An Economic Replacement Model With Probabilistic Asset Utilization

Joseph C. Hartman
Lehigh University

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IMSE Department, Lehigh University, 200 W. Packer Ave., Bethlehem, PA 18015-1522

Abstract

Traditional economic replacement analysis provides asset purchase and sale decisions over a given horizon based on expected purchase, operating, maintenance and salvage costs. As these costs are dependent on asset utilization, a constant or predetermined usage is generally assumed. However, due to randomness in operations, such as customer demand, these expected utilization schedules may not be realized in practice, thus invalidating the replacement schedule. This paper examines the effect of probabilistic asset utilization on replacement decisions through the use of dynamic programming. The solution determines minimum expected cost decisions for each state defined by the asset's age and cumulative utilization in each period. These decisions generalize the definition of the economic life of an asset to include age and cumulative utilization. Assumptions common to replacement analysis allow the model to grow linearly, avoiding dynamic programming's 'curse of dimensionality.' Examples with time invariant and variant economics illustrate the model's robustness when compared to traditional solution procedures.

1 Introduction

The economic life of an asset defines the optimal replacement age that minimizes acquisition, operating, maintenance and salvage costs over some (infinite) horizon. Under the assumption of time invariant costs, the economic life may be determined by minimizing equivalent annual costs. Otherwise, methods such as dynamic programming are used to determine replacement schedules.

Traditional equipment replacement models assume constant or predetermined utilization schedules when examining the life cycle costs. However, the periodic usage of these assets is generally uncertain as the operational environments are characterized by randomness, including customer demand. As operating, maintenance and salvage costs are directly dependent on the asset's utilization, fluctuations in usage may alter these costs and subsequently, the economic life. To account for the uncertainty of asset utilization, this paper presents a stochastic dynamic programming formulation which assumes that the periodic usage of an asset is probabilistic. This formulation defines an asset according to its age and cumulative utilization, leading to a generalized definition of economic life.

As firms and government entities rely on capital assets for the production of goods or delivery of services, the timely replacement of this equipment is imperative to uninterrupted, economical operations. Replacement is generally motivated by deterioration of the current asset or the availability of newer, more efficient assets on the market. This decision is central to an entity's long-range planning and capital allocation decisions and has led to voluminous research on serial replacement analysis. This analysis determines acquisition and disposal decisions for a single asset or groups of independent assets over some time horizon. Early research focused on cash flow modeling for deterioration and technological change (e.g. Terborg [15]) and efficient solution procedures, including dynamic programming. First introduced by Bellman [3], this method has been generalized to consider multiple challengers over an infinite horizon by Oakford, Lohmann and Salazar [14] and Bean, Lohmann and Smith [1].

These dynamic programming solutions assume that all costs are known with certainty at time zero. A number of alternatives have been developed to consider the uncertainty inherent in these replacement decisions. To model risky cash flows, the dynamic programming approach has been modified with the use of Monte Carlo simulation (Lohmann [11]), probabilistic cash flows (Brown [4] and Fleischer [7]) and fuzzy set theory (Hearnes [8]).

These methods have been concerned with the uncertainty in payoffs or cash flows over time, including purchase, operating, maintenance and salvage values. They define the state of the asset as its age. However, in this research, we are concerned with the uncertainty associated with the state of the asset. Derman [5] initiated considerable research under the assumption that asset states were defined by levels of Markovian deterioration. With some probability, the asset would move to a further degraded state over time until it had to be replaced (if not replaced earlier). Under assumptions of no technological change, a variety of closed form solutions were developed.

Further research considered constant technological change (Bean, Lohmann and Smith [2]) and discontinuous technological change (Nair and Hopp [13] and Hopp and Nair [9]). Hopp and Nair [10] have considered stochastic technological change and Markovian deterioration simultaneously.

This paper examines the equipment replacement problem under uncertain utilization. While researchers have acknowledged that utilization effects replacement schedules (e.g. Meyers [12]), they have not explicitly defined replacement strategies based on utilization or examined the case of uncertain utilization. In this paper, the state of the asset is defined by its age *and* cumulative utilization, such as total miles traveled or parts produced. This allows for further classification of assets, as costs may be dependent on age, time, periodic utilization and/or cumulative utilization, and also allows for the utilization to vary over time. Due to randomness in the operating environment, the asset may be utilized at one of many feasible levels with given probability in a given period, leaving the resulting state of the asset uncertain. A stochastic dynamic programming formulation is presented to solve the problem and examine replacement strategies under certain cost assumptions. The solutions are shown to be more robust than those assuming constant or predetermined utilization schedules. Although dynamic programming formulations are subject to the 'curse of dimensionality' in that the state space grows exponentially with time, the presented method is shown to grow linearly in time under common replacement analysis assumptions. This allows for the efficient solution of long finite horizon problems which generally approximate infinite horizons to assure optimal time zero decisions (which are implemented immediately). The solution procedure also generalizes the definition of economic life to include age and cumulative usage.

