Linear Optimization (LO) problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0, \end{aligned} \tag{P}$$

where  $x \in \mathbb{R}^n$  is the *primal* variable. We assume  $\|A_{1,\cdot}\|, \dots, \|A_{m,\cdot}\|, \|b\|, \|c\| \leq 1$  for normalization purposes, where  $A_{i,\cdot}$  denotes the  $i$ -th row of  $A$ . Normalization of the input data is a common assumption in the literature. The primal problem (P) has an associated *dual* problem, given by

$$\begin{aligned} \max_{y \in \mathbb{R}^m} \quad & b^\top y \\ \text{subject to} \quad & A^\top y \leq c, \\ & y \geq 0, \end{aligned} \tag{D}$$

where  $A^\top \in \mathbb{R}^{n \times m}$  and  $y \in \mathbb{R}^m$  is the dual variable. Whenever the *canonical* primal and dual LO problems (P)-(D) are feasible, strong duality holds: there exists a primal-dual optimal solution  $(x_*, y_*)$  to (P)-(D) with vanishing duality gap, i.e.,  $c^\top x_* - b^\top y_* = 0$ .

Linear Optimization is routinely used to model and solve fundamental problems in economics, finance, and engineering. LO was brought to the forefront of computer science and applied mathematics following the advent of digital computer along with Dantzig's Simplex method [Dan48, DOW<sup>+</sup>55]. The first polynomial time algorithm for LO was Kachiyan's ellipsoid method [Kha80], though it failed to elicit an efficient practical implementation. Shortly thereafter, Karmarkar introduced his projective method for LO problems in the seminal work [Kar84], which improved the complexity of the ellipsoid method. Although interior point algorithms had been studied since at least the 1950's [Dik67, Dik74, FM64, FM90, Fri54, Fri55, Fri56], Karmarkar's algorithm was the first to run in polynomial time, and the so-called Interior Point Method (IPM) revolution was underway.

Enhancements over Karmarkar's algorithm followed closely; Renegar [Ren88] improved the iteration complexity, Vaidya [Vai87] and Gonzaga [Gon89] simultaneously reduced the overall complexity, and Nesterov and Nemirovskii [NN88, NN94] introduced the paradigm of *self-concordant barrier functions*. Today, the IPM literature constitutes a vast and celebrated line of research, making it a fruitless endeavor to provide a comprehensive review here. Rather, we point the reader to the excellent texts [Ren01, RTV05, Ter13, Wri97, Ye11] and the references therein. Many recent developments can be found in [vdB20, vdBLSS20, CLS21, JKL<sup>+</sup>20, JSWZ20, LS14, LS15, Vla23], and the current state of the art running time results are obtained by the randomized IPM of Cohen, Lee and Song [CLS21] and the deterministic IPM from van den Brand [vdB20]. These IPMs can solve LO problems to precision  $\varepsilon \in (0, 1)$  in  $\tilde{\mathcal{O}}_{\frac{2}{\varepsilon}}((m+n)^\omega)$  time<sup>1</sup>, where  $\omega \in [2, 2.38)$  is the exponent of the running time for matrix multiplication (i.e., we can multiply two  $n \times n$  matrices in time  $\mathcal{O}(n^\omega)$ ),

---

<sup>1</sup>The authors in [vdB20, CLS21] solve the standard form LO problem which requires the assumption  $\text{rank}(A) = m$  (and in particular,  $m \leq n$ ), so their complexity result is typically written as  $\tilde{\mathcal{O}}_{\frac{2}{\varepsilon}}(n^\omega)$ . We make the dependence on  $m$  explicit because this assumption is unnecessary in the canonical (or, symmetric) form we consider: introducing  $m$  slack variables transforms the canonical problem into standard form, and the augmented coefficient matrix is

and the notation  $\tilde{\mathcal{O}}_{\alpha,\beta}(f(x))$  suppresses polylogarithmic factors in  $f(x)$ ,  $\alpha$  and  $\beta$  appearing in the overall running time.

One can view IPMs as a homotopy approach for Newton’s method. IPMs are initialized to a strictly feasible point  $x^{(0)} \in \{x \in \mathbb{R}^n : Ax > b, x > 0\}$ , and approximately track an analytic curve known as the *central path* towards optimality. Specifically, letting  $\mu > 0$  and defining

$$\phi : \{x \in \mathbb{R}^n : Ax > b, x > 0\} \mapsto \mathbb{R},$$

a generic path-following scheme solves a sequence of *barrier problems* of the form

$$x(\mu) := \arg \min_{\{x \in \mathbb{R}^n : Ax > b, x > 0\}} c^\top x + \mu\phi(x), \quad (1)$$

using Newton’s method. The *barrier function*  $\phi$  is chosen to be *self-concordant*: the value  $\phi(x)$  diverges to  $\infty$  upon approaching the boundary of  $\{x \in \mathbb{R}^n : Ax > b, x > 0\}$ , and at a high level, the norms of higher-order derivatives of  $\phi$  can be bounded in terms of its Hessian.<sup>2</sup> The value of  $\mu$  is decreased in each iteration, and the optimal solution is reached upon tracing the central path  $\{x(\mu) : \mu > 0\}$  as  $\mu \rightarrow 0$ . The use of the self-concordant barrier function  $\phi$  in (1) ensures that we are always in the region of rapid local convergence enjoyed by Newton’s method.

It is also standard in the literature to define the central path as the set of minimizers associated with

$$x(t) := \arg \min_{\{x \in \mathbb{R}^n : Ax > b, x > 0\}} tc^\top x + \phi(x). \quad (2)$$

The value of  $t$  is increased in each iteration, and so tracing the central path  $\{x(t) : t > 0\}$ , we approach an optimal solution to (P) as  $t \rightarrow \infty$ . Note however, there is no meaningful difference between (1) and (2). Indeed, taking  $\mu = 1/t$  and applying Newton’s method to (1) yields the same sequence of minimizers as one would obtain from applying Newton’s method to (2). Yet, the perspective offered by (1) can be attractive for both theoretical and practical reasons; tracking  $t \rightarrow \infty$  in (2) would require computation involving huge numbers.

In every iteration of the classical IPM, one must obtain the solution to a linearized set of perturbed Karush-Kuhn-Tucker (KKT) optimality conditions known as the Newton linear system. The computation of this so-called *Newton step* is usually performed by solving dense linear systems of equations on a very large scale. The IPMs that achieve the best complexity results [vdB20, CLS21] perform this step exactly in the first iteration, and subsequently utilize sophisticated data structures to maintain an approximation of the Hessian inverse, amortizing the cost of solving the linear systems over the run of the algorithm. To the best of our knowledge, the improved theoretical guarantees of the IPMs in [vdB20, CLS21] have not led to a practical implementation at the time of writing. Additionally, while the previous decade saw improvements in the iteration complexity for the specific applications of maximum flow [M13, M16, AKL<sup>+</sup>24], minimum cost flow [CMSV17], matrix scaling [CMTV17], and  $\ell_p$ -regression [BCLL18], whether the asymptotic iteration count of  $\mathcal{O}(\sqrt{n} \log(1/\varepsilon))$  (or,  $\mathcal{O}(\sqrt{m} \log(1/\varepsilon))$  if  $m \ll n$ ) can be reduced in the general setting remains an important open question in optimization. Quantum IPMs (QIPMs) currently found in the literature

---

of full row-rank. We also remark that the complexity of the algorithm in [CLS21] is more accurately expressed as  $\tilde{\mathcal{O}}_{\frac{1}{\varepsilon}} \left( n^\omega + n^{2.5 - \frac{\alpha}{2}} + n^{2 + \frac{1}{6}} \right)$ , where  $\alpha$  is the dual-exponent of  $\omega$ . This simplifies to  $\tilde{\mathcal{O}}_{\frac{1}{\varepsilon}}(n^\omega)$  for the current value  $\omega \approx 2.38$ . Jiang et al. [JSWZ20] improved the  $n^{2 + \frac{1}{6}}$  term to  $n^{2 + \frac{1}{18}}$ .

<sup>2</sup>To put it another way, the second-order Taylor series approximation of (1) is highly accurate.

seek to accelerate the solution of the Newton linear system via quantum subroutines, hence they do not reduce the iteration count, and it is unclear whether or not this approach can provide an overall speedup. We review these works in detail next.

## 1.1 Related work

QIPMs were first introduced by Kerenidis and Prakash [KP20], who proposed a quantum algorithm for solving LO and Semidefinite Optimization (SDO) problems. The main idea of their approach is to solve the Newton linear system at each iterate using a quantum linear systems algorithm (QLSA) [HHL09, CW12, CGJ19], and obtain a classical estimate of the resulting quantum state via quantum state tomography. This approach limits one to inexactly solving the Newton system at each iterate, and therefore, additional safeguards need to be taken in order to guarantee convergence. To reconcile the quantum noise introduced into the Newton steps, Augustino et al. [ANTZ23] proposed two convergent QIPMs for SDO and LO. The first closely quantized the classical Inexact-Infeasible IPM of [TK02]. The second framework is a novel Inexact-Feasible QIPM (IF-QIPM) that uses a *nullspace representation* of the Newton system to ensure the sequence of iterates maintain primal-dual feasibility, in spite of using an inexact linear system subroutine. These ideas were specialized to LO by Mohammadisiahroudi et al. [MFT22, MFT24], who use the iterative refinement algorithm for LO from Gleixner et al. [GSW12, GSW16, GS20] to exponentially improve the dependence on  $\varepsilon^{-1}$ , the inverse precision to which we seek to solve the primal and dual LO problems (P)-(D).

While quantizing IPMs in this manner has led to polynomial speedups in the problem dimension  $n$ , the current approach has two major drawbacks: (i) existing QIPMs assume a strong input model (i.e., the quantum RAM model) that is not always justified in a practical setting, and speedups are heavily dependent on this model; (ii) the iterative uses of QLSAs rely on an expensive tomography procedure, prohibiting a conclusive overall speedup. While it is possible that there is a gap between the algorithm’s theoretical worst-case running time and its practical performance, the latter cannot yet be studied: the data structures and computational primitives utilized by existing QIPMs are beyond the capabilities of near-term quantum devices. A detailed resource analysis by Dalzell et al. [DCS<sup>+</sup>23] for the specific application of portfolio optimization found that a QIPM based on the combined use of QLSA and tomography would require a  $T$ -gate count of  $10^{28}$  when  $n = 100$ . Informally, this implies that it would take a quantum computer millions of years to solve a problem that could be solved in seconds on a personal laptop. Thus, QIPMs based on the QLSA paradigm do not provide a convincing end-to-end speedup over classical IPMs, and their estimated resource cost is a negative result in relation to their viability even on large-scale devices.

A day before the first version of this paper appeared on the arXiv, Apers and Gribling [AG23] gave a QIPM that does not make use of QLSAs, thereby avoiding dependence on condition number. This framework leads to asymptotic speedups over classical IPMs for “tall” LO problems (i.e., for LO problems with many redundant constraints), in which the number of constraints is much larger than the number of variables, i.e.,  $m \gg n$ . That said, there is no speedup over classical IPMs unless  $m = \Omega(n^{10})$ , and the authors assume that each entry of the problem data of  $(A, b, c)$  is of the order  $\mathcal{O}(\text{polylog}(m, n))$ : we do not have this assumption. Like earlier QIPMs, their framework relies on QRAM and block-encodings, and due to results from Clader et al. [CDS<sup>+</sup>22], the resources needed to block-encode classical data are a key contributor to the resource estimates of QIPMs [DCS<sup>+</sup>23].

More generally, the use of quantum subroutines in QIPMs only serves as an attempt to accelerate a single step of the classical IPM. An inherent drawback of this approach is that QIPMs produce a sequence of iterates that exactly mimics the classical IPM trajectory, and therefore lack the potential

to exploit quantum effects or improve on the worst-case iteration complexity bound. This raises the question as to whether there exists a “more naturally quantum” algorithm that solves optimization problems by tracking the central path.

In the recent work [LHLW23], Leng, Hickman, Li and Wu proposed a truly quantum analogue to classical (accelerated) gradient descent named *Quantum Hamiltonian Descent* (QHD). Their work can be viewed as a quantization of the Bregman-Lagrangian framework for analyzing the continuous-time dynamics of gradient descent introduced by Wibisono, Wilson, and Jordan [WWJ16]. The QHD framework admits an implementation on current quantum devices; this is a (practical) advantage over existing QIPMs, for which the underlying quantum primitives are out of reach for current quantum computers. The guiding principle of QHD is to first recast the task of optimization as a dynamical system, and then quantize the resulting continuous-time dynamics. This novel approach offers a new paradigm for the design of quantum algorithms for optimization, which previously, amounted to using quantum subroutines to accelerate a single step of a classical optimization algorithm. Interestingly, the study of dynamical systems arising from IPMs was an active area of research following the introduction of Karmarkar’s algorithm in 1984. Many works [Ans88, BL89a, BL89b, Fay91, Fay95, Meg89] studied the trajectories generated by the so-called *Newton barrier flow*. Megiddo [Meg89] and Bayer and Lagarias [BL89a, BL89b] provided (classical) Lagrangian and Hamiltonian dynamical systems for path-following methods that use a logarithmic barrier function. Faybusovich [Fay95] later generalized these ideas to convex optimization. Yet, the design and complexity analysis of an algorithm based on these ideas was not considered in these works: although these papers precisely characterize the central path, these efforts did not translate into new algorithms.

## 1.2 Contributions

We use a previously unexplored connection between the central path and the Schrödinger equation to develop a new quantum algorithm for LO, that we call the *Quantum Central Path Method* (QCPM). This is achieved by proposing a 1-parameter family of Hamiltonian operators over the positive orthant that encodes the behavior of the central path. Specifically, we show that one can approximately follow the central path by simulating a Schrödinger equation associated to a certain Hamiltonian. The ground state of this Hamiltonian encodes a squeezed Gaussian distribution centered at an  $\varepsilon$ -optimal solution to the primal dual LO pair (P)-(D). Our main result is informally stated as follows:

**Theorem 1** (Main result, informal). *Let  $\mathcal{H}(\mu(t))$  be a time-dependent quantum Hamiltonian with potential function  $f(\mu(t))$ . Suppose that for every value of  $\mu(t)$ , the ground state of  $\mathcal{H}(\mu(t))$  encodes a probability distribution centered on the central path of (P)-(D). For  $\varepsilon > 0$  there is a quantum algorithm that returns classical vectors  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$  satisfying:*

$$\begin{aligned} A_{i,\cdot} x &\leq b_i \quad \forall i \in [m], \quad x \geq 0, \\ \left(A^\top\right)_{\cdot,i} y &\leq c_i \quad \forall i \in [n], \quad y \geq 0, \end{aligned}$$

and

$$c^\top x - b^\top y \leq \varepsilon.$$

The algorithm can be implemented using

$$\mathcal{O}\left(\sqrt{m+n} \cdot \frac{R_1}{\varepsilon}\right)$$

queries to an evaluation oracle for  $f$ . If the problem data  $(A, b, c)$  is stored in binary, one can implement an evaluation oracle for  $f$  in the standard gate model using  $\mathcal{O}(\text{nnz}(A))$  elementary gates.

