Parallel Replacement Analysis Under Variable Asset Utilization and Stochastic Demand: The Two Asset Case

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Abstract

The economic life of an asset is dependent on a variety of factors, including deterioration and obsolescence. While obsolescence is generally a result of changes external to the asset, such as technological change, deterioration is generally a result of how the asset is utilized over its lifetime. If multiple assets are available to meet demand and the assets must not continually operate at maximum capacity, then a decision-maker may have some control over asset utilization patterns by allocating workload. These utilization patterns directly impact operating costs and salvage values and thus have a strong influence on the optimal replacement time of the assets. In this paper, we examine asset replacement decisions, based on age and cumulative utilization, under variable periodic utilization with multiple, parallel assets under various cost and demand assumptions. We provide an efficient optimal solution procedure through the use of stochastic dynamic programming for the finite horizon, two-asset case and provide insight into the n-asset case.

Keywords: Replacement (equipment), asset utilization, dynamic programming

1 Introduction

Replacement analysis is concerned with determining the optimal (1) time to remove a current asset (defender) from service and (2) selection of another asset (challenger) to take its place. The serial replacement problem, which analyzes the replacement of a single asset or multiple independent assets, is well studied in the literature. (See, for example, Bellman [3], Fraser and Posey [8], Hopp and Nair [13], Oakford et al. [19], Nair and Hopp [18] and Terborg [26], among others.) In the single asset case, a deterministic utilization pattern is generally assumed and decisions are made periodically, based on the age of the asset. Authors have included the effects of utilization (e.g. Meyers [17] and Taylor [25]), but not treated it as a variable.

Recently, authors have examined the serial replacement problem under the assumption that the periodic utilization of the asset is not fixed. As operating and maintenance costs and salvage values are dependent on periodic and cumulative utilization levels (in addition to age), replacement schedules are highly dependent on the assumed use of the asset over time. Bethuyne [5] showed that the economic life of an asset could be prolonged if the periodic utilization level of an asset could be reduced over time. Hartman [10] considered the case where the utilization level of the asset was probabilistic and thus the resulting state of the asset, defined by both age and cumulative utilization, was also probabilistic. A threshold replacement policy in

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which an asset was kept until it reached certain combinations of age and cumulative utilization existed under mild cost assumptions (Hartman [9]).

In the case of a single asset, the utilization of an asset is generally not a controllable variable as its usage must be a reaction to the demand environment. However, in the case of multiple assets, one has the ability to set utilization patterns by allocating demand among the available assets, assuming all assets must not continually operate at their maximum capacity. This allocation decision has a direct impact on each asset's operating costs and salvage values and thus the optimal replacement schedule.

The study of multiple asset replacement problems is not as prevalent in the literature. The models may be categorized as either parallel replacement problems, or, as we term them here, series replacement problems. In both of these problems, the assets are economically interdependent in that they are subject to demand and/or budgeting constraints and/or have costs that are not linear with respect to the number of assets, such as economies of scale in purchase price. In parallel replacement models, it is assumed that the assets operate in parallel and thus contribute to demand independently. An example would be a fleet of trucks that service a distribution center. The total capacity available is the sum of the individual capacities of the trucks. In series replacement analysis, the assets operate in series, and thus, demand is served by the group of assets which operate in sequence. An example of this situation is a production line in which multiple machines must operate together to meet a demand or service constraint. Generally, the capacity of the system is defined by the smallest capacity asset in the line. Obviously, situations exist with assets both in series and parallel with capacity definitions following suit.

Research in the area of parallel replacement analysis has examined problems in which a fixed charge was included in asset purchases (Jones et al. [14], Chen [7]), fixed charge plus service constraints (Rajagopalan [20]), capital budgeting constraints (Karabakal et al. [15]) and demand and rationing constraints (Hartman and Lohmann [12]). In series replacement work of Tanchoco and Leung [24] use price-volume relationships to determine optimal machine output and replacements. Leung and Tanchoco [16] approach the multiple equipment replacement decision as a configuration selection problem, Suresh [23] evaluates multiple machines in a flexible manufacturing system and Stinson and Khumawala [22] solve a series problem with integer programming. Although these models consider multiple assets, replacement decisions are strictly age-based.

Hartman [11] determined both utilization and replacement decisions for a group of assets that operated in parallel, and thus simultaneously determined the optimal number of assets to be retained over a horizon. A linear-integer programming formulation was presented in which all parameters were assumed to be deterministic.