This paper proceeds as follows. The next section presents scenario analysis modeling for the possible periodic utilization levels. Methods in which to reduce the state space are discussed here. Section 3 incorporates the scenario analysis into a stochastic dynamic program. The recursion is formulated and illustrated through examples in Section 4 with results leading to a generalized definition of economic life. Conclusions and directions for future research are presented in the final section.

2 Probabilistic Utilization Analysis through Scenarios

When asset utilization schedules are fixed or predetermined over long time horizons, traditional dynamic programming solutions (Oakford et al. [14]) provide viable solution procedures. In this paper, we are interested in examining the change in the economic life of an asset that is a result of randomness in operations and thus variations in asset utilization. This is examined through the use of scenario analysis.

2.1 Scenario Analysis

Scenario analysis provides a straightforward approach to analyzing possible utilization levels in a given period. This analysis is common for characterizing uncertainty in stochastic optimization problems. The use of scenarios is appealing as it is common in practice. For instance, demand scenarios for a certain product line may be estimated according to "likely," "most likely" and "least likely," thereby defining three scenarios. The difficulty with this modeling approach is that the number of scenarios grows exponentially over the study horizon. Consider a single piece of equipment that may be utilized at one of three levels, u_1, u_2 or u_3, in a given period. As illustrated in the scenario tree in Figure 2.1, this results in 27 scenarios over a three-year horizon.

With an exponential growth in the number of scenarios over time, the problem becomes more difficult to analyze over long time horizons. Generally, long finite horizons are required to approximate infinite horizons and ensure optimal decisions at time zero. The exponential growth would be prohibitive to solving real sized problems. Consider the same asset from Figure 2.1 being examined over a 20 year horizon. With three utilization scenarios, this would result in $3^{20} = 3.487$ billion scenarios. Note that this is merely the number of scenarios resulting from keeping an asset each period and does not consider additional scenarios for replacing the asset.

2.2 Reduction in Total Scenarios

We turn to fundamental assumptions in replacement analysis to limit the total number of scenarios resulting from possible variances in asset utilization. Specifically, we make the following assumptions:

1. All decisions are dependent on the current state of the asset (age and cumulative utilization), the time period, possible future states and their associated costs. Previous states and

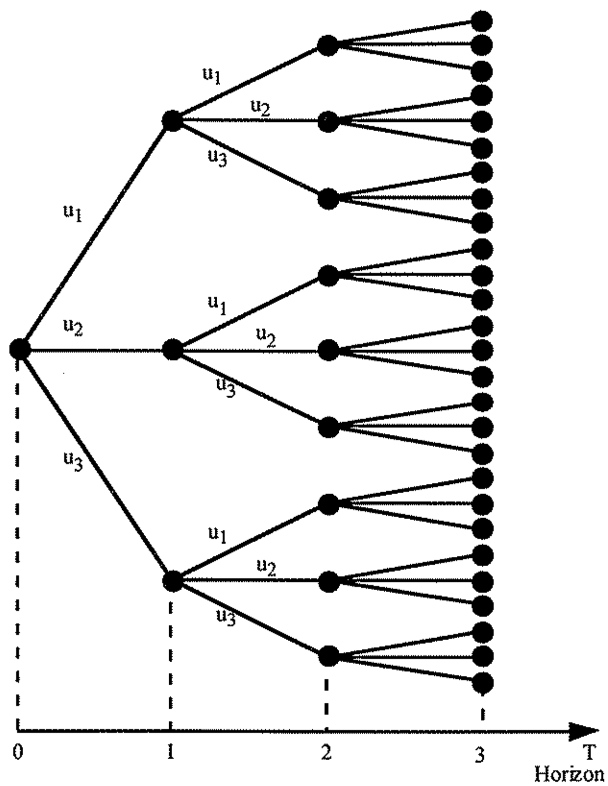

Figure 2.1: Scenarios resulting from three possible utilization levels.

associated costs are irrelevant in the analysis of current and future decisions.

2. The possible utilization levels in each period do not change over the time horizon.

3. Defining the possible utilization levels in a given period as u_1, u_2, ..., u_m, the difference between any two consecutive values is constant, or:

$$u_i - u_{i+1} = \Delta, i = 1, 2, \ldots, m - 1.$$

Thus, the utilization scenarios must be equally spaced.

The first assumption is common in replacement analysis in that sunk costs, or costs previously incurred, are ignored. Decisions are based on current and future costs associated with age, cumulative utilization, periodic utilization (current and future) and/or the time period. The decision does not depend on "how" an asset arrived at its current state. The assumptions concerning the possible periodic utilization levels in each period are not overly restrictive in that they do not limit the number of utilization levels to be analyzed. Similarly, there are no restrictions on the probabilities associated with these scenarios.

With these assumptions, the 27 scenarios in the three period example with three utilization levels of Figure 2.1 can be reduced to seven scenarios, as shown in Figure 2.2. The utilization assumptions allow for the "lattice" structure of the network such that different utilization scenarios over multiple periods may lead to the same asset state. A horizontal line represents the utilization level u while diagonal lines represent $u + \Delta$ (high) and $u - \Delta$ (low) levels.