The QCPM inherits many desirable properties from IPM theory (such as provable convergence to strictly complementary solutions), while only computing zero order information on the potential of the Hamiltonian. We emphasize that the QCPM, unlike (Q)IPMs, is not an iterative algorithm: the desired solution can be obtained in *one shot* by measuring the final state. This could potentially provide a significant advantage over the state of the art (Q)IPMs, which require  $\mathcal{O}(\min\{\sqrt{m}, \sqrt{n}\} \log(\frac{1}{\varepsilon}))$  iterations to reach an  $\varepsilon$ -optimal solution. Of course simulating the Schrödinger equation does not come for free, and this determines the complexity of our algorithm. For this simulation task we use a slightly improved version of a result in [CLL<sup>+</sup>22], tailored to our specific setup: this provides a fast quantum algorithm that has near-optimal dependence (with respect to the simulation task) on the dimension of the problem and the simulation error.

The idea for the QCPM is rooted in the early works on the Newton barrier flow discussed above, with some significant differences: we use the *self-dual embedding* model [YTM94] to guarantee an easy-to-prepare ground state for an initial Hamiltonian, and this is crucial to the eventual application of the adiabatic theorem to show convergence to the optimal solution (i.e., ground state of the final Hamiltonian). We also work *directly* with the nonlinear complementarity equations in the perturbed KKT conditions that define the central path, rather than using a local linearization as is done in IPMs. To the best of our knowledge, the QCPM is the first algorithm that takes this perspective.

Our contribution is a  $\mathcal{O}(\sqrt{m+n} \cdot \frac{R_1}{\varepsilon})$ -query algorithm for solving linear optimization problems involving  $m$  constraints and  $n$  variables to  $\varepsilon$ -optimality, where  $R_1$  is an  $\ell_1$ -norm upper bound on the size of the optimal solution. An implementation of our algorithm in the standard gate model (i.e., without access to QRAM) can identify an  $\varepsilon$ -optimal solution of an LO problem using at most

$$\mathcal{O}\left(\sqrt{m+n} \cdot \text{nnz}(A) \frac{R_1}{\varepsilon} \text{polylog}\left(m, n, \frac{1}{\delta}\right)\right)$$

elementary gates, where  $\text{nnz}(A)$  is the total number of non-zero entries found in  $A$  and  $1 - \delta$  is the probability of success. We stress that our algorithm does not utilize QRAM, and the stated complexity result only requires access to the binary representation of  $(A, b, c)$ .

To contextualize our results, we provide a comparison of our algorithm to the current state of the art algorithms for solving LO problems in both the classical and quantum models of computation in Table 1. A more detailed discussion is given in Section 3.5; here we report a summary. For the quantum algorithms, we report both query and gate complexities to highlight the impact of the QRAM input model, wherein query and gate complexity often coincide.

The QCPM achieves polynomial speedups in  $m$  and  $n$  over the state of the art classical and quantum IPMs whenever  $\text{nnz}(A) < (m+n)^{\omega-\frac{1}{2}}$ . Without further enhancements to our framework, an end-to-end speedup over these algorithms is only possible in the low-precision regime, and requires  $\text{nnz}(A)R_1 < (m+n)^{\omega-\frac{1}{2}} \cdot \text{polylog}(m, n)$ . These regimes for speedup are *not* reliant on access to a classical-write/quantum-read RAM (QRAM): our algorithm only requires access to the natural binary description for the LO problem data  $(A, b, c)$ . Like the QIPM from Apers and Gribling [AG23], our algorithm avoids a condition number dependence. We also emphasize that the QIPM found in [AG23] assumes access to QRAM.

Another approach to solve linear optimization problems, besides IPMs, is the Primal-Dual Hybrid Gradient (PDHG) algorithm with restarts [AHL23]. The PDHG algorithm achieves polylogarithmic dependence on precision using restarts. The number of restarts depends on the Hoffman



Table 1: Complexity to solve the primal-dual pair (P)-(D) to precision  $\varepsilon$ 

Classical Algorithms		Time complexity			
IPM	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\sqrt{m}(\text{nnz}(A) + n^2))$	[LS15]			
IPM	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}((m+n)^\omega)$	[CLS21, vdB20]			
PDHG	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\kappa \cdot \text{nnz}(A))$	[AHL23]		( $\kappa$ is the Hoffman constant, see discussion.)	
Quantum Algorithms		Query complexity		Gate complexity	QRAM
QMMWU	$\tilde{\mathcal{O}}(\sqrt{m+n}R_1^{2.5}\varepsilon^{-2.5} + R_1^3\varepsilon^{-3})$	[BGJ <sup>+</sup> 23, GJLW24]	$\tilde{\mathcal{O}}(\sqrt{m+n}R_1^{2.5}\varepsilon^{-2.5} + R_1^3\varepsilon^{-3})$	[BGJ <sup>+</sup> 23, GJLW24]	✓
QIPM	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\sqrt{mn}^5)$	[AG23, Section 7]	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\sqrt{mn}(n^{6.5}s^2 + n^{\omega+2}))$	[AG23, Theorem 1.1]	✓
QCPM (this work)	$\tilde{\mathcal{O}}_{m,n,\alpha}(\sqrt{m+n}R_1\varepsilon^{-1})$	Theorem 5	$\tilde{\mathcal{O}}_{m,n,\alpha}(\sqrt{m+n}\text{nnz}(A)R_1\varepsilon^{-1})$	Corollary 1	✗

constant<sup>3</sup>  $\kappa$  of the KKT system associated to (P)-(D). This constant is notoriously difficult to compute and rigorously bound [Peñ24]; for general problems, it is often left “as is” and not expressed as a function of the input size. Our algorithm’s performance matches that of PDHG when  $\kappa = \mathcal{O}(\frac{\sqrt{m+n}R_1}{\varepsilon})$ , but we are not aware of explicit bounds for  $\kappa$  in the general case so a direct comparison with our algorithm is difficult.

Like the QCPM, quantum algorithms for zero-sum games [vAG19, BGJ<sup>+</sup>23, GJLW23] also achieve sublinear running times in  $m$  and  $n$ , and depend polynomially in  $R_1$  and the inverse precision. There are notable differences in the query and input models used by both algorithms. The queries made by the QCPM are evaluation queries for a potential function, whereas algorithms for zero-sum games make queries to block-encodings of the problem data. As for the input model, the algorithms found in [vAG19, BGJ<sup>+</sup>23, GJLW23] rely on QRAM, while the QCPM does not. Algorithms for zero-sum games employ different definitions of optimality than (Q)IPMs and our QCPM, and see Section 3.5 for details. With these differences in mind, our algorithm achieves super-linear savings in  $\frac{R_1}{\varepsilon}$  over the current state of the art [BGJ<sup>+</sup>23, GJLW23].

The QCPM proposed in this paper provides (for the first time, as far as we are aware) convincing evidence that an end-to-end quantum speedup for LO is possible even in the circuit model without QRAM. While this would require further enhancements, we view our framework as promising since it has no direct classical analogue. We view the main contribution of our paper as conceptual, rather than technical: the technical ingredients for our algorithm and its analysis were largely present in the literature, but we believe that the QCPM idea could provide a new, potentially interesting avenue for constrained convex optimization on quantum computers. It may be useful to perform a detailed resource analysis of the QCPM framework, in the same spirit as the analysis of Dalzell et al. [DCS<sup>+</sup>23] for QIPMs; we leave this for future work.

Next, we provide a more detailed summary of the technical overview of the paper’s main results.

### 1.3 Technical summary and roadmap

Hamiltonian simulation is a fundamental task in quantum computation as it separates the computational power between quantum and classical [Osb12]. For a given time-dependent Hamiltonian operator  $\mathcal{H}(t)$  and an initial state  $|\psi_0\rangle$ , the Hamiltonian simulation task is to simulate the dynamics governed by the *Schrödinger equation*:

$$i \frac{d}{dt} |\psi(t)\rangle = \mathcal{H}(t) |\psi(t)\rangle, \text{ subject to } |\psi(0)\rangle = |\psi_0\rangle,$$

where  $i$  denotes the imaginary unit. In this paper, we propose a Quantum Central Path Method (QCPM) relying on the efficient simulation of Schrödinger dynamics on quantum computers (see

<sup>3</sup>This is not the same constant as the condition number used in IPMs.

Theorem 4). Specifically, we design a quantum Hamiltonian (known as the central-path Hamiltonian) that *encodes* the behavior of the central path in its ground state. By preparing an initial state centered at a point on the central path and simulating the associated Schrödinger equation for a sufficiently long evolution time  $T$ , the quantum adiabatic theorem (Theorem 3) guarantees that the quantum register remains in the instantaneous ground state of the central-path Hamiltonian. At the end of the Hamiltonian simulation, we obtain an (approximate) optimal solution to the primal-dual LO pair (P)-(D) by simply measuring the final state.

Our QCPM shares some features with Quantum Hamiltonian Descent (QHD) [LHLW23] in that (i) both can be regarded as *quantizations* of classical dynamical processes (path-following in QCPM, accelerated gradient descent in QHD), (ii) both are formulated as quantum dynamics governed by Schrödinger equations, and (iii) in both cases the ground state of the Hamiltonian operator encodes the solution to an optimization problem. The central-path Hamiltonian can be interpreted as a *global* linearization of the central path: this is essentially different from any existing QIPMs, where each Newton step represents a *local* linearization of the central path. Consequently, our algorithm can be more efficient because there is no need for iterative uses of QLSA and state tomography.

The adiabatic theorem is predicated on the assumption that we start in a ground state of the initial Hamiltonian  $\mathcal{H}(0)$ . In general, this ground state is hard to prepare because it would require the exact location of the starting point on the central path for *any* LO problem, a problem known as hard as the linear optimization itself. We overcome this challenge by casting the primal-dual pair (P)-(D) as a slightly larger problem (of dimension  $m + n + 2$ ) known as the *self-dual embedding model* (first proposed by Ye, Todd and Mizuno [YTM94]), a standard approach employed by classical IPMs both in theory and in practice. This allows us to make a step towards providing a practical algorithm because the self-dual embedding model always admits the all-ones vector as an interior-feasible solution. By using the self-dual embedding model as a foundation for the design of our central-path Hamiltonian, we ensure that the ground state of the initial Hamiltonian is trivial to prepare. A review of the self-dual embedding model and some important lemmas are provided in Section 2.1.2.

Under the self-dual embedding framework, the central path is (uniquely) characterized by element-wise complementarity and positivity constraints on the variable  $z$  and its associated slack  $s(z)$ :

$$\mathcal{C} := \{z(\mu) : \mu \in (0, 1]\} = \{z \in \mathbb{R}^{m+n+2} : z_i \cdot s(z)_i = \mu, z_i, s(z)_i > 0, \forall i \in \{1, \dots, m+n+2\}\}.$$

Classical IPMs track the central path by iteratively applying Newton’s method, which requires locally linearizing the nonlinear complementarity equation

$$z_i \cdot s(z)_i = \mu \quad \forall i \in \{1, \dots, m+n+2\},$$

and successively decreasing  $\mu$  by a constant factor after each Newton step. We propose a different approach: a time-dependent Schrödinger operator on the positive orthant that directly incorporates the nonlinear complementarity equation as a potential field,

$$\mathcal{H}(\mu(t)) := -\frac{h(t)}{2}\nabla^2 + \|z \odot s(z) - \mu(t)e\|^2,$$

where for a fixed  $t > 0$ , the global minimum of the potential function  $\|z \odot s(z) - \mu(t)e\|^2$  corresponds to a point on the central path. Here,  $h(t)$  and  $\mu(t)$  are monotonically decreasing functions in  $t$ ,  $e$  is

the all-ones vector of dimension  $m+n+2$  and  $\odot$  denotes the Hadamard (or, element-wise) product. The central-path Hamiltonian  $\mathcal{H}(\mu(t))$  is inspired by the canonical time-independent Hamiltonian in quantum mechanics:

$$\mathcal{H} := -\nabla^2 + V,$$

where  $\nabla^2$  denotes the Laplacian operator in  $\mathbb{R}^n$ , accounting for kinetic energy, and  $V: \mathbb{R}^n \mapsto \mathbb{R}$  is a potential function, accounting for potential energy. In our time-dependent Hamiltonian  $\mathcal{H}(\mu(t))$ , the role of  $h(t)$  is to control the variance of the probability distribution encoded in the ground state of the central-path Hamiltonian. In the usual setting, a (small) fixed value  $\hbar$  known as *Planck's constant* is used, but this choice is insufficient to guarantee convergence in our setting: by decreasing  $h(t)$  over the course of the time evolution, we ensure that we concentrate on a probability mass centered at the optimal solution. The detailed construction of the central-path Hamiltonian is provided in Section 3.1. Properly designing  $h(t)$  is a crucial technical insight of our approach, and  $\mu(t)$  is constructed using ideas introduced in [AL22].

Equipped with the Hamiltonian  $\mathcal{H}(\mu(t))$  defined above, the remaining challenge is two-fold: we need to (i) verify that simulating the Schrödinger equation associated to our central-path Hamiltonian does in fact solve our optimization problem; and (ii) rigorously analyze the cost of doing so. We establish the correctness of our approach through a combination of results already found in the literature and new results of our own. In Section 3.1, we apply the harmonic approximation theory of Schrödinger operators in order to certify that, for fixed  $\mu \in (0, 1]$ , the ground state of the central-path Hamiltonian gives rise to a Gaussian distribution centered at the corresponding point on the central path  $z(\mu)$ . This suggests another interesting connection to classical IPM theory: the harmonic approximation of the low-energy spectrum is obtained via a second-order Taylor series approximation of  $\mathcal{H}(\mu)$  at  $z(\mu)$ , which is precisely how one applies Newton's method to barrier problems of the form (1) in the classical IPM framework. We also provide a lower bound on the minimum spectral gap: this is an important component of the complexity of simulating our Schrödinger equation, and we derive the lower bound by again interpreting the system governed by  $\mathcal{H}(\mu(t))$  as a quantum harmonic oscillator, whose spectral gap is well understood. In Section 3.2, we demonstrate that properly defining  $h(t)$  and  $\mu(t)$  allows one to solve (P)-(D) via a *single* simulation of our Schrödinger equation: the final output is a classical description of an  $\varepsilon$ -optimal solution, and no intermediate measurements are required. This is in contrast with existing quantum algorithms for LO [vAG19, ANTZ23, BGJ<sup>+</sup>23, KP20, MFT24], which require either  $\mathcal{O}(\sqrt{n} \log \frac{1}{\varepsilon})$  or  $\mathcal{O}\left(\frac{\log(n)}{\varepsilon^2}\right)$  calls to a Hamiltonian simulation subroutine (to either solve quantum linear systems or prepare Gibbs states).

We analyze the complexity of the QCPM in Section 3.3. To achieve our complexity result, we combine the requirements for our convergence guarantees with the cost of the simulation algorithm from [CLL<sup>+</sup>22]. We prove that the QCPM returns a classical description of an  $\varepsilon$ -precise solution to (P)-(D) using at most

$$\tilde{\mathcal{O}}_{m,n,\frac{1}{\delta}}\left(\sqrt{m+n} \cdot \text{nnz}(A) \frac{R_1}{\varepsilon}\right)$$

elementary gates. The overall complexity suggests that in the low-precision regime, the size of a potential quantum speedup is largely determined by the problem sparsity and solution size. We stress that the speedups achieved here are independent of QRAM. We also believe our techniques should readily generalize to other classes of constrained convex optimization problems.

The rest of this paper is organized in the following manner. In Section 2 we define some notation and review the theory of classical interior point methods applied to solving LO problems,

before providing results on the adiabatic theorem for unbounded Hamiltonians and algorithms for simulating Schrödinger equations. Section 3 concerns the design and analysis of the Quantum Central Path Method, and compares our overall complexity result to the current state-of-the-art. Section 4 concludes the paper.