In this paper, we are concerned with examining the optimal replacement and utilization schedules for a number of assets over a finite horizon with stochastic demand and gaining some insight into optimal decisions under different cost assumptions. As an asset's utilization is also a variable, both age and cumulative utilization are treated as state variables for replacement decisions. With the flexibility of dynamic programming, we examine the impact of economies of scale in purchase prices (fixed charge) and stochastic demand. We provide explicit analysis of the two asset case and also illustrate the correspondence to the n asset case. Although typical dynamic programming solutions generally suffer from the 'curse of dimensionality' with an increased number of state variables [4], we show that the proposed method is efficient despite having four or five state variables, depending on cost assumptions. Additionally, we provide an efficient method for examining a solution of the program as it is shown when replacement decisions for an asset can be made independently of the other asset.

2 Utilization Based Cost Considerations

As mentioned earlier, replacement models generally assume a given level of utilization for each asset in each period over the life of the asset. In multiple asset problems, these levels are generally assumed to be the same for similar assets (same age and technology). There are a variety of cases where this assumption is valid in that skewed loads among assets leads to a higher overall total cost solution. However, it is possible that unevenly distributing the workload among assets may lead to lower overall cost solutions. Here, we examine a number of costs cases to illustrate this possibility.

Let us first examine the operating and maintenance (O&M) costs for an asset which occur in each period of its physical life. Given an asset of age i and cumulative utilization j, it is a common assumption that the O&M costs are non-decreasing in utilization (and age). This function could take on various shapes, such as those in Figure 2.1. In this figure, all O&M costs are increasing. However, (a) has a decreasing derivative, (b) is linear and (c) has an increasing derivative. Case (a) might represent assets that must "warm-up" for use, such that costs actually decrease (on a per unit basis) with increased utilization. Case (c) might represent assets that deteriorate with increased use, leading to exponentially increasing costs.

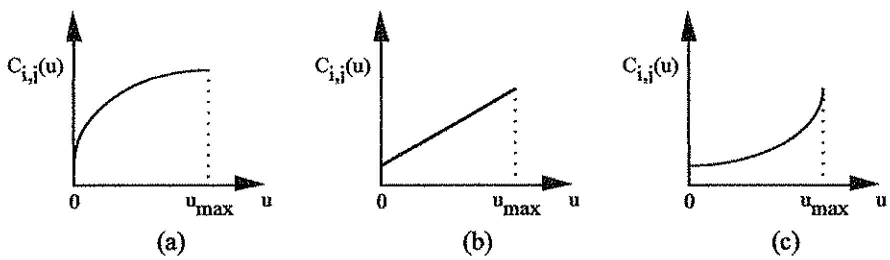

Figure 2.1: O&M cost in a given period for an asset of age i and cumulative utilization j utilized at level u.

These costs may be represented by the following polynomial:

$$C_{i,j}(u) = \alpha u^{\beta} + \gamma,$$

where α and γ are non-negative and $\beta < 1$ for (a), $\beta = 1$ for (b) and $\beta > 1$ for (c).

For motivation, consider the simple case with two identical assets being utilized for one period. Further assume that the sum of their maximum utilization levels for a given period ($\bar{u}$) is greater than demand. For feasibility, the sum of the individual utilization levels of each asset must be d_t, the demand in period t. If the salvage values are negligible at the end of the period, then the optimal utilization schedule for each asset to minimize total cost is strictly dependent on the individual O&M costs. As the assets are identical, then their costs functions are identical. If we restrict our discussion to the smooth functions defined according to (a), (b) and (c) in Figure 2.1, then the optimal decisions are straightforward. For (a), the optimal decision is to operate one asset at $\bar{u}$, as unit costs decrease with increased use, and the other asset at $d_t - \bar{u}$. For the linear case (b), any feasible combination of utilization levels results in the same total cost. In case (c), the optimal decision is to operate each asset at $d_t/2$ to minimize total costs.

Now consider salvage values, which are assumed to be revenues at the end of the one-period problem. Again, restricting the functions to continuous and smooth, we have the three cases in Figure 2.2. Again, (a) has a decreasing derivative, (b) is linear and (c) has an increasing derivative.
Ignoring the O&M cost, the optimal decisions for utilization would be operating the assets at $d_t/2$ for case (a), any combination for case (b) and operating at the extremes ($\bar{u}$ and $d_t - \bar{u}$) for case (c).