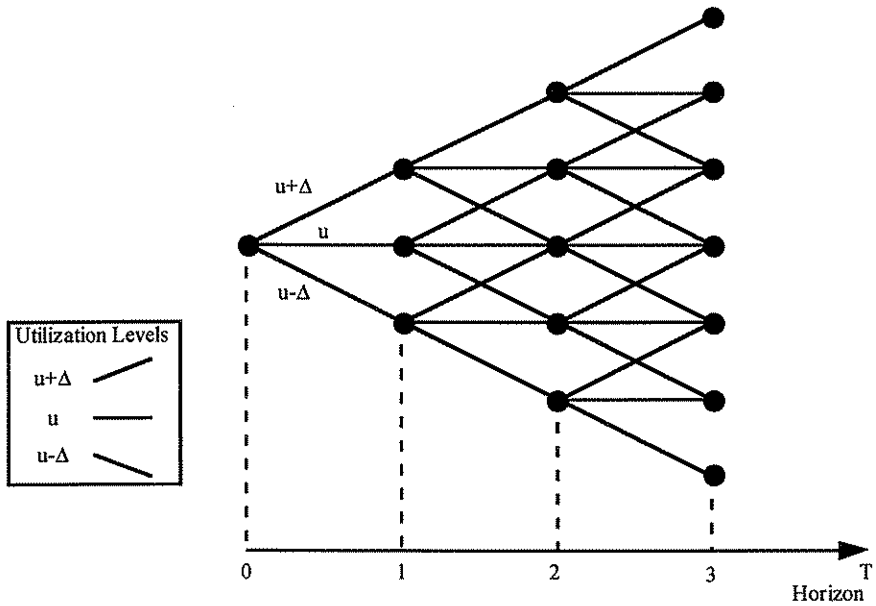

Figure 2.2: Reduction of total utilization scenarios with assumptions.

With T defining the decision horizon and m the number of possible utilization levels in a given period, the total number of scenarios (S) without employing the assumptions is the exponential function:

$$S = m^T.$$

With the assumptions, the total number of scenarios does not grow exponentially, but rather linearly, as follows:

$$S = (m-1)T + 1.$$

To illustrate the drastic reduction in total scenarios, the 3.487 billion scenarios resulting from the 20 horizon problem with three utilization levels reduces to 41. With these assumptions, the dynamic program is now formulated.

3 Stochastic Dynamic Programming Formulation

A stochastic dynamic program (Dreyfus and Law [6]) is formulated to solve the economic replacement problem. The following notation is utilized:

$P_t(i,j)$ = purchase cost of an i-period old asset with cumulative utilization j at time t;
$S_t(i,j)$ = salvage value of an i-period old vehicle with cumulative utilization j at time t;
$C(u)_t(i,j)$ = operating and maintenance cost of an i-period old asset with cumulative utilization j utilized during time t at level u;

$p_t(u)$ = probability of asset being utilized at level u in period t;
m = number of possible utilization levels in a period;
α = one period discount factor;
N = maximum allowable age of an asset;
M = maximum allowable cumulative utilization of an asset;
T = horizon time.

The stages of the dynamic program are the time or decision periods. At each time $t = 0, 1, 2, \ldots, T-1$, the decision maker decides whether to salvage the existing asset and replace it with the purchase of another asset or to keep the current asset. As this is a finite horizon problem, the asset is sold after period T for its salvage value. It is assumed that purchases and sales occur at the beginning of the period while the remaining costs occur at the end of the period.

The states of the dynamic program refer to the state of the asset which is defined by the age, i, and cumulative utilization, j. For time variant economic problems, the time period may also be included in the state definition. An asset may be retained until it reaches age N or cumulative utilization M, at which time it must be replaced.

Under the three earlier assumptions regarding asset utilization, the periodic usage levels may be indexed as $1, 2, \ldots, m$. Regardless of the utilization measure, such as miles driven or parts produced per period, they may be indexed accordingly. For notational purposes, the minimum utilization level for a period is u_1 and the maximum is u_m. Note that this assumes that $u_1 = 1$ and $u_m \leq M$. For problems which assume that an asset may not be utilized in a period, the scale may run from 0 to M. An inventory fee may be charged for the $u_0 = 0$ case. (For this paper, we assume that the asset is utilized each period.)

It should be noted that as the periodic usage of an asset is probabilistic, it is possible that an asset may be replaced with a cumulative utilization greater than M. To enforce this rule, an asset must be replaced in the first period after it reaches or eclipses the cumulative level M. For example, if an asset has two possible periodic utilization levels and has a cumulative total of $M-1$ before the period, it may be utilized for the ensuing period upon which it must be replaced as it will have a cumulative utilization of M or $M+1$.

The stochastic dynamic program is solved with backwards dynamic programming as the uncertainty in future states prohibits reaching (forward dynamic programming). Define the functional equation as follows:

$f_t(i,j)$ = minimum expected net present value of costs when starting with an asset of age i and cumulative utilization level j at time t and choosing optimal decisions through time T.