## 2 Preliminaries

We distinguish the quantity  $a$  to the  $k$ -th power and the value of  $a$  at iterate  $k$  using round brackets, writing  $a^k$  and  $a^{(k)}$  to denote these quantities, respectively. We write  $[n]$  to represent the set of elements  $\{1, \dots, n\}$ .

We denote the  $i$ -th element of a vector  $x \in \mathbb{R}^n$  by  $x_i$  for  $i \in [n]$ , and the  $ij$ -th element of a matrix  $A \in \mathbb{R}^{m \times n}$  by  $A_{ij}$  for  $i \in [m]$  and  $j \in [n]$ . To refer to the  $i$ -th row of a matrix  $A$ , we write  $A_{i\cdot}$  and write  $A_{\cdot j}$  when referring to its  $j$ -th column. For a vector  $x \in \mathbb{R}^n$ , the matrix  $\text{diag}(x) \in \mathbb{R}^{n \times n}$  takes the values of  $x$  along its diagonal and zero elsewhere. For vectors  $u, v \in \mathbb{R}^n$ ,  $u \odot v$  denotes their *Hadamard* (or, entry-wise) product:

$$u \odot v = \begin{bmatrix} u_1 \cdot v_1 \\ \vdots \\ u_n \cdot v_n \end{bmatrix}.$$

We write  $0_n \in \mathbb{R}^n$  when referring to an all-zeros vector of length  $n$ , and  $0_{m \times n} \in \mathbb{R}^{m \times n}$  denotes the  $m \times n$  all-zeros matrix.

The smallest and largest singular values of a matrix  $A$  are denoted  $\sigma_{\min}(A), \sigma_{\max}(A)$ , and the smallest and largest eigenvalues are denoted  $\lambda_{\min}(A), \lambda_{\max}(A)$ . The operator norm of  $A$  is defined as  $\|A\| := \sigma_{\max}(A)$ .

We let  $\mathcal{S}_+^n$  and  $\mathcal{S}_{++}^n$  represent the spaces of symmetric positive semidefinite, and symmetric positive definite matrices in  $\mathbb{R}^{n \times n}$ , respectively. For  $U, V \in \mathcal{S}^n$ , we write  $U \succeq V$  ( $U \succ V$ ) to indicate that the matrix  $U - V$  is symmetric positive semidefinite (symmetric positive definite), i.e.,  $U - V \in \mathcal{S}_+^n$  ( $U - V \in \mathcal{S}_{++}^n$ ).

### Order estimates

We define  $\mathcal{O}(\cdot)$  as

$$f(x) = \mathcal{O}(g(x)) \iff \exists \ell \in \mathbb{R}, \alpha \in \mathbb{R}_+, \text{ such that } f(x) \leq \alpha g(x) \quad \forall x > \ell.$$

We write  $f(x) = \Omega(g(x)) \iff g(x) = \mathcal{O}(f(x))$ . If there exists positive constants  $\alpha_1$  and  $\alpha_2$  such that

$$\alpha_1 g(x) \leq f(x) \leq \alpha_2 g(x) \quad \forall x > 0,$$

then we write  $f(x) = \Theta(g(x))$ .

We also define  $\tilde{\mathcal{O}}(f(x)) = \mathcal{O}(f(x) \cdot \text{polylog}(f(x)))$  and when the function depends poly-logarithmically on other variables we write

$$\tilde{\mathcal{O}}_{\alpha, \beta}(f(x)) = \mathcal{O}(f(x) \cdot \text{polylog}(\alpha, \beta, f(x))).$$

## 2.1 Interior Point Methods for Linear Optimization

In this section we outline the classical IPM theory applied to solving linear optimization problems.

### 2.1.1 Primal and dual LO problems

Recall that we are interested in the primal LO problem and its dual

$$\min_{x \in \mathcal{P}} c^\top x, \quad \max_{y \in \mathcal{D}} b^\top y,$$

where  $\mathcal{P}$  and  $\mathcal{D}$  are the *primal and dual feasible sets*, defined as

$$\mathcal{P} := \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}, \quad \mathcal{D} := \{y \in \mathbb{R}^m : A^\top y \leq c, y \geq 0\}.$$

Likewise, the sets of *interior feasible solutions* of (P) and (D) are given by

$$\text{int}(\mathcal{P}) := \{x \in \mathbb{R}^n : Ax > b, x > 0\}, \quad \text{int}(\mathcal{D}) := \{y \in \mathbb{R}^m : A^\top y < c, y > 0\}.$$

The well known property of *weak duality* always holds, and asserts that any primal feasible  $x \in \mathcal{P}$  provides an upper bound  $c^\top x$  on the value  $b^\top y$ , and conversely, any dual feasible  $y \in \mathcal{D}$  provides a lower bound  $b^\top y$  on the value  $c^\top x$ . The nonnegative quantity

$$c^\top x - b^\top y$$

is referred to as the *duality gap* associated to the pair  $(x, y) \in \mathcal{P} \times \mathcal{D}$ .

Whenever  $(x_*, y_*) \in \mathcal{P} \times \mathcal{D}$  exhibit vanishing duality gap, i.e.,  $c^\top x_* - b^\top y_* = 0$ , then  $x_*$  is an optimal solution to (P), and  $y_*$  is an optimal solution to (D). Hence, we define the optimal set of (P)-(D) to be

$$\mathcal{PD}_* := \{(x, y) \in \mathcal{P} \times \mathcal{D} : c^\top x = b^\top y\}.$$

Clearly,  $\mathcal{PD}_*$  is nonempty if and only if the inequality system

$$\begin{aligned} Ax &\geq b, & x &\geq 0, \\ -A^\top y &\geq -c, & y &\geq 0, \\ b^\top y - c^\top x &\geq 0, \end{aligned} \tag{3}$$

is solvable. Introducing a *homogenizing variable*  $\beta$ , system (3) is equivalent to the homogeneous system

$$\begin{bmatrix} 0_{m \times m} & A & -b \\ -A^\top & 0_{n \times n} & c \\ b^\top & -c^\top & 0 \end{bmatrix} \begin{bmatrix} y \\ x \\ \beta \end{bmatrix} \geq \begin{bmatrix} 0_m \\ 0_n \\ 0 \end{bmatrix}, \quad x \geq 0, y \geq 0, \beta \geq 0, \tag{4}$$

when  $\beta = 1$ . Indeed, one can verify that any  $(x, y, \beta)$  which solves (4) with  $\beta > 0$  gives rise to a solution  $\left(\frac{x}{\beta}, \frac{y}{\beta}, 1\right)$  to (3). More concisely, letting

$$\overline{M} := \begin{bmatrix} 0_{m \times m} & A & -b \\ -A^\top & 0_{n \times n} & c \\ b^\top & -c^\top & 0 \end{bmatrix}, \quad \overline{z} := \begin{bmatrix} y \\ x \\ \beta \end{bmatrix},$$

solving (P) and (D) to optimality is equivalent [RTV05, see, Theorem I.3] to obtaining a feasible solution to the system

$$\overline{M}\bar{z} \geq 0, \quad \bar{z} \geq 0, \quad \beta > 0. \quad (5)$$

Next, we discuss the condition under which (5) is solvable, which is intimately related to the practical concern of starting from an interior feasible solution.

### 2.1.2 The self-dual embedding model

We begin with a formal definition of the *interior point condition* (IPC) from [RTV05].

**Definition 1** (Definition I.4 in [RTV05]). *We say that any system of (linear) equalities and (linear) inequalities satisfies the interior-point condition (IPC) if there exists a solution that satisfies all inequality constraints in the system.*

The IPC ensures that the primal-dual pair (P) and (D) has an optimal solution  $(x_*, y_*) \in \mathcal{PD}_*$  whenever there exists a strictly feasible solution  $(x, y) \in \text{int}(\mathcal{P}) \times \text{int}(\mathcal{D})$ . Being able to easily determine an interior point  $(x, y) \in \text{int}(\mathcal{P}) \times \text{int}(\mathcal{D})$  is also a practical concern: (feasible) IPMs are initialized to a strictly feasible starting point, and naïvely determining a strictly feasible solution to (P) and (D) is as challenging as solving these problems to optimality. Here we discuss a way to embed the problems (P) and (D) into a slightly larger one that always has a trivial strictly feasible solution, and can be readily solved with an IPM, following a strategy first introduced by Ye, Todd and Mizuno [YTM94].

While system (5) does not satisfy the IPC, this can be reconciled upon introducing another auxiliary variable  $\vartheta \geq 0$ , and adding one additional row and column to  $\overline{M}$  as follows:

$$M := \begin{bmatrix} \overline{M} & r \\ -r^\top & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0_{m \times m} & A & -b \\ -A^\top & 0_{n \times n} & c \\ b^\top & -c^\top & 0 \end{bmatrix} & r \\ 0 & 0 \end{bmatrix}, \quad z := \begin{bmatrix} \bar{z} \\ \vartheta \end{bmatrix} = \begin{bmatrix} y \\ x \\ \beta \\ \vartheta \end{bmatrix},$$

where

$$r = \bar{e} - \overline{M}\bar{e},$$

and  $\bar{e}$  is the all-ones vector of length  $m + n + 1$ . Note that  $M$  is a *skew-symmetric* matrix of dimension  $\bar{n} := n + m + 2$ , i.e.,  $M^\top = -M$ . Defining  $q \in \mathbb{R}^{\bar{n}}$  to be the vector

$$q = \begin{bmatrix} 0_{\bar{n}-1} \\ \bar{n} \end{bmatrix},$$

one can also see that the system

$$Mz \geq -q, \quad z \geq 0, \quad (6)$$

admits the all-ones vector of length  $\bar{n}$  as a strictly feasible solution, which we denote by  $e$ . If  $z$  solves (6) with  $\vartheta = 0$ , i.e., we have  $z = (\bar{z}, 0)$ , then  $\bar{z}$  must be a solution to (5). Hence, solving (P) and (D) can be reduced to finding a solution to (6) with  $\vartheta = 0$  and  $\beta > 0$ .

The *self-dual embedding* of (P) and (D) is defined as

$$\min \left\{ q^\top z : Mz \geq -q, \quad z \geq 0 \right\}. \quad (\text{SP})$$

This problem is called “self-dual” because its dual problem

$$\max \left\{ -q^\top u : M^\top u \leq q, u \geq 0 \right\},$$

can be recognized as equivalent to (SP), upon recalling that  $M$  is a skew-symmetric matrix with  $M^\top = -M$ . Like system (6), the self-dual embedding problem (SP) also admits the strictly feasible solution  $e$ , and thus trivially satisfies the IPC.

Moving forward, we will write  $\mathcal{SP}$  when referring to the set of feasible solutions to (SP). We denote the sets of interior feasible, and optimal solutions by  $\text{int}(\mathcal{SP})$  and  $\mathcal{SP}_*$ , respectively.

The next result from [RTV05] asserts that we obtain optimal solutions to (P) and (D) by solving (SP).

**Theorem 2** (Theorem I.6 in [RTV05]). *The system (6) has a solution with  $\vartheta = 0$  and  $\beta > 0$  if and only if the problem (SP) has an optimal solution with  $\beta = z_{\bar{n}-1} > 0$ .*

### 2.1.3 The central path and its neighborhood

For any vector  $z \in \mathbb{R}^{\bar{n}}$ , we define its slack vector<sup>4</sup>

$$s(z) := Mz + q.$$

It follows that

$$z \text{ is a feasible solution to (SP)} \iff z \geq 0 \text{ and } s(z) \geq 0.$$

Recalling that  $e$  denotes the all-one vector of length  $\bar{n}$ , note that  $s(e) = e$  and  $e \odot s(e) = e$ . Hence,  $\mu = 1$  for the point  $(z, s(z)) = (e, e) \in \text{int}(\mathcal{SP})$ .

For every positive  $\mu$  there exists a *unique* non-negative vector  $z$  such that

$$z \odot s(z) = \mu e, \quad z \geq 0, \quad s(z) \geq 0, \tag{CP}$$

see, [RTV05, Lemma I.13]. We denote the unique non-negative solution to the quadratic system (CP) as  $z(\mu)$ , which we refer to as the  $\mu$ -center. Using this notation, we may write  $z(1) = e$ . The set of  $\mu$ -centers  $\mathcal{C} = \{z(\mu) : \mu > 0\}$  is the *central path* of the problem (SP), and an optimal solution of (SP) is the limiting point of the central path as  $\mu \rightarrow 0^+$ . The central path constitutes an analytic curve  $\xi(\mu) : (0, \infty) \mapsto \mathbb{R}^{\bar{n}}$ , and its graph  $(\mu, \xi(\mu))$  satisfies

$$z \odot s(z) = \mu e, \quad z > 0, \quad s(z) > 0. \tag{7}$$

Since IPC is satisfied, the central path is guaranteed to exist and is uniquely determined from the starting point  $(z(1), s(z(1))) = (e, e)$ . Put another way,  $(e, e)$  is the point *on* the central path corresponding to  $\mu = 1$ .

The standard IPM can be interpreted as an algorithm that approximately follows the central path by locally linearizing the quadratic system (CP). The solution of each locally linearized system is the so-called *Newton step*. To see this, suppose that  $z$  is a positive solution to (SP) such that its slack vector  $s(z)$  is also positive, i.e.,  $z$  and  $s(z)$  satisfy (7). To find the displacement  $\Delta z$  such that  $z^+ := z + \Delta z$  is the  $\mu$ -center, we want to solve the equation:

$$(z + \Delta z) \odot s(z + \Delta z) = \mu e.$$

---

<sup>4</sup>As we have defined it here,  $s$  really is a *surplus* variable.

Defining  $s = s(z)$  and  $\Delta s = M\Delta z$ , we obtain a nonlinear equation with quadratic term  $\Delta z \odot \Delta s$ , namely,

$$Zs + Z\Delta s + S\Delta z + \text{diag}(\Delta z)\Delta s = \mu e, \quad (8)$$

where  $Z := \text{diag}(z)$  and  $S := \text{diag}(s)$ . If we further assume  $z^+$  is in a small neighborhood of  $z$ , i.e.,  $\Delta z$  is small (and hence, so is  $\Delta s = M\Delta z$ ), the quadratic term in (8) can be omitted, and the nonlinear equation reduces to a linear system in the unknowns  $\Delta z$  and  $\Delta s$ .

The foundation for IPM theory is that performing the local linearization we just described at each iterate, and solving the resulting equation system for  $\Delta z$  and  $\Delta s$  allows us to make sufficient progress towards the optimal solution (and remain in  $\text{int}(\mathcal{SP})$ ), provided that the current iterate  $(z, s(z))$  is in some sense close to the central path. To quantify a notion of closeness, one defines a proximity measure that gives rise to a *neighborhood* of the central path. In this paper, we consider the distance metric

$$d_2(z, s; \mu) := \|z \odot s(z) - \mu e\|_2,$$

which for fixed  $\gamma \in (0, 1)$  gives rise to a narrow neighborhood of the central path:

$$\mathcal{N}_2(\gamma) := \{(z, s(z)) \in \text{int}(\mathcal{SP}) : d_2(z, s; \mu) \leq \gamma\mu\}.$$

One can observe that for every  $\gamma \in (0, 1)$ , the following set of inclusions hold:

$$\mathcal{C} \subset \mathcal{N}_2(\gamma) \subset \text{int}(\mathcal{SP}).$$

## 2.2 Quantum adiabatic theorem for unbounded Hamiltonians

Given a quantum Hamiltonian  $\mathcal{H}(t)$ ,  $t \in [0, 1]$ , we consider the dynamics described by the Schrödinger equation:

$$\mathbf{i}\eta \frac{d}{dt} |\psi(t)\rangle = \mathcal{H}(t) |\psi(t)\rangle, \quad (9)$$

where  $\mathbf{i}$  denotes the imaginary unit and  $\eta$  is a positive real number. We suppose that  $U(t)$  is the propagator of the dynamics, such that the solution to (9) at time  $t$  is given by

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle.$$

If  $|\psi(0)\rangle$  is a nondegenerate ground state of  $\mathcal{H}(0)$ , then the quantum adiabatic theorem [Mes58] asserts that in the limit  $T \rightarrow \infty$ , the state  $|\psi(T)\rangle$  obtained from (9), will be close to the ground state of  $\mathcal{H}(T)$ .