To complete the one period problem, the optimal utilization allocation among the two identical assets is

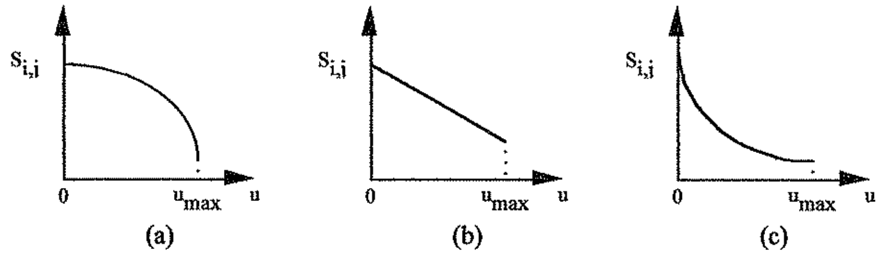

Figure 2.2: Salvage value in a given period for an asset of age i and cumulative utilization j.

dependent on both the O&M costs and salvage values. Table 1 gives the nine possible scenarios under these costs assumptions and their optimal allocations.

Table 1: Optimal asset utilization levels for single period, two-asset problem.

Case	Salvage Value Function	O&M Cost Function	Allocation (Asset 1, Asset 2)
1	a	a	–
2	a	b	$(\frac{d_t}{2}, \frac{d_t}{2})$
3	a	c	$(\frac{d_t}{2}, \frac{d_t}{2})$
4	b	a	$(\bar{u}, d_t - \bar{u})$
5	b	b	$(u_1, d_t - u_1)$
6	b	c	$(\frac{d_t}{2}, \frac{d_t}{2})$
7	c	a	$(\bar{u}, d_t - \bar{u})$
8	c	b	$(\bar{u}, d_t - \bar{u})$
9	c	c	–

Thus, for the nine possible combinations of cost functions examined here, we can make definitive assignments when the assets are the same and the derivatives of the cost functions work in a similar fashion. There are four cases, 2, 3, 5 and 6, where the optimal decision is to always allocate demand among the two identical assets equally (includes linear case). That is, when (1) both the operating cost and salvage value functions are linear; (2) salvage value is decreasing with a decreasing derivative and operating costs are linearly increasing or (3) increasing with an increasing derivative; or (4) salvage value is decreasing linearly and operating costs are increasing with an increasing slope, the optimal decision is to allocated demand equally among assets. This translates to problems with n identical assets assuming constant demand (no greater than $2\bar{u}$), time-invariant costs and no capital budgeting constraints for t period problems.

For the other cases, it is better to skew demand in favor of an asset such that additional costs may be saved. For cases 4, 7, and 8, one asset is operated at its maximum while the other asset operates to fulfill remaining demand. For cases 1 and 9, the optimal decisions are based on the cost functions as opposing derivatives may drive the decision in opposite directions.

This one period problem provides motivation for a solving a more realistic problem over several periods. This problem can be modeled as a dynamic program with two decisions occurring each period: replacement decisions and utilization allocation decisions. We provide a dynamic programming solution to the two-asset case in the following section and make some observations about the optimal solution in the ensuing section.

3 Dynamic Programming Formulation for The Two-Asset Case

In the single asset replacement problem, the decision is whether to keep or replace the asset at the end of each period. Here, the decision is twofold. First, a combination of assets may either be kept or replaced. Once this decision is made, the utilization level of each individual asset must be determined. Due to the growth in states, we examine the two-asset case in detail to gain insight into larger problems. In the two asset case, the decisions are whether to keep both assets (KK), replace both assets (RR) or keep one and replace the other (KR and RK), totalling four possibilities. Once this decision has been made, the allocation of demand to the two assets must be made. This two-stage decision process is repeated at each period over the decision horizon.

We assume that there is only one challenger available for replacement in each period t and it may be used to replace either or both assets. The dynamic programming formulation utilizes the following notation:

$P_t(i,j)$	= purchase cost of an i-period old asset with cumulative utilization j at time t;
$S_t(i,j)$	= salvage value of an i-period old asset with cumulative utilization j at time t;
$C_t(u,i,j)$	= operating and maintenance cost of an i-period old asset with cumulative utilization j utilized during time t at level u;
K_t	= fixed cost charge if asset purchased at time t;
$d_{m,t}$	= demand level m in period t;
$p(d_{m,t})$	= probability of demand $d_{m,t}$ in period t;
D	= number of demand levels in each period;
α	= one period discount factor;
N	= maximum allowable age of an asset;
M	= maximum allowable cumulative utilization of an asset;
T	= horizon time.

If the assets are not homogeneous, then subscripts may be added to the purchase and O&M costs and salvage values to differentiate assets. For homogeneous assets, the i, j and u values are subscripted for each asset.

Decisions occur at the end of each time period, labeled $t = 0, 1, 2, \ldots, T-1$. At time T, both assets are sold for their salvage values. It is assumed that purchases and sales occur at the beginning of the period while the remaining costs occur at the end of the period. All costs are discounted to time zero.