As this is a finite horizon problem, a boundary condition is assigned to period T representing the sale of the asset after the final period. The reward function is equal to the discounted salvage value of the asset. Focusing on the situation where only a new asset may be purchased such that $P_t(0,0) = P_t$, the recursive functional equation is written as follows:

$$f_T(i,j) = \alpha S_T(i,j)$$

$$f_t(i,j) \quad = \quad \min \left\{ \begin{array}{ll} \text{Keep:} & \alpha \sum_{u=1}^{m} p_t(u)\left[C(u)_t(i,j) + f_{t+1}(i+1, j+u)\right], \\ \text{Replace:} & P_t - S_t(i,j) + \alpha \sum_{u=1}^{m} p_t(u)\left[C(u)_t(0,0) + f_{t+1}(1,u)\right] \end{array} \right\}, \ t < T$$

Solving this recursion given values of α, p, P, S and C results in keep and replace decisions for each feasible asset state i, j and t over the horizon T. Note that this formulation can be easily generalized to the case of allowing used asset purchases or multiple challengers. To model these situations, additional notation labeling possible challengers with respect to age, cumulative utilization and asset type would be required. The solution procedure would not be altered; however, the representative network would be larger.

The recursion provided with the single (new) challenger in each period is depicted graphically in Figure 3.3. Each square node represents the decision to keep the asset or replace it with a new asset. This defines the age (i) of the asset in the next time period. The circular nodes represent chance nodes as the utilization level experienced by the asset is uncertain. The path from the chance node defines the cumulative utilization (j) of the asset in the next period. Although not shown in the figure, all nodes at time period T are connected to a dummy node at time period $T+1$ to represent the sale of the asset after of the final period of the finite horizon problem.

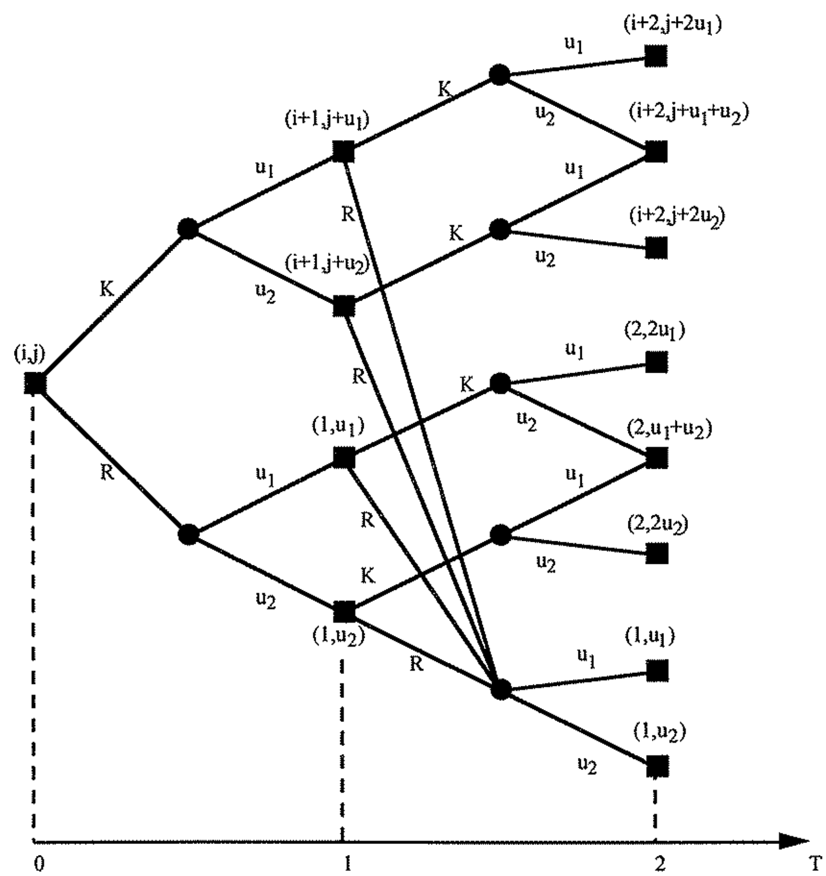

Figure 3.3: Dynamic programming network for stochastic formulation with one challenger.