We assume  $\mathcal{H}(t)$  has a non-degenerate ground state for all  $t \in [0, 1]$ , and we define  $P(t)$  as a rank-1 projector onto the ground-energy subspace of  $\mathcal{H}(t)$ .

**Theorem 3** (Exponential estimate). *Define  $\mathcal{H}^{(k)}(t) := \frac{d^k}{dt^k} \mathcal{H}(t)$ . Let  $\mathcal{H}(t)$  be a quantum Hamiltonian for  $t \in [0, 1]$  such that the following conditions hold:*

- (a)  $\mathcal{H}(t)$  admits an analytic continuation to some strip on  $\mathbb{C}$  containing  $[0, 1]$ .
- (b) For  $t \in [0, 1]$ , the spectral gap of  $\mathcal{H}(t)$  is greater than a constant  $\Delta_0 > 0$ .
- (c) For any  $k = 1, 2, \dots$ , we have that  $\mathcal{H}^{(k)}(0) = \mathcal{H}^{(k)}(1) = 0$ .



Then, there exists a constant  $C$  such that

$$\|\psi(1) - \phi_*\| \leq Ce^{-1/\eta},$$

where  $\psi(1)$  is the solution to (9) at  $t = 1$  and  $\phi_*$  is a ground state of  $\mathcal{H}(1)$ , i.e.,  $P(1)\phi_* = \phi_*$ .

*Proof.* Combining the main theorem in [HJ02] and the vanishing derivative condition at  $t = 0, 1$ .  $\square$

### 2.3 Quantum algorithms for Schrödinger equations

The primary subroutine of our algorithm is simulating the Schrödinger equation, otherwise known as *quantum simulation*. Our work here will consider the Schrödinger equation over the time interval  $[t_0, t_1]$  for a given time-dependent potential  $V(t, x)$ ,

$$i\frac{\partial}{\partial t} |\Psi(x, t)\rangle = \left[ -\frac{1}{2}\nabla^2 + V(x, t) \right] |\Psi(x, t)\rangle, \quad (10)$$

where we specify  $\Omega = [-R, R]^d$  for a sufficiently large  $R$  and  $V(x, t): \Omega \times [t_0, t_1] \mapsto \mathbb{R}$  is a time-dependent potential function. Accordingly,  $\Psi(x, t): \Omega \times [t_0, t_1] \mapsto \mathbb{C}$  is the wave function subject to certain initial data  $\Psi(x, t_0) = \Psi_0(x)$  and the periodic boundary condition.

We utilize the quantum simulation algorithm of Childs, Leng, Li, Liu, and Zhang [CLL<sup>+</sup>22], which exhibits near-optimal dependence in the dimension  $d$ , and precision  $\epsilon$  to which the simulation is carried out. However, the gate complexity of that quantum simulation algorithm also involves a parameter  $g'$  that depends on the higher-order derivatives of the wave function. An accurate upper bound of  $g'$  relies on a refined *a priori* estimate of the wave function. By leveraging the fact that the initial condition is always *analytic* in our setting, we eschew the regularity parameter  $g'$  in [CLL<sup>+</sup>22, Theorem 8], resulting in an enhanced complexity result.

**Theorem 4** (Improved version of Theorem 8 in [CLL<sup>+</sup>22]). *Suppose the potential field  $V(x, t)$  is bounded, smooth in  $x$  and  $t$ , and periodic in  $x$ . Define the  $\|\cdot\|_{\infty,1}$ -norm of  $V(x, t)$  as*

$$\|V\|_{\infty,1} := \int_{t_0}^{t_1} \|V(\cdot, t)\|_{\infty} dt.$$

*We assume that we have access to the zeroth-order oracle of  $V$ , which is a unitary map  $O_V$  on  $\Omega \otimes [t_0, t_1] \mapsto \mathbb{R}$  such that for any  $|x\rangle \in \Omega$  and  $|s\rangle \in [t_0, t_1]$ ,*

$$O_V(|x\rangle \otimes |s\rangle \otimes |0\rangle) = |x\rangle \otimes |s\rangle \otimes |V(x, s)\rangle.$$

*Moreover, we assume the initial data  $\Psi_0(x)$  is analytic on  $\Omega$  and  $V$  is  $G$ -Lipschitz in  $t$ . Then, the Schrödinger equation (10) can be simulated for time  $t \in [t_0, t_1]$  up to accuracy  $\epsilon$  with the following cost:*

1. Queries to  $O_V$ :  $\mathcal{O}\left(\|V\|_{\infty,1} \frac{\log(\|V\|_{\infty,1}/\epsilon)}{\log \log(\|V\|_{\infty,1}/\epsilon)}\right)$ ,
2. 1- and 2-qubit gates:

$$\mathcal{O}\left(\|V\|_{\infty,1} (\text{poly}(z) + \log^{2.5}(G\|V\|_{\infty,1}/\epsilon) + d \log \log(1/\epsilon)) \frac{\log(\|V\|_{\infty,1}/\epsilon)}{\log \log(\|V\|_{\infty,1}/\epsilon)}\right).$$

Note that the simulation accuracy  $\epsilon$  referred to in Theorem 4 corresponds to the  $\ell_2$ -distance between the actual final state and the state returned by the quantum simulation algorithm. This theorem is an improved version of the original Theorem 8 in [CLL<sup>+</sup>22]. We provide the proof of this theorem in Appendix C. We also refer the readers to [LZW23, Section 2.4] for a detailed discussion.

### 3 A quantum algorithm that traces the central path

In this section, we propose a Hamiltonian formalism for the central path of linear optimization problems. We show how one can solve LO problems by simulating the Schrödinger equation associated with our Hamiltonian, and provide a rigorous complexity analysis of the resulting scheme.

#### 3.1 Quantum representation of the central path

In this subsection, we construct a family of quantum Hamiltonian operators  $\mathcal{H}(\mu)$ , where  $\mu \in (0, 1]$  is a positive parameter. We will show that, for each  $0 < \mu \leq 1$ , the ground state of the operator  $\mathcal{H}(\mu)$  gives rise to a Gaussian distribution centered at the  $\mu$ -center  $z(\mu)$ . Therefore, by measuring the ground state of  $\mathcal{H}(\mu)$ , we can approximate the  $\mu$ -center.

Recall that we define  $F(z) := z \odot s(z) \in \mathbb{R}^{\bar{n}}$ , where  $s(z) = Mz + q$ . For any  $0 < \mu \leq 1$ , we define the function

$$f_\mu(z) := \frac{1}{2} (F(z) - \mu e)^\top (F(z) - \mu e) = \frac{1}{2} \|F(z) - \mu e\|^2. \quad (11)$$

**Definition 2** (Central-path Hamiltonian). *For  $0 < \mu \leq 1$  and  $h > 0$ , we define a 1-parameter family of elliptic operators over the positive orthant  $\mathbb{R}_{++}^{\bar{n}}$ ,*

$$\mathcal{H}(\mu) = -\frac{h^2}{2} \nabla^2 + f_\mu(z), \quad (12)$$

where  $\nabla^2 = \sum_{j=1}^{\bar{n}} \frac{\partial^2}{\partial z_j^2}$  is the Laplacian operator.

As defined in (11), the function  $f_\mu(z)$  is non-negative. The  $\mu$ -center  $z(\mu)$  is the unique zero of  $f_\mu(z)$  in the positive orthant  $z > 0$ . When the parameter  $h$  is sufficiently small, the harmonic approximation theory of Schrödinger operators [HS12, Section 11] allows us to approximate the low-energy spectrum of  $\mathcal{H}(\mu)$  by that of the following operator,

$$\tilde{\mathcal{H}}(\mu) = -\frac{h^2}{2} \nabla^2 + \frac{1}{2} [z - z(\mu)]^\top H(\mu) [z - z(\mu)], \quad (13)$$

where  $H(\mu)$  is the Hessian of  $f_\mu(z)$  at  $z = z(\mu)$ . A quick calculation yields that

$$H(\mu) = \mathcal{J}(z(\mu))^\top \mathcal{J}(z(\mu)) \succ 0, \quad (14)$$

where  $\mathcal{J}(z) := \frac{\partial F(z)}{\partial z} = ZM + S$ , see, Lemma 4 in Appendix A.

In the following lemma, we summarize some important properties of the Hessian matrix  $H(\mu)$ .

**Lemma 1.** *Let  $0 < \mu \leq 1$ . Letting  $R_\infty > 0$  be an  $\ell_\infty$ -upper bound on the  $z$  and  $s(z)$ , we have*

$$\lambda_0(H(\mu)) \geq \left( \frac{\mu}{R_\infty} \right)^2.$$

*Proof.* From the proof of Lemma 5 in Appendix A, the smallest eigenvalue of  $H(\mu)$  satisfies

$$\lambda_{\min}(H(\mu)) = \lambda_{\min} \left( \mathcal{J}(z(\mu))^\top \mathcal{J}(z(\mu)) \right) \geq \left( \frac{\mu}{R_\infty} \right)^2.$$

□

The Hamiltonian operator  $\tilde{\mathcal{H}}(\mu)$  describes a quantum harmonic oscillator. The eigenvalues and eigenstates of a quantum harmonic oscillator are well understood [GS18]. Let  $0 < \lambda_0(\mu) \leq \lambda_1(\mu) \leq \dots \leq \lambda_{\bar{n}-1}(\mu)$  be the eigenvalues of the Hessian matrix  $H(\mu)$ . The spectral gap (i.e., the difference between the first two eigenvalues) of the operator  $\tilde{\mathcal{H}}(\mu)$  is

$$\Delta = h\lambda_0^{1/2}(\mu). \quad (15)$$

Moreover, let  $\Phi_0(\mu)$  be the ground state of the operator  $\tilde{\mathcal{H}}(\mu)$ . It turns out that  $|\Phi_0(\mu)|^2$  is the probability density function of the multivariate normal distribution

$$|\Phi_0(\mu)|^2 \sim \mathbf{N}\left(z(\mu), \frac{h}{2}(H(\mu))^{-1/2}\right). \quad (16)$$

**Proposition 1.** Fix  $\delta > 0$ . For any  $\mu \in (0, 1]$ , choose

$$h := \frac{\mu^2}{\sqrt{2(m+n)}R_1}, \quad (17)$$

where  $R_1$  is an  $\ell_1$ -norm upper bound on the size of the solution to (SP). Then,

$$\Pr_{x \sim |\Phi_0(\mu)|^2} [x \notin \mathcal{N}_2(\gamma)] \leq \delta.$$

The proof of Proposition 1 is available in Appendix B.

### 3.2 Quantum simulation of the central path

The key idea of our quantum algorithm is to simulate a quantum evolution for some  $t \in [0, 1]$  in which the quantum state  $|\Psi(t)\rangle$  is (approximately) the ground state of  $\mathcal{H}(\mu(t))$ . Here, the function  $\mu(t)$  is a monotonically decreasing function such that

$$\mu(0) = 1, \quad \mu(1) = \mu_f \ll 1.$$

In this way, the quantum state  $|\Psi(t)\rangle$  follows the central path as  $\mu(t)$  decreases. At  $t = 1$ , if we measure the final state  $|\Psi(1)\rangle$ , Proposition 1 guarantees that we will obtain an approximate solution that is in the neighborhood of  $z(\mu_f)$  with high probability.

First, we introduce a function  $g: [0, 1] \mapsto [0, 1]$  such that (i)  $g$  is analytic in  $(0, 1)$ , and (ii) for any  $k = 1, 2, \dots$ , we have  $g^{(k)}(0) = g^{(k)}(1) = 0$ , where  $g^{(k)}(t) := \frac{d^k}{dt^k}g(t)$ . The construction of this function  $g(t)$  follows [AL22, Equation 8]. Namely, we choose

$$g(t) = c_e^{-1} \int_0^t \exp\left(-\frac{1}{\tau(1-\tau)}\right) d\tau, \quad (18)$$

where  $c_e = \int_0^1 \exp\left(-\frac{1}{\tau(1-\tau)}\right) d\tau$  is a normalization constant such that  $g(1) = 1$ . Then, given a fixed  $\mu_f \in (0, 1)$ , we define the function  $\mu(t)$ ,

$$\mu(t) = 1 - (1 - \mu_f)g(t). \quad (19)$$

Clearly,  $\mu(t)$  is a monotonically decreasing function such that  $\mu(0) = 1$ ,  $\mu(1) = \mu_f$ , and we have  $\mu^{(k)}(0) = \mu^{(k)}(1) = 0$  for any positive integer  $k$ .

**Lemma 2.** Let  $\mu(t)$  be the same as in (19) and define  $\mathcal{H}^{(k)}(t) := \frac{d^k}{dt^k} \mathcal{H}(t)$ . Then, the Hamiltonian  $\mathcal{H}(t) := \mathcal{H}(\mu(t))$  is analytic in  $t \in (0, 1)$ . Moreover, for any  $k = 1, 2, \dots$ , we have  $\mathcal{H}^{(k)}(0) = \mathcal{H}^{(k)}(1) = 0$ .

*Proof.* The analyticity immediately follows from our definition of  $\mathcal{H}$  (see (12)) because  $f_\mu$  is a quadratic function in  $\mu$ . By the chain rule, we have

$$\frac{d\mathcal{H}(t)}{dt} = -2\dot{\mu} (F(z) - \mu e)^\top e.$$

Since  $\dot{\mu}(0) = \dot{\mu}(1) = 0$ , we have  $\mathcal{H}'(0) = \mathcal{H}'(1) = 0$ . Similarly, using an induction argument, we can prove  $\mathcal{H}^{(k)}(0) = \mathcal{H}^{(k)}(1) = 0$  for any  $k \geq 1$ .  $\square$

**Proposition 2.** Let  $\mathcal{H}(t) := \mathcal{H}(\mu(t))$ , where  $\mu(t)$  is defined in (19). For sufficiently small  $h$ , we consider the following Schrödinger equation,

$$i\eta \frac{\partial}{\partial t} \Psi^\eta(t) = \frac{1}{h\mu(t)} \mathcal{H}(t) \Psi^\eta(t), \quad (20)$$

where the initial state  $\Psi(0)$  is the ground state of  $\mathcal{H}(0)$ . Then, there exists a constant  $C$  such that

$$\|\Psi^\eta(1) - \Phi_0(\mu_f)\| \leq C e^{-1/\eta}, \quad (21)$$

where  $\Phi_0(\mu_f)$  is a ground state of the final Hamiltonian  $\mathcal{H}(1)$ .