The states of the dynamic program refer to the state of each asset, defined by their age, i, and cumulative utilization, j. Once an asset reaches its maximum service life, age N or cumulative utilization M, it must be replaced.

Periodic usage levels are indexed as $0, 1, \ldots, \bar{u}_t$, with $\bar{u}_t$ representing the maximum utilization in period t and level zero representing an asset being offline. Note that minimum and maximum utilization values may be different for each asset. Regardless of the utilization measure, such as miles driven or parts produced per period, they may be indexed accordingly.

Define the functional equation as follows:

$f_t(i_1, j_1, i_2, j_2)$ = minimum expected net present value of costs when starting with two assets of age i_1 and i_2, respectively, and cumulative utilization levels of j_1 and j_2, respectively, at time t and choosing optimal decisions through time T.

The dynamic program determines optimal keep and replace decisions in addition to optimal asset utilization levels in each period. With utilization in period t denoted as $u_{1,t}$ and $u_{2,t}$ for assets 1 and 2, respectively, the following constraints must hold in each period:

$$\begin{aligned} u_{1,t} + u_{2,t} &= d_{m,t} \quad \forall m = 1, 2, \ldots, D, \ \forall t = 0, 1, \ldots, T-1 \\ u_{1,t} &\leq \bar{u}_t \quad \forall t = 0, 1, \ldots, T-1 \\ u_{2,t} &\leq \bar{u}_t \quad \forall t = 0, 1, \ldots, T-1 \end{aligned}$$

These constraints hold implicitly in the dynamic program as $u_2 = d_{m,t} - u_1$ and the utilization decisions are restricted to being feasible.

For the case where a new asset is the only challenger available each period (such that $P_t(0,0) = P_t$), the recursion for periods $t < T$ may be written as follows. In the interest of space, u_2 remains in the formulation although $d_{m,t} - u_1$ is substituted when solving the dynamic program and the search for the optimal utilization levels is over u_1.

$$f_t(i_1, j_1, i_2, j_2) = \min \Bigg\{$$

$$\begin{aligned} &\text{KK: } \alpha \sum_{m=1}^{D} p(d_{m,t}) \left[\min_{u_1,u_2} \{ C_t(u_1, i_1, j_1) + C_t(u_2, i_2, j_2) + f_{t+1}(i_1 + 1, j_1 + u_1, i_2 + 1, j_2 + u_2) \} \right], \\ &\text{RK: } P_t - S_t(i_1, j_1) + \alpha \sum_{m=1}^{D} p(d_{m,t}) \left[\min_{u_1,u_2} \{ C_t(u_1, 0, 0) + C_t(u_2, i_2, j_2) + f_{t+1}(1, u_1, i_2 + 1, j_2 + u_2) \} \right], \\ &\text{KR: } K_t + P_t - S_t(i_2, j_2) + \alpha \sum_{m=1}^{D} p(d_{m,t}) \left[\min_{u_1,u_2} \{ C_t(u_1, i_1, j_1) + C_t(u_2, 0, 0) + f_{t+1}(i_1 + 1, j_1 + u_1, 1, u_2) \} \right], \\ &\text{RR: } K_t + 2P_t - S_t(i_1, j_1) - S_t(i_2, j_2) + \alpha \sum_{m=1}^{D} p(d_{m,t}) \left[\min_{u_1,u_2} \{ C_t(u_1, 0, 0) + C_t(u_2, 0, 0) + f_{t+1}(1, u_1, 1, u_2) \} \right] \end{aligned} \tag{3.1}$$

As this is a finite horizon problem, a boundary condition is assigned to period T representing the sale of the assets after the final period, as follows:

$$f_T(i_1, j_1, i_2, j_2) = -S_T(i_1, j_1) - S_T(i_2, j_2) \tag{3.2}$$

We refer to the recursion defined in Equations (3.1) and (3.2) as DP. Solving DP given values of α, P, S, C, d and $p(d)$ results in keep or replace and utilization level decisions for each combination of feasible asset state i and j in each period t over the horizon. Note that this formulation can be easily generalized to the case of allowing used asset purchases or multiple challengers. To model these situations, additional notation labeling possible challengers with respect to age, cumulative utilization and asset type would be required. The solution procedure would not be altered; however, the representative network would be larger.

3.1 Representative Network and State Space Size

Although utilization is a continuous variable, we discretize it here as decisions are made according to a discrete state space. With the two asset, single challenger case, one stage of decisions may be illustrated as in Figure 3.3. Here, an asset may be utilized at one of two levels (u_1^1 and u_1^2 for asset 1) in a given period. For the purposes of this discussion, we examine the case of deterministic demand.