In this formulation with one challenger, the maximum number of states at time t, where $t = 1, 2, \ldots, T$, is:

$$(m-1)t + 1 + \sum_{n=1}^{t} [(m-1)n + 1] .$$

With N and M defining the maximum age and cumulative utilization level for an asset, this will generally be truncated lower. Consider the case depicted in Figure 3.3. For the original asset, the number of states in each period grows from the current state (i, j) to higher aged states until the maximum age or cumulative utilization is reached and the asset must be replaced. The resulting number of states is:

$$\sum_{t=1}^{\min(N-i,T)} \left[\min\left\{ M + (u_m - u_1), j + (u_m \cdot t) \right\} - (j + (u_1 \cdot t)) + 1 \right] + 1. \tag{3.1}$$

With a new challenger also available in each period, the resulting additional number of states for the challengers is:

$$\sum_{t=1}^{\min(N,T)} \left[\left(\min\left\{ M + (u_m - u_1), (u_m \cdot t) \right\} - (u_1 \cdot t) + 1 \right) \left(T - t + 1 \right) \right] . \tag{3.2}$$

Thus, the total number of states in the dynamic program with one new available challenger each period is the sum of Equations (3.1) and (3.2). Under the assumption of multiple challengers which are all new, the number of states increases by Equation (3.2) for each challenger. For instance, a 20-period problem with an asset that may undergo one of three utilization levels in a period with $N = 10$ and $M = 30$ and an initial asset defined as $(i, j) =$ (8,27) has 1,703 states in the dynamic program with one new challenger available each period. The new challenger in each period accounts for 1,695 of these states which would be replicated under the assumption of multiple challengers.

The fact that the state space grows linearly with time is critical as long horizons are generally used to simulate infinite horizon problems. Empirical evidence has shown (Bean et al. [1]) that finite horizons approximately twice the length of the maximum physical age of the asset minimize end of study effects and assure consistent time zero decisions. These decisions are most critical as they are implemented immediately. As shown later in this paper, long horizon problems are solved efficiently with this formulation and thus can assure consistent time zero decisions.

4 Examples and Insights

This section provides two types of examples to illustrate the model and its capabilities with respect to certain analyses. The first example assumes time invariant economics in that it does not consider inflation or technological change. These assumptions lead to a characterization of economic life for the replacement problem under probabilistic utilization. The second example considers time variant economics to illustrate the full capability of the model in considering more relevant replacement problems with technological change.

4.1 Time Invariant Economic Examples and Observations

A utility firm wants to determine when to replace a 6-year old bucket truck with 65,000 miles. The truck is generally driven 10,000 miles per year, but may travel from 5,000 to 15,000 miles in a given year. A new truck costs \$20,000 with its salvage value dependent on age and cumulative utilization. It has been estimated that the salvage value drops 2.5 percent in value each year, regardless of the amount it is utilized and falls an additional 2.5 percent for each increment of 5,000 miles surpassed.

Operating costs, including fuel, oil, tires and preventive maintenance are estimated at \$0.15 per mile. Due to deterioration, this cost is expected to rise three percent every 5,000 miles driven. Maintenance costs of \$200 are expected for new trucks, rising \$100 per year of service and \$50 for each 5,000 miles eclipsed. Insurance costs are estimated at \$800 per year, rising \$50 per year the vehicle is kept in service. Under these assumptions, the cost function is as follows:

$$C(u)(i,j) = 1000 + 150i + 50j + 750(1.03)^j u,$$

and the salvage value is calculated as follows:

$$S(i,j) = 15,000(1 - 0.025i - 0.025j).$$

It is assumed that a truck must be replaced once it reaches 10 years of age or 150,000 miles. As noted earlier, a truck is generally driven between 5,000 and 15,000 miles in a year. For this problem, one of three utilization levels, 5,000 (Low or L), 10,000 (Middle or M) or 15,000 (High or H) miles, is possible per year. With these increments, the initial asset state is $(i,j) = (6,13)$.

Taxes and the effect of depreciation (age related) are ignored in this problem, although they may be easily incorporated. A discount rate of 10 percent is assumed. Solutions for various probability assumptions, including fixed utilization schedules, are presented in Table 1. The table shows the time zero decisions and the expected net present value cost of all decisions over the horizon.

Table 1: Time invariant economic example solutions with differing utilization probabilities.

Trial	P(L)	P(M)	P(H)	Cost	Time Zero Decision
1	1	0	0	\$43,592.18	Keep
2	0	1	0	\$57,073.49	Replace
3	0	0	1	\$71,077.09	Replace
4	.50	.25	.25	\$53,610.90	Replace
5	.25	.50	.25	\$57,046.56	Replace
6	.25	.25	.50	\$60,510.67	Replace
7	.335	.335	.33	\$57,031.53	Replace

The solutions were found over a 50-year horizon with decisions being made at the beginning of every year. This long horizon was chosen to simulate an infinite horizon and minimize end of study effects. The solutions were found using the spreadsheet software Microsoft Excel and its Visual Basic macro language on a personal computer. The longest solution time (including input and output) for these six problems was 21 seconds.

Note that the solutions assuming constant utilization schedules (low, medium or high) do not agree on the time zero decision. This decision is most critical as it is to be implemented immediately with the ability to run the model again at later periods for subsequent decisions. This illustrates the need for a stochastic model when the utilization level of an asset is uncertain as it may lead to different keep or replace decisions.

To illustrate the replacement policies that were determined in the probabilistic models, trial 5 is expanded in Table 2 with each decision according to the state (age and cumulative utilization) of the asset in the respective period. In the interests of space, a state labeled as (1,1,1-3) refers to states (1,1,1), (1,1,2) and (1,1,3) when similar decisions have been combined.