*Proof.* This is a direct consequence of Theorem 3. In Lemma 2, we proved that  $\mathcal{H}(t)$  is analytic in  $t \in (0, 1)$  and  $\mathcal{H}^{(k)}(0) = \mathcal{H}^{(k)}(1) = 0$  for any positive integer  $k$ . It remains to show that the spectral gap of  $\mathcal{H}(t)$  has a lower bound  $\Delta_0$ . For sufficiently small  $h$ , the spectral gap of  $\mathcal{H}(t)$  is approximately the same as its harmonic approximation  $\tilde{\mathcal{H}}(\mu(t))$  (see (13)). The operator  $\tilde{\mathcal{H}}(\mu(t))$  describes a quantum harmonic oscillator and its spectral gap is precisely  $\Delta(t) = h\lambda_0^{1/2}(\mu(t))$ , see Equation (15). It follows from Lemma 1 that

$$\Delta(t) \geq h \sqrt{\left(\frac{\mu(t)}{R_\infty}\right)^2} = h \frac{\mu(t)}{R_\infty}.$$

Therefore, the spectral gap of the Hamiltonian operator in (20) has a lower bound  $\Delta_0 = \frac{1}{R_\infty}$ .  $\square$

### 3.3 Quantum central path algorithm

We present the quantum central path method for linear optimization in full detail in Algorithm 1. The algorithm takes as input: (i) the self-dual embedding formulation (SP) of the LO problem data  $(A, b, c)$ ; (ii) the optimality tolerance  $\varepsilon \in (0, 1)$  to which we seek to solve (P)-(D); (iii) the neighborhood opening parameter  $\gamma \in (0, 1)$  that specifies  $\mathcal{N}_2(\gamma)$ , the  $d_2$ -neighborhood of the central path; and (iv) the failure rate  $\delta \in (0, 1)$  to which we allow the quantum algorithm returns a point that is not in  $\mathcal{N}_2(\gamma)$ .

In Algorithm 1, we simulate the Schrödinger equation (20) to error  $\frac{\delta}{8}$ , which can be accomplished through the use of

$$\tilde{\mathcal{O}}_{m,n,\frac{1}{8}} \left( \sqrt{m+n} \cdot \frac{R_1}{\varepsilon} \right)$$

queries to an evaluation oracle for  $f_\mu(z)$  and

$$\tilde{\mathcal{O}}_{m,n,\frac{1}{\delta}} \left( (m+n) \frac{R_1}{\varepsilon} \right)$$

elementary gates. We address the construction an evaluation oracle for  $f_\mu(z)$  and its associated cost in Section 3.4. Letting  $\Phi_0(\mu_f)$  denote the ground state associated to the Hamiltonian  $\mathcal{H}(\mu_f)$ , and choosing  $h$  according to Equation (17), simulating the Schrödinger equation suffices to ensure that  $|\Phi_0(\mu_f)|^2$  is the probability density function of the multivariate normal distribution  $\mathcal{N}(z(\mu_f), \frac{h}{2}(H(\mu_f))^{-1/2})$ , with  $\mu_f \leq \varepsilon$ . A sketch of this process is visualized in Figure 3.3.

---

**Algorithm 1** Quantum central path algorithm for linear optimization

---

**Input:** Matrix  $A \in \mathbb{R}^{m \times n}$ , vectors  $b \in \mathbb{R}^m$  and  $c \in \mathbb{R}^n$ , optimality tolerance  $\varepsilon \in (0, 1)$ , neighborhood opening  $\gamma \in (0, 1)$ , failure rate  $\delta \in (0, 1)$

**Output:**  $\varepsilon$ -optimal solution  $z \in \mathbb{R}^{\bar{n}}$  to the LO problem (SP)

**Initialize:**  $\mu_f \leftarrow \varepsilon$

1. Initialize the quantum register to  $|\Phi(0)\rangle$ , the ground state of  $\mathcal{H}(0)$ .
2.  $|\Psi_{\text{sim}}^\eta(1)\rangle \leftarrow$  simulate the Schrödinger equation (20) to error  $\delta/8$  with

$$h(t) = \frac{\gamma^2 \mu(t)^2}{2R_1(\sqrt{\bar{n}}/2 + 3 \log(2/\delta)/4)}, \quad \eta = \frac{1}{\log(8C/\delta)},$$

where  $\mu(t)$  is defined in (19) and the constant  $C$  is independent of  $m, n$  and  $\delta$  (see Proposition 2).

3.  $z \leftarrow$  sample from the quantum state  $|\Psi^\eta(1)\rangle$ .
- 

We may now establish the correctness and complexity of Algorithm 1.

**Theorem 5.** *Let  $\varepsilon \in (0, 1)$ . Choose  $\gamma \in (0, 1)$  and  $\delta \in (0, 1)$ . Given access to a quantum oracle  $O_{f_\mu}$  that evaluates the potential function  $f_\mu(z, t)$  in a superposition:*

$$O_{f_\mu}(|z\rangle \otimes |t\rangle \otimes |0\rangle) = |z\rangle \otimes |t\rangle \otimes |f_\mu(z, t)\rangle,$$

*with probability at least  $1 - \delta$  Algorithm 1 returns a classical vector  $z \in \mathbb{R}_{++}^{m+n+2}$  such that  $(z, s(z)) \in \mathcal{N}_2(\gamma)$  with  $d_2(z, s(z); \varepsilon) \leq \gamma\varepsilon$ . Moreover, Algorithm 1 can be implemented with*

$$\mathcal{O} \left( \sqrt{m+n} \cdot \frac{R_1}{\varepsilon} \cdot \text{polylog} \left( m, n, \frac{1}{\delta} \right) \right)$$

*queries to  $O_{f_\mu}$  and*

$$\mathcal{O} \left( (m+n) \cdot \frac{R_1}{\varepsilon} \cdot \text{polylog} \left( m, n, \frac{1}{\delta} \right) \right)$$

*elementary gates.*

*Proof.* We obtain a quantum state  $|\Psi_{\text{sim}}^\eta(1)\rangle$  by simulating the Schrödinger equation (9) to error  $\delta/8$ . By Proposition 2, if we choose  $\eta = 1/\log(8C/\delta)$  (where  $C$  is a constant independent of  $d$  and  $\delta$ ), we have  $\|\Psi^\eta(1) - \Phi_0(\mu_f)\| \leq \delta/8$ . By the triangle inequality, it turns out that

$$\|\Psi_{\text{sim}}^\eta(1) - \Phi_0(\mu_f)\| \leq \frac{\delta}{4}. \quad (22)$$

We define  $B_{\gamma,\varepsilon} := \{z \in \mathbb{R}^{\bar{n}} : f(z) \leq \frac{1}{2}\gamma^2\varepsilon^2\}$ , which is a compact set containing the point  $z(\varepsilon)$ . Let  $\mathbb{1}_{B_{\gamma,\varepsilon}}(z)$  be the indicator function of the set  $B_{\gamma,\varepsilon}$ , then we have

$$\begin{aligned} \Pr_{z \sim |\Psi_{\text{sim}}^\eta(1)\rangle^2} \left[ f_\mu(z) > \frac{1}{2}\gamma^2\varepsilon^2 \right] &= 1 - \langle \Psi_{\text{sim}}^\eta(1) | \mathbb{1}_{B_{\gamma,\varepsilon}} | \Psi_{\text{sim}}^\eta(1) \rangle, \\ \Pr_{z \sim |\Phi_0(\mu_f)\rangle^2} \left[ f_\mu(z) > \frac{1}{2}\gamma^2\varepsilon^2 \right] &= 1 - \langle \Phi_0(\mu_f) | \mathbb{1}_{B_{\gamma,\varepsilon}} | \Phi_0(\mu_f) \rangle. \end{aligned}$$

Then, it follows from Lemma 8 that

$$\Pr_{z \sim |\Psi_{\text{sim}}^\eta(1)\rangle^2} \left[ f_\mu(z) > \frac{1}{2}\gamma^2\varepsilon^2 \right] \leq \Pr_{z \sim |\Phi_0(\mu_f)\rangle^2} \left[ f_\mu(z) > \frac{1}{2}\gamma^2\varepsilon^2 \right] + \frac{\delta}{2} \leq \delta,$$

where the last step follows from Proposition 1. This proves the first part of the theorem.

Next, we discuss the complexity of Algorithm 1. The Schrödinger equation (20) can be written as follows:

$$i\eta \frac{\partial}{\partial t} \Psi^\eta(t) = \left[ -\frac{\theta}{2} \nabla^2 + \frac{1}{\mu(t)h(t)} f_\mu(z) \right] \Psi^\eta(t),$$

where the coefficient

$$\theta := \frac{h(t)}{\mu(t)} = \frac{\gamma^2 \mu(t)}{R_1 \left( \frac{\sqrt{\bar{n}}}{2} + 2 \log \left( \frac{2/\delta}{4} \right) \right)}. \quad (23)$$

We introduce the change of variable  $t \mapsto \frac{\eta}{\theta} \tau$  for  $\tau \in [0, \theta/\eta]$ , and we define a new wave function

$$\tilde{\Psi}^\eta(\tau) := \Psi^\eta \left( \frac{\eta\tau}{\theta} \right).$$

It turns out that simulating the Schrödinger equation (20) is equivalent to simulate the following time-dilated Schrödinger equation

$$i \frac{\partial}{\partial \tau} \tilde{\Psi}^\eta(\tau) = \left[ -\frac{1}{2} \nabla^2 + \frac{1}{h^2(\tau)} f_\mu(z) \right] \tilde{\Psi}^\eta(\tau) \quad (24)$$

for  $0 \leq \tau \leq \frac{\theta}{\eta}$ , where  $h(\tau) := h \left( \frac{\theta\tau}{\eta} \right)$ .

The time-dilated Schrödinger equation (24) can be simulated using Theorem 4. Note that in the quantum simulation, we need to truncate the time-dependent potential function  $f_\mu(z)/h^2(\tau)$  over a compact domain  $\Omega = [0, D]^{\bar{n}}$ , where  $D$  is a large number such that  $\Omega$  contains the neighborhood of the central path (e.g.,  $\mathcal{N}_2(\gamma)$ ). Usually,  $D$  can be fixed as an absolute constant. We denote the truncated time-dependent potential function as

$$V(z, \tau) = \frac{1}{h^2(\tau)} V_\mu(z),$$

where  $V_\mu(z) = f_\mu(z)$  for  $z \in \Omega$ . The  $\|\cdot\|_{\infty,1}$ -norm of  $V(z, \tau)$  is defined as

$$\|V\|_{\infty,1} := \int_0^{\theta/\eta} \frac{1}{h^2(\tau)} \|V_\mu(\cdot)\|_\infty d\tau.$$

At each  $\tau$ , the  $\ell_\infty$ -norm of  $V_\mu(z)$  could be large because the function increases when  $z$  is far away from the central path. However, as we are simulating the quantum dynamics in the *adiabatic* regime and the quantum state is approximately the ground state of the quantum Hamiltonian, we can give an improved estimate on  $\|V_\mu\|_\infty$  based on our knowledge of the ground state. Thanks to our choice of  $h(\cdot)$ , the ground state is concentrated in a narrow neighborhood  $\mathcal{N}_2(\gamma)$  on which the function  $f_\mu(z)$  is controlled by  $\gamma^2\mu^2/2$ . Thus, we may assume there is a small constant  $K > 0$  such that for any  $\mu \in [\varepsilon, 1]$ ,

$$\|V_\mu\|_\infty \leq K\gamma^2\mu^2.$$

It follows that

$$\|V\|_{\infty,1} \leq \int_0^{\theta/\eta} \frac{K\gamma^2\mu^2(\tau)}{h^2(\tau)} d\tau \leq \frac{K\gamma^2}{\theta\eta} = \mathcal{O}\left(\frac{R_1}{\mu} \log\left(\frac{1}{\delta}\right) \cdot \left(\sqrt{\bar{n}} + \log\left(\frac{1}{\delta}\right)\right)\right),$$

where  $\theta$  is defined according to (23). Similarly, we estimate the Lipschitz constant of  $V(x, s)$ ,

$$\|\dot{V}\|_\infty \leq \frac{2\dot{\mu}}{\theta^2\mu^2} \|V_\mu\|_\infty \leq \frac{2K\gamma^2}{\theta^2} = \mathcal{O}\left(\left(\frac{R_1}{\mu}\right)^2 \left(\sqrt{\bar{n}} + \log\left(\frac{1}{\delta}\right)\right)^2\right).$$

Noting that  $\mu \geq \varepsilon$ , and applying Theorem 4, we can simulate the Schrödinger equation (24) using

$$\mathcal{O}\left(\frac{R_1}{\varepsilon} \left(\sqrt{\bar{n}} + \log\left(\frac{1}{\delta}\right)\right) \log\left(\frac{1}{\delta}\right) \log\left(\frac{\bar{n}}{\delta}\right)\right) = \tilde{\mathcal{O}}_{m,n,\frac{1}{\delta}}\left(\sqrt{m+n}\frac{R_1}{\varepsilon}\right)$$

queries to  $O_{V_\mu}$ , and

$$\tilde{\mathcal{O}}\left(\frac{R_1}{\varepsilon} \left(\sqrt{\bar{n}} + \log^{2.5}\left(\frac{\bar{n}}{\delta}\right)\right) \left(\sqrt{\bar{n}} + \log\left(\frac{1}{\delta}\right)\right) \log\left(\frac{1}{\delta}\right) \log\left(\frac{\sqrt{\bar{n}}}{\delta}\right)\right) = \tilde{\mathcal{O}}_{m,n,\frac{1}{\delta}}\left((m+n)\frac{R_1}{\varepsilon}\right)$$

additional gates. Note that we ignore  $\log\log(\bar{n})$  and  $\log\log(1/\delta)$  factors in the big- $\mathcal{O}$  notation.  $\square$

### 3.4 Constructing an evaluation oracle for $f_\mu$

We now discuss the cost of constructing a quantum oracle  $O_{f_\mu}$  that queries the function value of  $f_\mu(z)$  for  $|z\rangle \in \Omega$ . To construct  $O_{f_\mu}$ , we only need the *classical* binary description of the matrix  $A \in \mathbb{R}^{m \times n}$  and the vectors  $b \in \mathbb{R}^m$ ,  $c \in \mathbb{R}^n$ . We also stress that our construction is only one possible way to do so.