As the figure illustrates, there are four keep or replace combinations and two possible asset utilization combinations in each period. From this one stage illustration, it would appear that each ensuing stage would result in eight new states from each given asset state. However, there are three situations that drastically

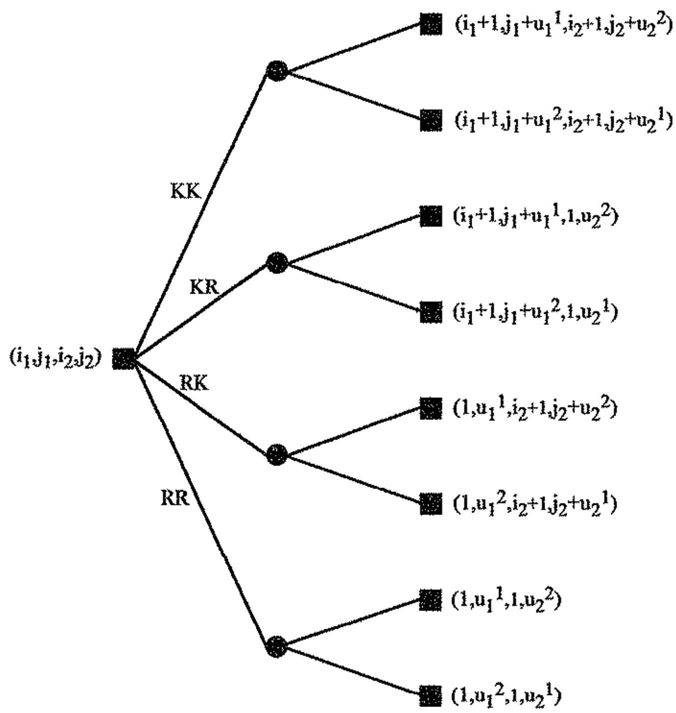

Figure 3.3: Network of decisions for one stage.

reduce the growth in the state space. The first situation requires an assumption about the periodic utilization levels of the individual assets. We assume that the possible periodic utilization levels for a given asset u_1, u_2, ..., $\bar{u}$ are equally spaced such that the difference between any two consecutive values is constant. This difference is constant over the horizon, even if the range of possible levels changes periodically. With this assumption, assets may achieve similar cumulative utilization states although their previous periodic utilization levels may have been different. This lattice structure for utilization allows for the reduction in state space growth.

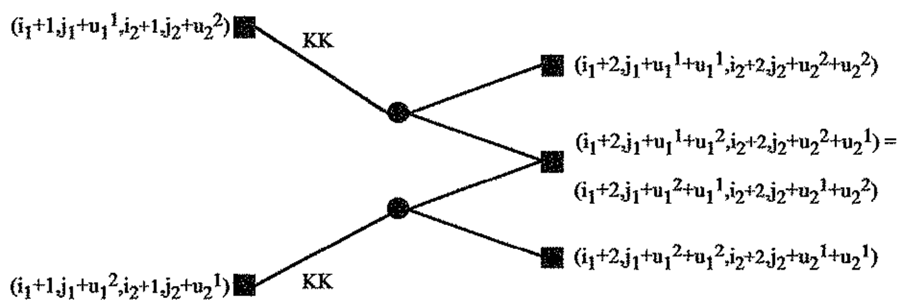

Figure 3.4: State reduction due to periodic utilization symmetry.

Figure 3.4 shows the lattice structure of the periodic utilization levels eliminating a possible state. This is the result of an asset being used at high and then low utilization levels in continuous periods is equivalent to an asset being used at low then high levels as cumulative utilization (and age) define the state. In this example (Figure 3.3), this lattice structure leads to the elimination of four states in the second period.

Second, if single assets have similar states in the same period, a reduction of states is achieved based on replacing the other asset. Consider asset 1 in Figure 3.3. Asset 1 has four possible states in the second period (age 1 or $i_1 + 1$ with one of two associated feasible utilization levels). For the states in which asset 1 has the same age and cumulative utilization, the resulting states from a KR decision are the same, as shown in Figure 3.5. This results in the elimination of 8 states in the next period. Similar reductions occur when examining asset 2 states and the replacement of asset 1.