Table 2: State dependent decisions for trial 5.

State (t,i,j)	Decision	State (t,i,j)	Decision	State (t,i,j)	Decision
(0,6,13)	R	(6,6,13-17)	R	(8,3,3-9)	K
(1,1,1-3)	K	(6,1,1-3)	K	(8,2,2-6)	K
(2,2,2-6)	K	(7,7,7-11)	K	(8,1,1-3)	K
(3,3,3-9)	K	(7,7,12-15)	R	(9,9,9-12)	R
(4,4,4-12)	K	(7,2,2-6)	K	(9,4,4-12)	K
(5,5,5-14)	K	(7,1,1-3)	K	(9,3,3-9)	K
(5,5,15)	R	(8,8,8-9)	K	(9,2,2-6)	K
(6,6,6-12)	K	(8,8,10-14)	R	(9,1,1-3)	K

For this problem, the asset was replaced in a variety of (i,j) states, including (6,13-17), (5,15), (7,12-15), (8,10-14) and (9,9-10). With traditional time invariant cost problems, a repeating replacement schedule defines the economic life according to the age of the asset. In this problem, the asset advances both in age and cumulative utilization and thus the definition of economic life is unclear.

To define economic life in this context, the replacement schedules are more closely examined. For the first three trials, a replacement pattern was determined easily as there was only one utilization level possible each period. As given in Table 3, the economic life of the asset decreased by two years for each increased level of utilization.

For the last four trials, there are three possible resulting states after an asset undergoes a period of utilization. However, because the problem was solved over a long horizon (approximating an infinite horizon) nearly all feasible asset states are reached and thus nearly all keep/replace decisions for each state have been determined. The states that have not been reached are mathematically feasible; however, they are never attained in this problem instance because the asset is replaced at

Table 3: Economic life given by age and cumulative utilization for constant utilization trials.

Trial	P(L)	P(M)	P(H)	Economic Life	
				Age	Cumulative Utilization
1	1	0	0	9	9
2	0	1	0	7	14
3	0	0	1	5	15

a lower cumulative utilization or age. Under assumptions where operating and maintenance costs rise and salvage values fall with age and cumulative utilization, the decisions associated with states that have not been attained should be easily determined. This notion is formalized in the following two intuitive lemmas which lead to the definition of economic life in this instance.

Lemma 1 *If the optimal decision to the stochastic dynamic program is to replace an asset in state* (i, j)*, then the optimal decision is to replace any asset in state* (i, j') *where* $j' > j$ *or state* (i', j) *where* $i' > i$ *under the assumptions of time invariant economics,* $C(u)(i, j)$ *non-decreasing in* i*,* j *and* u*, and* $S(i, j)$ *non-increasing in* i *and* j*.*

Proof. Assume that the optimal solution to the stochastic dynamic program is such that the decision is to replace an asset in state (i, j) and keep an asset in state (i, j'), $j' > j$. Consider the (i, j) case. The minimum expected cost path from this node (see Figure 3.3) begins with "replace." By the principle of optimality, the expected cost on this path is lower than the expected cost of the path starting with a "keep" decision. Now consider the (i, j') case. If a "replace" decision is made, the minimum expected cost path follows that of the (i, j) case. As costs are non-decreasing in i, j and u, this path must be less than the minimum cost path resulting from a "keep" decision. Thus, the decisions to replace an asset in state (i, j) and keep an asset in state (i, j'), $j' > j$ cannot be optimal. A similar argument holds for the (i', j), $i' > i$ case. □

Lemma 2 *If the optimal decision to the stochastic dynamic program is to keep an asset in state* (i, j)*, then the optimal decision is to keep any asset in state* (i, j') *where* $j' < j$ *or state* (i', j) *where* $i' < i$ *under the assumptions of time invariant economics,* $C(u)(i, j)$ *non-decreasing in* i*,* j *and* u*, and* $S(i, j)$ *non-increasing in* i *and* j*.*

Proof. The proof follows the logic of the previous Lemma. □

Under the assumptions of non-decreasing operating and maintenance costs in both age and cumulative utilization and non-increasing salvage values in both age and cumulative utilization, we can define the "economic life frontier." This frontier defines the combinations of age and cumulative utilization in which an asset should be replaced for a given problem instance. Consider trial 5 of the example problem. A plot of the decisions to replace an asset for each i and minimum j is given in Figure 4.4.