**Lemma 3.** *Suppose that the total number of non-zeros in  $A$  is  $\text{nnz}(A)$ . Then, the quantum oracle  $O_{f_\mu}$  can be constructed with  $\ell$  bits of precision using  $\mathcal{O}(\text{nnz}(A) \cdot \text{poly}(\ell))$  elementary quantum gates and at most  $\mathcal{O}(\text{nnz}(A) \cdot \text{poly}(\ell))$  ancilla qubits.*

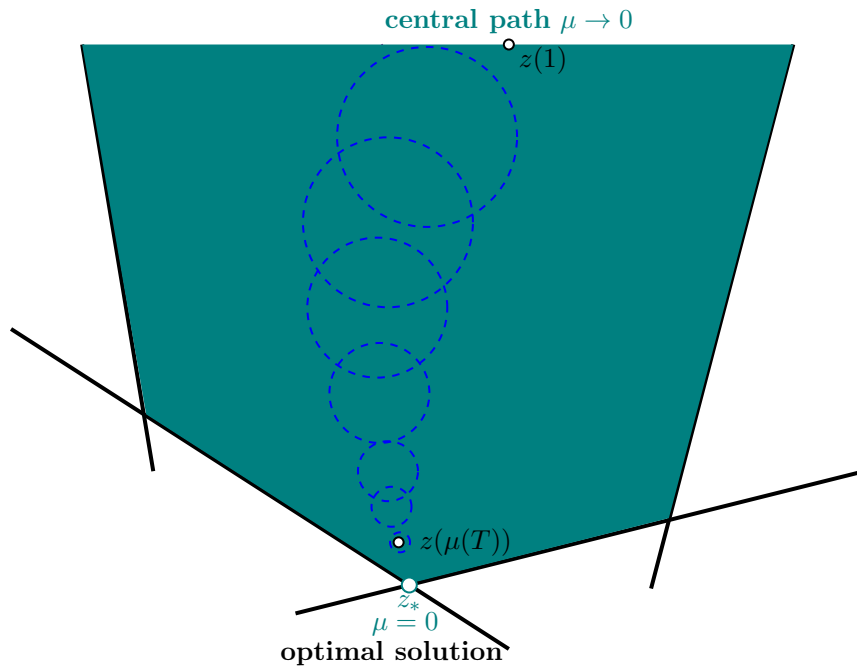


Figure 1: Visualization of the quantum central path algorithm. The dotted lines define the boundary of a neighborhood of the central path. The dashed circles indicate the progression of the wave packet from time  $t = 0$  to  $t = T$ . The wave packet begins to concentrate on a small ball centered at  $z(\mu(T))$ , near the optimal solution to the linear optimization problem  $z_*$ .



*Proof.* For any  $|z\rangle \in \Omega$  and  $|t\rangle \in [0, 1]$ , the quantum oracle is defined as

$$\mathcal{O}_{f_\mu}(|z\rangle \otimes |t\rangle \otimes |0\rangle) = |z\rangle \otimes |t\rangle \otimes |f_\mu(z, t)\rangle,$$

where the function  $f_\mu(z, t)$  is given by

$$f_\mu(z, t) = \frac{1}{2} \sum_{j=1}^{\bar{n}} (z_j s_j - \mu(t))^2.$$

Recall that vector  $s$  is given by  $s(z) = Mz + q$ . We write  $M = (m_{j,k})_{j,k=1}^{\bar{n}}$ , so

$$s_j = \sum_{k=1}^{\bar{n}} m_{j,k} z_k + q_j \quad \forall j \in [\bar{n}].$$

Note that the input  $|z\rangle$  and  $|t\rangle$  are in binary representation. To maintain  $\ell$  bits of precision,  $|z\rangle$  is represented using  $\ell\bar{n}$  qubits (in which each entry uses  $\ell$  qubits), and  $|t\rangle$  is represented by  $\ell$  qubits.

Without loss of generality, we may assume  $\text{nnz}(b), \text{nnz}(c) = \mathcal{O}(\text{nnz}(A))$ . For  $j = 1, \dots, \bar{n}$ , we denote  $\tilde{N}_j$  as the number of non-zeros in the  $j$ -th row of  $M$ . Due to our definition of  $M$  (see Section 2.1.2), we have that  $\sum_{j=1}^{\bar{n}} \tilde{N}_j = \mathcal{O}(\text{nnz}(A))$ . To compute each  $s_j$  (for  $j = 1, \dots, \bar{n}$ ), we need  $\tilde{N}_j$  uses of quantum multipliers and quantum adders, respectively.

Since we have a closed-form formula for the function  $\mu(t): [0, 1] \rightarrow \mathbb{R}$ , we assume the value of  $\mu(t)$  can be computed by a quantum circuit with  $\mathcal{O}(\text{poly}(\ell))$  elementary gates and at most  $\mathcal{O}(\text{poly}(\ell))$  ancilla qubits, where  $\ell$  is the digit length of the floating point number  $t$ .

Counting the arithmetic operations in the definition of  $f_\mu$ , we find that the function value of  $f_\mu(z, t)$  can be computed using

$$1 + \bar{n} + \sum_{j=1}^{\bar{n}} \tilde{N}_j = \mathcal{O}(\text{nnz}(A))$$

quantum multipliers and  $2\bar{n}$  quantum adders. In this process, we also need to query  $\mu(t)$  for  $\bar{n}$  times. A quantum adder with input size  $\ell$  can be implemented using  $\mathcal{O}(\ell)$  elementary gates and  $\mathcal{O}(1)$  ancilla qubits [VBE96]. A quantum multiplier with input size  $\ell$  can be implemented using  $\mathcal{O}(\ell^{\log(3)})$  elementary gates and  $\mathcal{O}(\ell)$  ancilla qubits [Gid19]. Therefore, we can construct the oracle  $\mathcal{O}_{f_\mu}$  using  $\mathcal{O}(\text{nnz}(A) \cdot \text{poly}(\ell))$  elementary gates and  $\mathcal{O}(\text{nnz}(A) \cdot \text{poly}(\ell))$  ancilla qubits.  $\square$

Now, we are ready to give an end-to-end complexity result of our quantum algorithm in the standard gate model. We point out that we made no attempt to optimize the implementation of the evaluation oracle; it may be possible to improve the gate cost further, particularly for highly LO problems.

**Corollary 1.** *Let  $\varepsilon \in (0, 1)$ . Assume that  $\text{nnz}(A) \geq \sqrt{m+n}$ . Choose  $\gamma \in (0, 1)$  and  $\delta \in (0, 1)$ . Then, with probability at least  $1 - \delta$ , Algorithm 1 returns a classical vector  $z \in \mathbb{R}_{++}^{\bar{n}}$  such that  $(z, s(z)) \in \mathcal{N}_2(\gamma)$  with  $d_2(z, s(z); \varepsilon) \leq \gamma\varepsilon$ . Moreover, if the LO problem data  $(A, b, c)$  is specified in a classical data structure (e.g., a sparse matrix list, etc.), Algorithm 1 can be implemented with*

$$\mathcal{O}\left(\sqrt{m+n} \cdot \text{nnz}(A) \frac{R_1}{\varepsilon} \cdot \text{polylog}\left(m, n, \frac{1}{\delta}\right)\right)$$

*elementary gates, where  $\text{nnz}(A)$  is the number of non-zeros in the matrix  $A$ .*

*Proof.* Without loss of generality, we assume  $\text{nnz}(A) \geq \sqrt{m+n}$ . By Lemma 3, the quantum evaluation oracle  $O_{f_\mu}$  can be constructed up to fixed digit precision using  $\mathcal{O}(\text{nnz}(A))$  elementary gates. Therefore, the overall gate complexity follows from Theorem 5.  $\square$

### 3.5 Comparison to existing LO solvers

In Table 2 we provide a comparison of our algorithms to the current state of the art algorithms for solving Linear Optimization problems in both the classical and quantum models of computation. For the classical setting, we state the overall cost of the algorithms in both time complexity and bit complexity. In the quantum setting, the costs for each algorithm are provided in the gate model and the QRAM input model.

The QCPM achieves polynomial speedups over the state of the art classical and quantum IPMs in  $m$  and  $n$  under the mild requirement  $\text{nnz}(A) < (m+n)^{\omega-\frac{1}{2}}$ . Without further enhancements to our framework, an end-to-end speedup over these algorithms is only possible in the low-precision regime, and requires  $\text{nnz}(A)R_1 < (m+n)^{\omega-\frac{1}{2}} \cdot \text{polylog}(m, n)$ . Strikingly, these speedups are not reliant on QRAM: our algorithm only requires access to the natural binary description for the LO problem data  $(A, b, c)$ . When compared to the QIPM in [AG23], our algorithm provides a speedup in  $n$  even when  $A$  is fully dense. However, the QIPM in [AG23] exhibits superior dependence on  $m$ , making it the favorable choice for very tall LO problems (with  $m \gg n$ ), or problems with very large solution size (in the  $\ell_1$ -norm). That said, we emphasize that the QIPM found in [AG23] assume access to a classical-write/quantum-read RAM (QRAM), and it is unclear whether the running time of their algorithm would remain competitive in the gate model (without QRAM).

Another approach to solve linear optimization problems, besides IPMs, is the Primal-Dual Hybrid Gradient (PDHG) algorithm with restarts [AHL23]. This algorithm can solve some huge-scale LO problems in practice, while offering interesting theoretical guarantees. PDHG is a first-order method, and only needs to compute matrix-vector products involving  $A$  at each iteration. This framework achieves polylogarithmic dependence on precision using restarts, and the number of restarts depends on the Hoffman constant  $\kappa$  of the KKT system associated to (P)-(D). The Hoffman constant is notoriously difficult to compute and rigorously bound; it is often left “as is” and not expressed as a function of the input size. The Hoffman constant is generally considered a pessimistic bound on the iteration complexity of restarted PDHG [XF23]. Our algorithm’s performance matches that of PDHG when  $\kappa = \mathcal{O}(\frac{\sqrt{m+n}R_1}{\varepsilon})$ , but we are not aware of explicit bounds for  $\kappa$  in the general case so a direct comparison with our algorithm is difficult.

There are also quantum algorithms based on zero-sum games and Gibbs sampling [vAG19, BGJ<sup>+</sup>23, GJLW24] that achieve sublinear running times in  $m$  and  $n$ , and polynomial dependence on precision and an  $\ell_1$ -norm upper bound  $R_1$  on the size of the solution to (P)-(D). Observe that the superquadratic dependence on  $R_1/\varepsilon$  for the algorithms in [vAG19, BGJ<sup>+</sup>23, GJLW24] is worse than the linear scaling enjoyed by the QCPM.

Algorithms for zero-sum games also operate under the assumption that the problem data is normalized: it is assumed that  $A \in [-1, 1]^{m \times n}$  in [vAG19, BGJ<sup>+</sup>23] and  $\|A\| \leq 1$  in [GJLW24]. These algorithms employ different definitions of optimality and feasibility than (Q)IPMs and our QCPM. The output of (Q)IPMs and the QCPM is a classical primal-dual pair  $(x, y)$  satisfying

$$\begin{aligned} A_{i,\cdot} x &\leq b_i \quad \forall i \in [m], \quad x > 0, \\ \left(A^\top\right)_{\cdot,i} y &\leq c_i \quad \forall i \in [n], \quad y > 0, \end{aligned}$$

Table 2: Complexity to solve the primal-dual pair (P)-(D) to precision  $\varepsilon$ 

Classical Algorithms	Time complexity		Gate complexity		QRAM
IPM	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\sqrt{n}(\text{nnz}(A) + n^2))$	[LS15]			
IPM	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}((m+n)^\omega)$	[CLS21, vdB20]			
PDHG	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\kappa \cdot \text{nnz}(A))$	[AHL23]			
Quantum Algorithms	Query complexity		Gate complexity		QRAM
QMMWU	$\tilde{\mathcal{O}}(\sqrt{m+n}R_1^{2.5}\varepsilon^{-2.5} + R_1^3\varepsilon^{-3})$	[BGJ+23, GJLW24]	$\tilde{\mathcal{O}}(\sqrt{m+n}R_1^{2.5}\varepsilon^{-2.5} + R_1^3\varepsilon^{-3})$	[BGJ+23, GJLW24]	✓
QIPM	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\sqrt{mn^5})$	[AG23, Section 7]	$\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\sqrt{mn}(n^{6.5}s^2 + n^{\omega+2}))$	[AG23, Theorem 1.1]	✓
QCPM (this work)	$\tilde{\mathcal{O}}_{m,n,\alpha}(\sqrt{m+n}R_1\varepsilon^{-1})$	Theorem 5	$\tilde{\mathcal{O}}_{m,n,\alpha}(\sqrt{m+n}\text{nnz}(A)R_1\varepsilon^{-1})$	Corollary 1	✗

with

$$c^\top x - b^\top y \leq \varepsilon.$$

Conversely, quantum algorithms for zero-sum games output a dual solution  $y \in \mathbb{R}^m$  such that the primal-dual pair  $(x, y)$  satisfies

$$\begin{aligned} A_i \cdot x &\leq b_i + \varepsilon_{\text{abs}} \quad \forall i \in [m], \quad x > 0, \\ (A^\top)_{i \cdot} \cdot y &\leq c_i + \varepsilon_{\text{abs}}, \quad \forall i \in [n], \quad y > 0, \end{aligned}$$

and the objective value attained by this solution is  $\text{OPT} \in [\varsigma - \mathcal{O}(\varepsilon_{\text{abs}}), \varsigma + \mathcal{O}(\varepsilon_{\text{abs}})]$ , where  $\varsigma$  is a bound on the optimal objective value determined using binary search. Observe that this is another difference between the QCPM/(Q)IPMs and quantum algorithms for zero-sum games: in contrast with the approximate infeasibility of solutions obtained from the zero-sum games approach, the output of the QCPM/(Q)IPMs always satisfies primal and dual feasibility *exactly*. We also point out that, like QIPMs, the state of the art running times for quantum algorithms for zero-sum games rely on access to QRAM, which is highlighted in Table 2 – the query and gate complexities are the same (at least up to polylogarithmic factors).

We note that there may exist a regime in which our algorithm is faster than known approaches. Assume that  $m \leq n$ , and that  $\text{nnz}(A) = \tilde{\mathcal{O}}(n)$ , e.g.,  $A$  has  $\mathcal{O}(\log(n))$ -row sparsity. Finally, assume  $R_1 = \mathcal{O}(n^\xi)$  for some  $\xi$ , and  $\xi$  is such that the gate complexity of the QCPM is

$$\tilde{\mathcal{O}}(n^{1.5+\xi}) = \tilde{\mathcal{O}}(n^{\omega-\delta}) \implies \xi \leq .87 - \delta,$$

where  $\delta > 0$  is independent of  $n$ . On the other hand, the gate complexity of the algorithms based on zero-sum games for this regime is  $\tilde{\mathcal{O}}(n^{0.5+2.5\xi})$ , so under these assumptions, for  $\xi \geq .67$  the QCPM has better asymptotic complexity than known classical and quantum state-of-the-art algorithms.

## 4 Conclusion and outlook

In this work, we proposed a new quantum algorithm that solves linear optimization problems by quantum evolution of the central path. Combining our approach with iterative refinement techniques, one can obtain an exact solution to a linear optimization problem involving  $m$  constraints and  $n$  variables using at most  $\tilde{\mathcal{O}}_{m,n,\frac{1}{\varepsilon}}(\sqrt{m+n}\text{nnz}(A)\frac{R_1}{\varepsilon})$  elementary gates, where  $\text{nnz}(A)$  is the total number of nonzero entries found in  $A$ . When the constraint matrix  $A$  is sufficiently sparse in the sense that  $\text{nnz}(A) < (m+n)^{\omega-\frac{1}{2}}$ , our results imply a polynomial speedup in  $m$  and  $n$  over all general-purpose classical and quantum algorithms for solving linear optimization problems that have polylogarithmic dependence on the error. For problems satisfying  $\text{nnz}(A)R_1 < (m+n)^{\omega-\frac{1}{2}} \cdot \text{polylog}(m, n)$

the QCPM attains an end-to-end speedup over these algorithms in the low-precision regime. We stress that these speedups are not reliant on QRAM, our algorithm only relies on data structures for the sparse binary representation of  $(A, b, c)$ .

Though we leave this for future work, our framework should readily generalize to more complex classes of convex optimization problems, such as semidefinite optimization and second-order conic optimization. Our work highlights a previously unexplored connection between Interior Point Methods and the Quantum Adiabatic Algorithm [FGGS00], and we believe further exploration into this relationship to be a worthy endeavor.