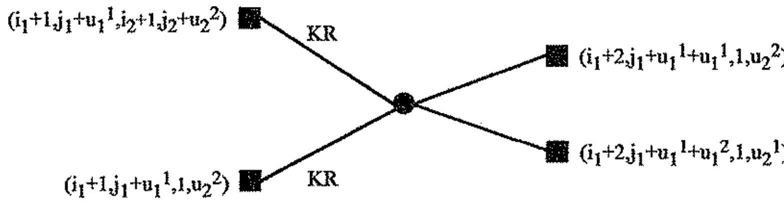

Figure 3.5: State reduction due to individual asset state symmetry.

Third, the decision to replace both assets from any state leads to the same ensuing states with assets of age 1 and their respective utilization levels. This eliminates 14 resulting states in the next period.

With these three reductions, there are only 30 feasible asset states at the end of period 2. This is a reduction from the possible $8 \times 8 = 64$ original possibilities. Continuing this analysis to period 3, there are 77 possible states which is considerably fewer than the $8 \times 30 = 240$ original possibilities.

In all, the symmetry of keep-replace decisions based on individual asset states and the lattice structure of cumulative utilization limits the growth in the state space. The maximum number of states that can be reached in a period is dependent on the maximum physical life of the individual assets, N and M. For an individual asset state (i, j), the total number of states that can be reached is $\{(i, j) : i = 1, 2, \ldots, N; j = 0, 1, \ldots, \min(i\bar{u}, M)\}$. Thus, for a problem where $N = 10$, $\bar{u} = 5$ and $M = 50$, the number of individual asset states is 305. As there are two assets in the problem, there are 93,025 possible combinations.

4 Examples and Insights

This section examines both time-invariant and time-variant economics examples to provide insight into utilization/replacement decisions. The time-invariant economics problems do not consider inflation or technological change but allow for repetitive replacement cycles to be determined. The time-variant examples illustrate the general capability of the model such that common problems with technological change and inflation may be evaluated over a finite horizon.

DP was programmed in C and executed on a SGI Octane/SI R10000 dual 250 Mhz Workstation with 384 MB memory. For the time-invariant examples, the average execution time for the 11 trials was 97.95 seconds, with the maximum being 98.8 seconds. For the time-variant examples, the average execution time was 136.32 seconds, with the maximum being 137.11 seconds. All of these examples assumed $N = 10$, $M = 50$, $D = \bar{u} = 5$ and $T = 50$.

4.1 Time-Invariant Economics and Stationary Demand Examples

In this example, two homogeneous assets are required to meet demand. The maximum utilization of each asset in any period (normalized) is five, or $\bar{u}_t = 5$ for all $t < T$. This utilization level may represent parts produced (e.g. 5,000 parts per level) or miles driven (e.g. 5,000 miles per level). Demand is assumed to take on values of $d_{1,t} = 6$ through $d_{5,t} = 10$ in each period t with known probabilities. Note that this assumes that both assets must be utilized in each period. The maximum life of an asset is defined by $N = 10$ and $M = 50$.

Although any cost functions may be used, we assume operating costs that increase with i, j and u and no salvage values, as follows:

$$\begin{aligned} P &= 15,000 \\ C(u,i,j) &= 500 + 300i + 50j\left((j+u)^{1.2} - (j)^{1.2}\right) \\ K, S(i,j) &= 0 \end{aligned}$$

It is assumed that at time zero, assets 1 and 2 are ages 2 and 4, respectively, with cumulative utilization levels of 8 and 15, respectively. Table 2 provides the solutions to 11 different trial runs involving these demands and costs. Each trial varies according to the probabilities of each demand level. Trials 1-5 represent the deterministic cases while the remaining trials have stochastic demand.

Table 2: Time-invariant economics solutions with differing stationary demand probabilities.

Trial	$p(d_{1,t})$	$p(d_{2,t})$	$p(d_{3,t})$	$p(d_{4,t})$	$p(d_{5,t})$	NPV Cost	Time Zero Decision					
							K or R	$u_{1,0}, u_{2,0}$				
1	1	0	0	0	0	$107,552.59	KK	5,1				
2	0	1	0	0	0	$121,271.59	KK	5,2				
3	0	0	1	0	0	$134,357.81	KK	5,3				
4	0	0	0	1	0	$146,659.91	KR	4,5				
5	0	0	0	0	1	$159,350.43	KR	5,5				
6	.50	.125	.125	.125	.125	$124,187.77	KK	5,1	5,2	5,3	5,4	5,5
7	.125	.50	.125	.125	.125	$129,334.32	KK	5,1	5,2	5,3	5,4	5,5
8	.125	.125	.50	.125	.125	$134,187.64	KR	1,5	2,5	3,5	4,5	5,5
9	.125	.125	.125	.50	.125	$138,751.19	KR	1,5	2,5	3,5	4,5	5,5
10	.125	.125	.125	.125	.50	$143,408.97	KR	1,5	2,5	3,5	4,5	5,5
11	.20	.20	.20	.20	.20	$134,008.48	KR	1,5	2,5	3,5	4,5	5,5