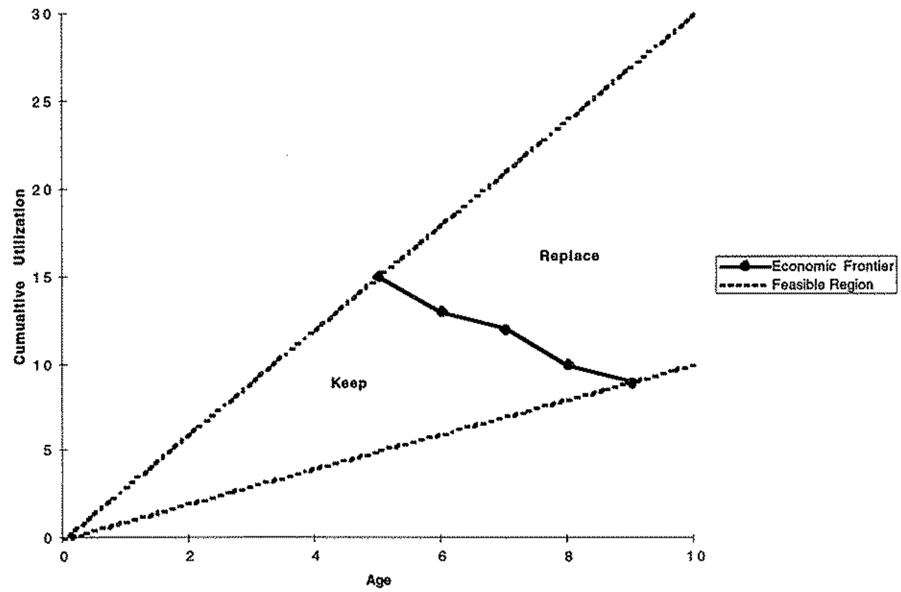

Figure 4.4: Plotted decisions for trial 5 of time invariant economics example.

As opposed to one age or cumulative utilization level, the combinations along the "economic life frontier" as given by the curve in Figure 4.4 define the economic life of the asset in the given problem scenario. The minimum cost decisions are to replace an asset if its state is on or above the frontier and retain the asset if it falls below the frontier.

This plot is easily constructed for any problem scenario as the data is made available from just one dynamic programming run. The run must be long enough such that one complete cycle equivalent to the physical age limit of the asset is reached. As noted earlier, empirical evidence has shown that this horizon should be around twice the economic life to ensure a consistent time zero decision. For this example, the asset may be retained until it reaches $N = 10$ years, thus, the solution must provide 10 consecutive years worth of decisions to determine which asset states are reached and the respective decisions. It is noted that the horizon must be larger such that end of study effects are eliminated. For this problem a 50-year horizon was sufficient as longer horizons were checked for verification. Large horizons are not a concern as they can be solved quickly, as in this example.

From the one cycle of decisions, Figure 4.4 can be generated from the output. For the states that are not reached because assets are replaced earlier, Lemmas 1 and 2 guarantee that the decisions do not change beyond the frontier.

Theorem 1 *Assuming time invariant economics, $C(u)(i,j)$ is rising in i, j and u and $S(i,j)$ is decreasing in i and j, an "economic life frontier" defined by (i,j) states exists such that the minimum cost decision is to replace an asset with state (k,l) where $k \geq i$ and $l \geq j$.*

Proof. Under these assumptions, the results from Lemma 1 and 2 hold. For each asset age i, define the minimum j such that the optimal decision from the stochastic dynamic program is to replace the asset. These states, one for each i, define the "economic life frontier" such that the optimal decision in any asset state (i', j') where $i' \geq i$ and/or $j' \geq j$ is to replace the asset for a given problem instance. □

With discrete solutions, the curve defined by the "economic life frontier" may not be convex or concave. However, there are tendencies depending on whether costs are driven by age or cumulative utilization. For problems in which age is the driving force, such that if:

$$\frac{\partial i}{\partial C(u)(i,j)} >> \frac{\partial j}{\partial C(u)(i,j)},$$

and/or:

$$\frac{\partial i}{\partial S(i,j)} >> \frac{\partial j}{\partial S(i,j)}$$

the curve will be more vertical as replacements will tend to occur at a given age. Similarly, if costs are driven by utilization and cumulative utilization, the curve will be more horizontal as replacements will tend to occur at a given cumulative utilization. The actual "economic life frontier" is defined by the problem instance.

4.2 Time Variant Economic Examples and Observations

The bucket truck is again analyzed here assuming a constant rate of technological change. To model this situation, the purchase costs are assumed to increase at 2 percent per year while the fixed periodic cost and initial operating cost per mile are expected to drop by 5 percent with each new model. The salvage values depreciate as before, starting from 80 percent of the respective purchase price. The costs for the defender are as given above, with costs of the challengers as follows:

$$\begin{aligned} C(u)_t(i,j) &= \frac{1000}{1.05^{t-i}} + 150i + 50j + \frac{750}{1.05^{t-i}}(1.03)^j u \\ S_t(i,j) &= (.80P_{t-i})(1 - 0.025i - 0.025j) \\ P_t &= 20,000(1.02)^t \end{aligned}$$

All other assumptions are the same as the previous example, including a 10 percent discount rate and an initial asset state of (6,13).

Note that trials 8 and 9 are dynamic in that the probability is dependent on the time period t, further illustrating the versatility of the model. These can be used to model product life cycles where demand may be expected to rise through early years, level off and finally decline in waning years of a project. Seasonality or cyclic demand may also be modeled easily.

Under the assumption of time-varying economics, the economic life of an asset may change over time as purchases may not be made at constant increments in order to take advantage of

Table 4: Time variant economic example solutions with differing utilization probabilities.