## Acknowledgements

This project has been carried out thanks to funding by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Quantum Systems Accelerator, the Defense Advanced Research Projects Agency (DARPA), ONISQ grant W911NF2010022, titled The Quantum Computing Revolution and Optimization: Challenges and Opportunities. G. Nannicini is supported by ONR award # N000142312585. J.L. and X.W. are partially supported by the U.S. National Science Foundation grant CCF-1816695 and CCF-1942837 (CAREER), and a Sloan research fellowship. J.L. is partially supported by the Simons Quantum Postdoctoral Fellowship and a Simons Investigator award through Grant No. 825053.

## References

- [AG23] Simon Apers and Sander Gribling. Quantum speedups for linear programming via interior point methods. *arXiv preprint arXiv:2311.03215*, 2023.
- [AHL23] David Applegate, Oliver Hinder, Haihao Lu, and Miles Lubin. Faster first-order primal-dual methods for linear programming using restarts and sharpness. *Mathematical Programming*, 201(1):133–184, 2023.
- [AKL<sup>+</sup>24] Arpit Agarwal, Sanjeev Khanna, Huan Li, Prathamesh Patil, Chen Wang, Nathan White, and Peilin Zhong. Parallel approximate maximum flows in near-linear work and polylogarithmic depth. In *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 3997–4061. SIAM, 2024.
- [AL22] Dong An and Lin Lin. Quantum linear system solver based on time-optimal adiabatic quantum computing and quantum approximate optimization algorithm. *ACM Transactions on Quantum Computing*, 3(2):1–28, 2022.
- [Ans88] Kurt M. Anstreicher. Linear programming and the Newton barrier flow. *Mathematical Programming*, 41(1-3):367–373, 1988.
- [ANTZ23] Brandon Augustino, Giacomo Nannicini, Tamás Terlaky, and Luis F. Zuluaga. Quantum interior point methods for semidefinite optimization. *Quantum*, 7:1110, 2023.
- [BCL18] Sébastien Bubeck, Michael B. Cohen, Yin Tat Lee, and Yuanzhi Li. An homotopy method for  $\ell_p$ -regression provably beyond self-concordance and in input-sparsity time. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1130–1137, 2018.

- [BGJ<sup>+</sup>23] Adam Bouland, Yosheb M. Getachew, Yujia Jin, Aaron Sidford, and Kevin Tian. Quantum speedups for zero-sum games via improved dynamic Gibbs sampling. In *International Conference on Machine Learning*, pages 2932–2952. PMLR, 2023.
- [BL89a] Dave A. Bayer and Jeffrey C. Lagarias. The nonlinear geometry of linear programming. I. Affine and projective scaling trajectories. *Transactions of the American Mathematical Society*, 314(2):499–526, 1989.
- [BL89b] Dave A. Bayer and Jeffrey C. Lagarias. The nonlinear geometry of linear programming. II. Legendre transform coordinates and central trajectories. *Transactions of the American Mathematical Society*, 314(2):527–581, 1989.
- [Bou99] Jean Bourgain. On growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potential. *Journal d’Analyse Mathématique*, 77(1):315–348, 1999.
- [CDS<sup>+</sup>22] B. David Clader, Alexander M. Dalzell, Nikitas Stamatopoulos, Grant Salton, Mario Berta, and William J. Zeng. Quantum resources required to block-encode a matrix of classical data. *IEEE Transactions on Quantum Engineering*, 3:1–23, 2022.
- [CGJ19] Shantanav Chakraborty, András Gilyén, and Stacey Jeffery. The power of block-encoded matrix powers: improved regression techniques via faster Hamiltonian simulation. In Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano Leonardi, editors, *46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, volume 132, pages 33:1–33:14, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
- [CLL<sup>+</sup>22] Andrew M. Childs, Jiaqi Leng, Tongyang Li, Jin-Peng Liu, and Chenyi Zhang. Quantum simulation of real-space dynamics. *Quantum*, 6:860, 2022.
- [CLS21] Michael B. Cohen, Yin Tat Lee, and Zhao Song. Solving linear programs in the current matrix multiplication time. *Journal of the ACM (JACM)*, 68(1):1–39, 2021.
- [CMSV17] Michael B. Cohen, Aleksander Mądry, Piotr Sankowski, and Adrian Vladu. Negative-weight shortest paths and unit capacity minimum cost flow in  $\tilde{O}(m^{10/7} \log W)$  time. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 752–771. SIAM, 2017.
- [CMTV17] Michael B. Cohen, Aleksander Mądry, Dimitris Tsipras, and Adrian Vladu. Matrix scaling and balancing via box constrained Newton’s method and interior point methods. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 902–913. IEEE, 2017.
- [CW12] Andrew M. Childs and Nathan Wiebe. Hamiltonian Simulation using linear combinations of unitary operations. *Quantum Information and Computation*, 12(11–12):901–924, Nov 2012.
- [Dan48] George B. Dantzig. Programming in a linear structure. In *Bulletin of the American Mathematical Society*, volume 54, pages 1074–1074. AMER MATHEMATICAL SOC 201 CHARLES ST, PROVIDENCE, RI 02940-2213, 1948.

- [DCS<sup>+</sup>23] Alexander M. Dalzell, B. David Clader, Grant Salton, Mario Berta, Cedric Yen-Yu Lin, David A. Bader, Nikitas Stamatopoulos, Martin J.A. Schuetz, Fernando G.S.L. Brandão, Helmut G. Katzgraber, et al. End-to-end resource analysis for quantum interior-point methods and portfolio optimization. *PRX Quantum*, 4(4):040325, 2023.
- [Dik67] I.I. Dikin. Iterative solution of problems of linear and quadratic programming. In *Doklady Akademii Nauk*, volume 174, pages 747–748. Russian Academy of Sciences, 1967.
- [Dik74] I.I. Dikin. On the convergence of an iteration process. *Upravlyaemya Sistemy*, (12):54–60, 1974.
- [DOW<sup>+</sup>55] George B. Dantzig, Alex Orden, Philip Wolfe, et al. The generalized simplex method for minimizing a linear form under linear inequality restraints. *Pacific Journal of Mathematics*, 5(2):183–195, 1955.
- [Fay91] Leonid Faybusovich. Hamiltonian structure of dynamical systems which solve linear programming problems. *Physica D: Nonlinear Phenomena*, 53(2-4):217–232, 1991.
- [Fay95] Leonid Faybusovich. A Hamiltonian formalism for optimization problems. *Journal of Mathematical Systems, Estimation, and Control*, 5(3):367–370, 1995.
- [FGGS00] Edward Farhi, Jeffrey Goldstone, Sam Gutmann, and Michael Sipser. Quantum computation by adiabatic evolution, 2000. arXiv:quant-ph/0001106.
- [FM64] Anthony V. Fiacco and Garth P. McCormick. The sequential unconstrained minimization technique for nonlinear programming, a primal-dual method. *Management Science*, 10(2):360–366, 1964.
- [FM90] Anthony V. Fiacco and Garth P. McCormick. *Nonlinear programming: sequential unconstrained minimization techniques*. SIAM, 1990. First published in 1968 by Research Analysis Corporation.
- [Fri54] Ragnar Anton Kittil Frisch. *Principles of linear programming: With particular reference to the double gradient form of the logarithmic potential method*. Universitets Socialøkonomiske Institutt, 1954.
- [Fri55] Ragnar Anton Kittil Frisch. The logarithmic potential method of convex programming. *Memorandum, University Institute of Economics, Oslo*, 5(6), 1955.
- [Fri56] Ragnar Anton Kittil Frisch. La résolution des problèmes de programme linéaire par la méthode du potentiel logarithmique. *Cahiers du Séminaire D’Econometrie*, pages 7–23, 1956.
- [Gid19] Craig Gidney. Asymptotically efficient quantum Karatsuba multiplication, 2019. arXiv:1904.07356.
- [GJLW23] Minbo Gao, Zhengfeng Ji, Tongyang Li, and Qisheng Wang. Logarithmic-regret quantum learning algorithms for zero-sum games. In A. Oh, T. Neumann, A. Globerson, K. Saenko, M. Hardt, and S. Levine, editors, *Advances in Neural Information Processing Systems*, volume 36, pages 31177–31203. Curran Associates, Inc., 2023.

- [GJLW24] Minbo Gao, Zhengfeng Ji, Tongyang Li, and Qisheng Wang. Logarithmic-regret quantum learning algorithms for zero-sum games. *Advances in Neural Information Processing Systems*, 36, 2024.
- [Gon89] Clóvis C. Gonzaga. An algorithm for solving linear programming problems in  $\mathcal{O}(n^3L)$  operations. In *Progress in Mathematical Programming: Interior-Point and Related Methods*, pages 1–28. Springer, 1989.
- [GS18] David J. Griffiths and Darrell F. Schroeter. *Introduction to quantum mechanics*. Cambridge University Press, 2018.
- [GS20] Ambros M. Gleixner and Daniel E. Steffy. Linear programming using limited-precision oracles. *Mathematical Programming*, 183(1):525–554, 2020.
- [GSW12] Ambros M. Gleixner, Daniel E. Steffy, and Kati Wolter. Improving the accuracy of linear programming solvers with iterative refinement. In Joris van der Hoeven and Mark van Hoeij, editors, *Proceedings of the 37th International Symposium on Symbolic and Algebraic Computation*, pages 187–194, 2012.
- [GSW16] Ambros M. Gleixner, Daniel E. Steffy, and Kati Wolter. Iterative refinement for linear programming. *INFORMS Journal on Computing*, 28(3):449–464, 2016.
- [HHL09] Aram W. Harrow, Avinatan Hassidim, and Seth Lloyd. Quantum algorithm for linear systems of equations. *Physical Review Letters*, 103(15):150502, 2009.
- [HJ02] George A. Hagedorn and Alain Joye. Elementary exponential error estimates for the adiabatic approximation. *Journal of Mathematical Analysis and Applications*, 267(1):235–246, 2002.
- [HKZ12] Daniel Hsu, Sham Kakade, and Tong Zhang. A tail inequality for quadratic forms of subgaussian random vectors. *Electronic Communications in Probability*, 17:1 – 6, 2012.
- [HS12] Peter D. Hislop and Israel Michael Sigal. *Introduction to spectral theory: With applications to Schrödinger operators*, volume 113. Springer Science & Business Media, 2012.
- [JKL<sup>+</sup>20] Haotian Jiang, Tarun Kathuria, Yin Tat Lee, Swati Padmanabhan, and Zhao Song. A faster interior point method for semidefinite programming. In Sandy Irani, Lisa O’Conner, and Patrick Kellenberger, editors, *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 910–918. IEEE, 2020.
- [JSWZ20] Shunhua Jiang, Zhao Song, Omri Weinstein, and Hengjie Zhang. Faster dynamic matrix inverse for faster LPs, 2020. arXiv:2004.07470.
- [Kar84] Narendra Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, (4):373–395, 1984.
- [Kha80] Leonid G. Khachiyan. Polynomial algorithms in linear programming. *USSR Computational Mathematics and Mathematical Physics*, 20(1):53–72, 1980.

- [KP20] Iordanis Kerenidis and Anupam Prakash. A quantum interior point method for LPs and SDPs. *ACM Transactions on Quantum Computing*, 1(1):1–32, 2020.
- [LHLW23] Jiaqi Leng, Ethan Hickman, Joseph Li, and Xiaodi Wu. Quantum Hamiltonian Descent, 2023. arXiv:2303.01471.
- [LS14] Yin Tat Lee and Aaron Sidford. Path finding methods for linear programming: Solving linear programs in  $\mathcal{O}(\sqrt{\text{rank}})$  iterations and faster algorithms for maximum flow. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 424–433. IEEE, 2014.
- [LS15] Yin Tat Lee and Aaron Sidford. Efficient inverse maintenance and faster algorithms for linear programming. In *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*, pages 230–249. IEEE, 2015.
- [LZW23] Jiaqi Leng, Yufan Zheng, and Xiaodi Wu. A quantum-classical performance separation in nonconvex optimization, 2023. arXiv:2311.00811.
- [Meg89] Nimrod Megiddo. *Pathways to the optimal set in linear programming*. Springer, 1989.
- [Mes58] Albert Messiah. *Quantum Mechanics*. John Wiley & Sons, New York, 1958.
- [MFT22] Mohammadhossein Mohammadisiahroudi, Ramin Fakhimi, and Tamás Terlaky. Efficient use of quantum linear system algorithms in interior point methods for linear optimization, 2022. arXiv:2205.01220.
- [MFT24] Mohammadhossein Mohammadisiahroudi, Ramin Fakhimi, and Tamás Terlaky. Efficient use of quantum linear system algorithms in inexact infeasible ipms for linear optimization. *Journal of Optimization Theory and Applications*, 202(1):146–183, June 2024.
- [M13] Aleksander Mądry. Navigating central path with electrical flows: From flows to matchings, and back. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 253–262. IEEE, 2013.
- [M16] Aleksander Mądry. Computing maximum flow with augmenting electrical flows. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 593–602. IEEE, 2016.
- [NN88] Yurii E. Nesterov and Arkadi Nemirovskii. A general approach to polynomial-time algorithms design for convex programming. *Report, Central Economical and Mathematical Institute, USSR Academy of Sciences, Moscow*, 1988.
- [NN94] Yurii E. Nesterov and Arkadi Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*, volume 13. SIAM, 1994.
- [Osb12] Tobias J. Osborne. Hamiltonian complexity. *Reports on progress in physics*, 75(2):022001, 2012.



- [Peñ24] Javier F. Peña. An easily computable upper bound on the hoffman constant for homogeneous inequality systems. *Computational Optimization and Applications*, 87(1):323–335, 2024.
- [Ren88] James Renegar. A polynomial-time algorithm, based on Newton’s method, for linear programming. *Mathematical Programming*, 40(1-3):59–93, 1988.
- [Ren01] James Renegar. *A mathematical view of interior-point methods in convex optimization*. SIAM, 2001.
- [RTV05] Cornelis Roos, Tamás Terlaky, and Jean-Phillipe Vial. *Interior Point Methods for Linear Optimization*. Springer Science & Business Media, 2005.
- [Ter13] Tamás Terlaky. *Interior Point Methods of Mathematical Programming*, volume 5. Springer Science & Business Media, 2013.
- [TK02] Kim-Chuan Toh and Masakazu Kojima. Solving some large scale semidefinite programs via the conjugate residual method. *SIAM Journal on Optimization*, 12(3):669–691, 2002.
- [vAG19] Joran van Apeldoorn and András Gilyén. Quantum algorithms for zero-sum games, 2019. arXiv:1904.03180.
- [Vai87] Pravin M Vaidya. An algorithm for linear programming which requires  $\mathcal{O}((m+n)n^2 + (m+n)^{1.5}n)L$  arithmetic operations. In *Proceedings of the nineteenth annual ACM symposium on Theory of computing*, pages 29–38, 1987.
- [VBE96] Vlatko Vedral, Adriano Barenco, and Artur Ekert. Quantum networks for elementary arithmetic operations. *Physical Review A*, 54(1):147, 1996.
- [vdB20] Jan van den Brand. A deterministic linear program solver in current matrix multiplication time. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 259–278. SIAM, 2020.
- [vdBLSS20] Jan van den Brand, Yin Tat Lee, Aaron Sidford, and Zhao Song. Solving tall dense linear programs in nearly linear time. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 775–788, 2020.
- [Vla23] Adrian Vladu. Interior point methods with a gradient oracle. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*, pages 1876–1889, 2023.
- [Wri97] Stephen J. Wright. *Primal-Dual Interior-Point Methods*, volume 54. SIAM, 1997.
- [WWJ16] Andre Wibisono, Ashia C. Wilson, and Michael I. Jordan. A variational perspective on accelerated methods in optimization. *Proceedings of the National Academy of Sciences*, 113(47):E7351–E7358, 2016.
- [XF23] Zikai Xiong and Robert Michael Freund. Computational Guarantees for Restarted PDHG for LP based on “Limiting Error Ratios” and LP Sharpness. *arXiv preprint arXiv:2312.14774*, 2023.