The table provides both the keep and replace decision for each asset and the level of utilization allocated to each asset for the first period ($u_{1,0}$ and $u_{2,0}$) given the demand for the time period. Thus, for Trial 8, the decision is to keep asset 1 with initial state (2,8) and replace asset 2 defined by initial state (4,15). Additionally, the new asset (asset 2) is allocated the maximum workload in the first period, as $u_{2,0} = 5$ for all demand levels and $u_{1,0} = 1, 2, \ldots, 5$ for demand levels six through 10 . The expected net present value cost over the 50-year horizon (used to solve this problem) when starting at that initial state and making optimal decisions over the horizon is $134,187.64. For Trials 1-3, 6 and 7, the optimal decision is to keep both assets and allocate the maximum demand to asset 1. For the remaining trials, asset 2 is replaced and this new asset is utilized at its maximum capacity.

While the solution for state $(i_1, j_1, i_2, j_2) = (2, 8, 4, 15)$ is provided in Table 2, the replacement cycle for any state can be determined from one solution of the dynamic program because the data is time-invariant. These problems were solved over a 50-year horizon in order to minimize end-of-study effects and simulate an infinite horizon. By solving one iteration of the dynamic program, the solutions for the time zero case can be examined for each possible asset state and determine the optimal decisions over time.

While examining 230 possible states for *each* asset may seem overwhelming, the symmetry of this problem (homogeneous assets) allows us to examine one type of decision for a complete picture. Consider the states (i_1, j_1, i_2, j_2) of DP where it is optimal to keep asset 1 and replace asset 2 (KR decisions). Define the states

(i_1, j_1) as Y states and (i_2, j_2) as Z states. Figure 4.6 gives a plot of sets Y and Z, with Y states labeled with a circle and Z states labeled with a dash.

Using the definitions of Y and Z, further define the following sets (Note that the state space of this problems is $S = \{(i,j) : i = 1, 2, \ldots, N; j = i, i+1, \ldots, i\bar{u}\}$):

$$\begin{aligned} E &= (Y \cup Z)^c \\ B &= (Y \cap Z) \cup E \\ A &= Y \cap B^c \\ C &= Z \cap B^c \end{aligned}$$

Examining Figure 4.6, set A is defined by states with optimal keep decisions that do not intersect with replace decisions (circles on plot), set C is defined by states with optimal replace decisions that do not intersect with keep decisions (dashes) and set B is defined by the intersecting states (circles with dashes). Specifically, B is defined by the states (3,14), (3,15), (4,14), (4,15), (4,16), (5,13), (5,14), (5,15), (6,13), (6,14), (7,12), (7,13), (8,12), (9,11) and (9,12) in Trial 8.