Trial	P(L)	P(M)	P(H)	Cost	Time Zero Decision
1	1	0	0	$43,590.65	Keep
2	0	1	0	$55,565.65	Replace
3	0	0	1	$67,989.79	Keep
4	.50	.25	.25	$52,546.75	Replace
5	.25	.50	.25	$55,547.96	Replace
6	.25	.25	.50	$58,590.59	Keep
7	.335	.335	.33	$55,483.78	Replace
8	.005(t+1)	.01(t+1)	1-P(L)-P(M)	$65,602.93	Keep
9	1-P(M)-P(L)	.01(t+1)	.005(t+1)	$45,760.76	Keep

technological change. However, under the assumption that operating and maintenance costs are non-decreasing in i, j and u for a given period t and salvage values are non-increasing in i and j for a given t, then the "economic life frontier" can be determined *for each period.* Figure 4.5 illustrates this concept for trial 5 with time variant economics for periods 0 and 1. Note that the "economic life frontier" shifts lower for period 1 as the effects of technological change promote earlier replacements. This will happen if productivity savings outpace inflation.

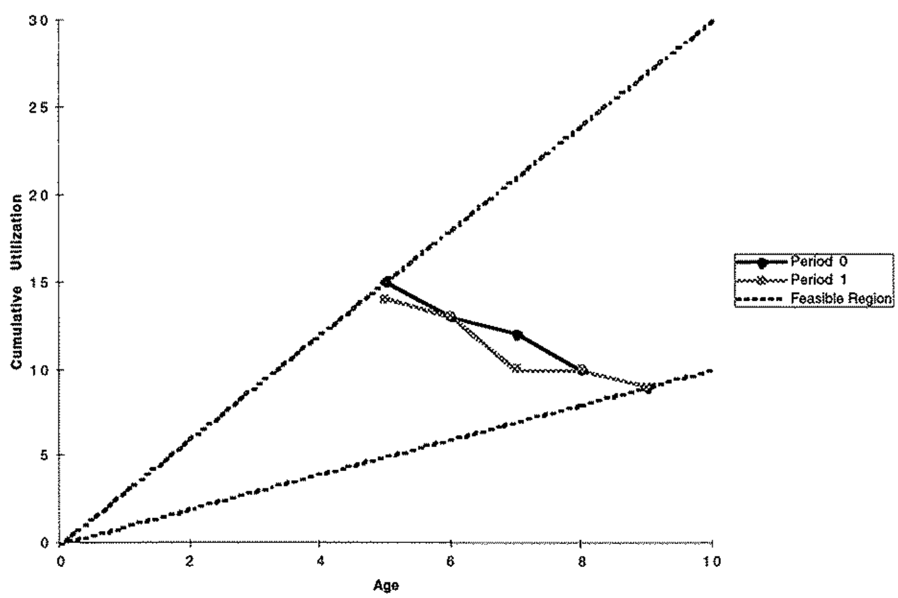

Figure 4.5: Plotted decisions for trial 5 of time variant economics example.

As this frontier is not constant over time, its construction is more difficult than the time invariant case which was constructed from one dynamic programming run. Here, the frontier must be constructed from several runs with different initial starting states. If developed for each period, the frontier can be used for decision making purposes, but it is easier to achieve desired solutions for specific problems from the dynamic program.

5 Conclusions and Directions for Future Research

The replacement of a capital asset is generally motivated by deterioration of the asset itself or the introduction of more technologically advanced assets in the marketplace, leading to cost reductions through productivity enhancements. To assure continued production and economical operations, the timely replacement of this equipment is critical.

Unfortunately, determining the economic life (replacement age and/or cumulative utilization level) can be difficult as randomness is operations may lead to fluctuations in utilization. Assumptions of constant utilization in these decision settings may lead to incorrect and costly replacement schedules. This paper has presented an efficient stochastic dynamic program to solve the equipment replacement problem under the assumption of probabilistic asset utilization. While traditional models assume that the state of an asset is defined by either its age or operating state, this model defines an asset's state by age and cumulative utilization. For the case of time invariant economics, the economic life is defined as a frontier of age and cumulative utilization combinations. For the time variant economics case, this definition holds for each time period.

Future research will examine this problem in the context of multiple assets and parallel replacement analysis. In this analysis, assets are economically interdependent and their replacements must be determined jointly. This combinatorial analysis is relatively new to the literature when compared to serial replacement analysis, but has a variety of applications in fleet management and multiple machine problems. As with serial replacement analysis, constant or predetermined utilization schedules are generally assumed. However, it is possible that lower overall cost savings can be achieved if both problems of asset replacement and utilization are examined simultaneously as different combinations of utilization levels among assets may alter investment decisions. Existing solutions to machine requirements and fleet sizing problems examine both tactical decisions (number of capital assets required) and operational decisions (utilization level requirements), however, influential aspects common to replacement analysis such as the economic modeling of deterioration and technological change over long horizons are generally not included. This will be the focus of future research.

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