- [Ye11] Yinyu Ye. *Interior point algorithms: theory and analysis*. John Wiley & Sons, 2011.
- [YTM94] Yinyu Ye, Michael J. Todd, and Shinji Mizuno. An  $\mathcal{O}(\sqrt{n}L)$ -iteration homogeneous and self-dual linear programming algorithm. *Mathematics of Operations Research*, 19(1):53–67, 1994.

# Appendices

## A Technical lemmas

Before proceeding further, we provide two results that will be useful in our analysis later in the paper. For ease of notation, we define  $F(z) := z \odot s(z) \in \mathbb{R}^{\bar{n}}$ . We begin by establishing that the Jacobian of  $F(z)$  is non-singular over the positive orthant.

**Lemma 4.** *Let  $Z := \text{diag}(z)$ ,  $S := \text{diag}(s(z))$  and  $\mathcal{J}(z) := \frac{\partial F(z)}{\partial z}$ . Then,*

$$\mathcal{J}(z) = ZM + S \tag{25}$$

*is non-singular whenever  $z, s(z) > 0$ .*

*Proof.* We use the following identities:

$$F(z) = z \odot s(z) = z \odot (Mz + q) = \text{diag}(z)Mz + q \odot z.$$

To show that  $\mathcal{J}(z) = ZM + S$  is non-singular whenever  $z, s > 0$ , note that in this case  $D = Z^{-1}S$  is a positive diagonal matrix, and hence

$$u^\top (M + D)u = u^\top Mu + u^\top Du = 0 \iff u^\top Du = 0 \iff u = 0,$$

as  $u^\top Mu = 0$  for all  $u \in \mathbb{R}^{\bar{n}}$  since  $M$  is skew-symmetric.  $\square$

The result in Lemma 4 allows us to analyze the singular values of the Jacobian  $\mathcal{J}(z)$  of  $F(z)$ , and thus the spectrum of the matrix  $\mathcal{J}(z)^\top \mathcal{J}(z)$ , which will play a crucial role in the quantum Hamiltonian we define later in Section 3.

**Lemma 5.** *Let  $\mu > 0$  and  $z(\mu)$  denote the  $\mu$ -center. Then, the smallest singular value of*

$$\mathcal{J}(z(\mu)) := \left. \frac{\partial F(z)}{\partial z} \right|_{z=z(\mu)}$$

*satisfies*

$$\sigma_{\min}(\mathcal{J}(z(\mu))) \geq \frac{\mu}{R_\infty} > 0,$$

*where  $R_\infty > 0$  is an  $\ell_\infty$  upper bound on  $z$  and  $s(z)$ .*

*Proof.* Let  $s(\mu) := s(z(\mu))$ . Note that if  $z(\mu)$  is the  $\mu$ -center, then we must have

$$\text{diag}(z(\mu)) \text{diag}(s(\mu)) = Z(\mu)S(\mu) = S(\mu)Z(\mu) = \mu I,$$

with  $z(\mu), s(\mu) > 0$ . Since  $Z(\mu)$  and  $S(\mu)$  are positive diagonal matrices, we may write

$$Z(\mu)^2 := \text{diag}\left((z(\mu))_1^2, \dots, (z(\mu))_{\bar{n}}^2\right), \quad S(\mu)^2 := \text{diag}\left((s(\mu))_1^2, \dots, (s(\mu))_{\bar{n}}^2\right).$$

Now, (SP) is strictly feasible, and so its solution set  $\mathcal{SP}$  is bounded. Without loss of generality, we may assume that there exists a positive constant  $R_\infty > 0$  such that

$$\frac{1}{R_\infty} \leq z_i, s(z)_i \leq R_\infty \quad \forall i \in [\bar{n}].$$

Hence,

$$\min_{i \in [\bar{n}]} \{s(\mu)_i\} = \min_{i \in [\bar{n}]} \left\{ \frac{\mu}{z(\mu)_i} \right\} = \frac{\mu}{\max_{i \in [\bar{n}]} \{z(\mu)_i\}} \geq \frac{\mu}{R_\infty}, \quad (26)$$

since  $z(\mu)_i s(\mu)_i = \mu$  for all  $i \in [\bar{n}]$ .

Given that  $z(\mu)$  and  $s(\mu)$  are strictly positive, Lemma 4 ensures that  $\mathcal{J}(z(\mu))$  is non-singular, and as a consequence,  $\mathcal{J}(z(\mu))^\top \mathcal{J}(z(\mu))$  is positive definite. Combining these facts, and noting that  $M$  is skew-symmetric, we obtain

$$\begin{aligned} \mathcal{J}(z(\mu))^\top \mathcal{J}(z(\mu)) &= (Z(\mu)M + S(\mu))^\top (Z(\mu)M + S(\mu)) \\ &= M^\top Z(\mu)^2 M + M^\top Z(\mu)S(\mu) + S(\mu)Z(\mu)M + S(\mu)^2 \\ &= M^\top Z(\mu)^2 M - MZ(\mu)S(\mu) + S(\mu)Z(\mu)M + S(\mu)^2 \\ &= M^\top Z(\mu)^2 M - \mu MI + \mu IM + S(\mu)^2 \\ &= M^\top Z(\mu)^2 M + S(\mu)^2 \succ 0. \end{aligned} \quad (27)$$

Combining equations (26) and (27) with Weyl's inequality, we have

$$\begin{aligned} \sigma_{\min}(\mathcal{J}(z(\mu))) &= \sqrt{\lambda_{\min}(M^\top Z(\mu)^2 M + S(\mu)^2)} \geq \sqrt{\lambda_{\min}(S(\mu)^2)} \\ &= \min_{i \in [\bar{n}]} \{(S(\mu))_{ii}\} \geq \frac{\mu}{R_\infty} > 0. \end{aligned} \quad (28)$$

The proof is complete.  $\square$

Next, we provide an upper bound for the row-norms of  $M$  in terms of the LO problem data.

**Lemma 6.** *Let  $M$  be the coefficient matrix of the self dual embedding model (SP). Then,*

$$\max_{i \in [\bar{n}-1]} \|M_{i,\cdot}\| \leq 3.$$

*Proof.* The result is a simple consequence of the assumption  $\|A_{1,\cdot}\|, \dots, \|A_{m,\cdot}\|, \|b\|, \|c\| \leq 1$  and the triangle inequality.  $\square$

## B Proof of Proposition 1

**Proposition 3** (Proposition 1, [HKZ12]). *Let  $V \in \mathbb{R}^{m \times n}$  be a matrix, and let  $\Sigma = V^T V$ . Let  $x = (x_1, \dots, x_n)$  be an isotropic multivariate Gaussian random vector with mean zero. For all  $t > 0$ ,*

$$\Pr \left[ \|Vx\|^2 > \text{Tr}(\Sigma) + 2\sqrt{\text{Tr}(\Sigma^2)t} + 2\|\Sigma\|t \right] \leq e^{-t}.$$

Now, we are ready to prove Proposition 1.

*Proof.* Recall that we define  $f(x; \mu) = \frac{1}{2} \|F(x) - \mu e\|^2 = \frac{1}{2} d_2^2(x; \mu)$ . So  $x \in \mathcal{N}_2(\gamma)$  is equivalent to

$$f(x) \leq \frac{\gamma^2 \mu^2}{2}.$$

Let  $W := \frac{h}{2}(H(\mu))^{-1/2}$ . We define  $y := W^{-1/2}(x - z(\mu))$ , then for  $x \sim |\Phi_0(\mu)|^2$ , we have  $y \sim \mathcal{N}(0, \mathbb{I})$ . In a small neighborhood of  $z(\mu)$ ,

$$f_\mu(x) \approx \frac{1}{2}(x - z(\mu))^T H(\mu)(x - z(\mu)) = \frac{1}{2}y^T (H(\mu)W)y = \|Vy\|^2,$$

where  $V := \frac{1}{2}\sqrt{h}(H(\mu))^{1/4}$ . Let  $\Sigma := V^2 = \frac{h}{4}(H(\mu))^{1/2}$ , we have one can write

$$\begin{aligned} \text{tr}[\Sigma^2] &= \frac{h^2}{16} \sum_{i=1}^{\bar{n}} \left( M^\top Z(\mu)^2 M + S(\mu)^2 \right)_{ii} = \frac{h^2}{16} \sum_{i=1}^{\bar{n}} z(\mu)_i^2 \|M_{i,\cdot}\|^2 + s(\mu)_i^2 \\ &\leq \frac{h^2}{16} \left( \max_{i \in [\bar{n}]} \|M_{i,\cdot}\|^2 \cdot \|z\|_1^2 + \|s(z)\|_1^2 \right) \\ &\stackrel{\text{Lemma 6}}{\leq} 2h^2 R_1^2. \end{aligned}$$

Therefore,

$$\text{tr}[\Sigma] \leq \sqrt{\bar{n}(2h^2 R_1^2)} = \sqrt{2}h\sqrt{\bar{n}}R_1.$$

By Proposition 3 we need to choose  $h$  such that

$$\Pr_{x \sim |\Phi_0(\mu)|^2} \left[ f(x; \mu) > \frac{h}{2\sqrt{2}}\sqrt{\bar{n}}R_1\alpha + 2\sqrt{2h^2\sqrt{\bar{n}}R_1^2} + 2R_1 \log\left(\frac{1}{\delta}\right) \right] \leq \delta.$$

Using the identity  $2\sqrt{ab} \leq a + b$  implies

$$\sqrt{2}h\alpha R_1 \left[ \sqrt{\bar{n}} + 2\sqrt{\sqrt{\bar{n}} \log\left(\frac{1}{\delta}\right)} + 2 \log\left(\frac{1}{\delta}\right) \right] \leq \sqrt{2}hR_1 \left[ 2\sqrt{\bar{n}} + 3 \log\left(\frac{1}{\delta}\right) \right].$$

Hence, in order to ensure

$$\Pr_{x \sim |\Phi_0(\mu)|^2} \left[ f(x; \mu) \geq \frac{1}{2}\gamma^2\mu^2 \right],$$

we must choose

$$h = \frac{\mu^2}{\sqrt{2\bar{n}}R_1}.$$

□

## C Additional Proofs

**Lemma 7.** Let  $\Psi(x, t)$  denote the exact solution of (10) and  $\tilde{\Psi}(x, t)$  denote the approximated solution by the Fourier spectral method (truncated up to frequency  $n$ ).<sup>5</sup> We assume that the initial data  $\Psi_0(x)$  is periodic and analytic in  $x \in \Omega$ . Then, for any integer  $n \geq 1$ , the error from the Fourier spectral method satisfies

$$\max_{x,t} |\Psi(x, t) - \tilde{\Psi}(x, t)| \leq 2r^{n/2+1}, \quad (29)$$

where  $0 < r < \min(\frac{1}{2}, \frac{1}{At})$ ,  $A$  is an absolute constant that only depends on  $\Psi_0(x)$ .

<sup>5</sup>More details on the Fourier spectral method can be found in Section 2.2 (in particular Lemma 1) in [CLL<sup>+</sup>22].

*Proof.* We give the proof in one dimension, as the same argument is readily generalized to arbitrary finite dimensions. We assume the initial data  $\Psi_0(x)$  is periodic over  $[0, 2\pi]$ . The analyticity implies that, for any  $x \in [0, 2\pi]$ , there is a constant  $C$  such that

$$\left| \Psi_0^{(k)}(x) \right| \leq C^{k+1}(k!). \quad (30)$$

Therefore, the function  $\Psi_0(x)$  admits an analytic continuation in the strip

$$\Gamma_0 = \left\{ z \in \mathbb{C} : |z - x| < \frac{1}{C}, x \in [0, 2\pi] \right\}.$$

Due to [Bou99, Proposition 1], there is an absolute constant  $A$  such that for any  $t \in [0, T]$ ,

$$\left\| \Psi^{(k)}(\cdot, t) \right\| \leq At \left\| \Psi_0^{(k)} \right\| \leq AC^{k+1}t(k!), \quad (31)$$

which implies that the wave function  $\Psi(x, t)$  is periodic and analytic for any finite  $t$ . The strip on which  $\Psi(x, t)$  admits an analytic continuation is

$$\Gamma_t = \left\{ z \in \mathbb{C} : |z - x| < \frac{1}{ACt}, x \in [0, 2\pi] \right\}.$$

Suppose that the function  $\Psi(x, t)$  allows an exact, infinite trigonometric polynomial representation (see [CLL<sup>+</sup>22, Lemma 16]),

$$\Psi(x, t) = \sum_{k=0}^{\infty} c_k(t) e^{ikx}. \quad (32)$$

Let  $\tilde{\Psi}(t, x)$  be the truncated Fourier series up to  $k = n/2$ , then the error from the Fourier spectral method satisfies

$$|\Psi(x, t) - \tilde{\Psi}(x, t)| \leq \sum_{k=n/2+1}^{\infty} |c_k(t)|. \quad (33)$$

Meanwhile, the function  $\Psi(x, t)$  has an analytic continuation defined by  $\Psi(x, t) = \sum_{k=0}^{\infty} c_k(t) z^k$  for  $z \in \Gamma_t$ . By Cauchy's integral formula, for any simply connected curve  $\gamma$  on the strip  $\Gamma_t$ , we have that

$$c_k(t) = \frac{1}{k!} \int_{\gamma} \Psi^{(k)}(z, t) dz. \quad (34)$$

Together with (31), it turns out that  $|c_k(t)| \leq r^k$  for some  $0 < r < \frac{1}{At}$ . Therefore, the error from the Fourier spectral method satisfies

$$|\Psi(x, t) - \tilde{\Psi}(x, t)| \leq \sum_{k=n/2+1}^{\infty} |c_k(t)| \leq \frac{r^{n/2+1}}{1-r}. \quad (35)$$

Moreover, if we force  $r < 1/2$ , the error is bounded by  $2r^{n/2+1}$ .  $\square$

**Proof of Theorem 4.** To ensure that the simulation error is bounded by  $\eta$ , we may choose the truncation number

$$n = \left\lceil 2 \left( \frac{\log(4/\eta)}{\log(1/r)} - 1 \right) \right\rceil \leq \mathcal{O}(\log(1/\eta)).$$

Next, by plugging this truncation number in [CLL<sup>+</sup>22, Equation 113], we prove Theorem 4.

**Lemma 8.** *Suppose that  $\psi_1, \psi_2$  are two unit vectors such that  $\|\psi_1 - \psi_2\| \leq \delta$ . Then, we have*

$$|\langle \psi_1 | O | \psi_1 \rangle - \langle \psi_2 | O | \psi_2 \rangle| \leq 2\|O\|\delta.$$

*Proof.* By the triangle inequality,

$$|\langle \psi_1 | O | \psi_1 \rangle - \langle \psi_2 | O | \psi_2 \rangle| \leq |(\langle \psi_1 | - \langle \psi_2 |)O | \psi_1 \rangle| + |\langle \psi_2 | O(|\psi_1\rangle - |\psi_2\rangle)| \leq 2\|O\|\delta.$$

□