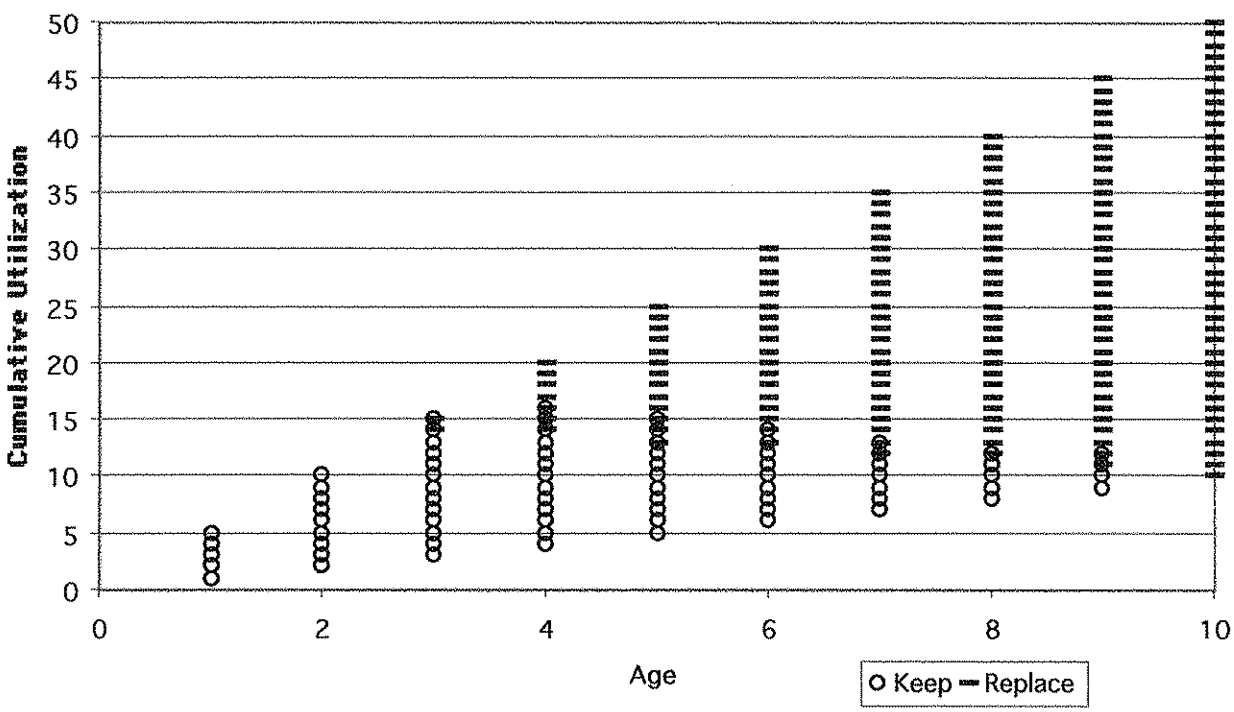

Figure 4.6: Individual asset states and decisions for KR solutions to Trial 8.

Although this plot only presents individual asset states for the KR solutions (or RK due to symmetry), it captures information for the entire solution. When an asset is defined by states in set A, the asset is kept regardless of the state of the other asset (Theorem 1). Similarly, an asset defined by a state in set C is replaced regardless of the state of the other asset (Theorem 2). Optimal decisions are only dependent on the state of the other asset if the asset is defined by a state in set B.

Theorem 1 *If state $(i,j) \in A$, then the optimal decision to DP is to keep an asset with age i and cumulative utilization j regardless of the state of the other asset.*

Proof. Consider the optimal solution to DP for state (i,j,i',j'). If the optimal decision is KK or KR, then the optimal decision for asset state (i,j) is keep. The optimal decision for (i,j,i',j') cannot be RK as $(i,j) \in A$. Thus, to prove the theorem, it must be shown that (i,j,i',j') cannot have an optimal decision of RR. This is a proof by contradiction.

Assume there exists a state (i,j,i',j') where $(i,j) \in A$ such that the optimal decision is RR. This implies that the cost of RR is less than that of KR, or:

$$K_t+2P_t-S_t(i,j)-S_t(i',j')+\alpha\sum_{m=1}^{D}p(d_{m,t})\left[\min_{u}\{C_t(u,0,0)+C_t(d_{m,t}-u,0,0)+f_{t+1}(1,u,1,d_{m,t}-u)\}\right] <$$

$$K_t+P_t-S_t(i',j')+\alpha\sum_{m=1}^{D}p(d_{m,t})\left[\min_{u_1}\{C_t(u_1,i,j)+C_t(d_{m,t}-u_1,0,0)+f_{t+1}(i+1,j+u_1,1,d_{m,t}-u_1)\}\right],$$

or:

$$P_t-S_t(i,j)+\alpha\sum_{m=1}^{D}p(d_{m,t})\left[C_t(u,0,0)+C_t(d_{m,t}-u,0,0)+f_{t+1}(1,u,1,d_{m,t}-u)\right] <$$

$$\alpha\sum_{m=1}^{D}p(d_{m,t})\left[C_t(u_1,i,j)+C_t(d_{m,t}-u_1,0,0)+f_{t+1}(i+1,j+u_1,1,d_{m,t}-u_1)\right]. \tag{4.3}$$

As $(i,j) \in A$, there exists at least one (I,J) such that the optimal decision for state (i,j,I,J) is KR. This implies that the cost of RR is greater than the cost of KR, or:

$$K_t+2P_t-S_t(i,j)-S_t(I,J)+\alpha\sum_{m=1}^{D}p(d_{m,t})\left[\min_{u}\{C_t(u,0,0)+C_t(d_{m,t}-u,0,0)+f_{t+1}(1,u,1,d_{m,t}-u)\}\right] >$$

$$K_t+P_t-S_t(I,J)+\alpha\sum_{m=1}^{D}p(d_{m,t})\left[\min_{u_1}\{C_t(u_1,i,j)+C_t(d_{m,t}-u_1,0,0)+f_{t+1}(i+1,j+u_1,1,d_{m,t}-u_1)\}\right],$$

which reduces to:

$$P_t-S_t(i,j)+\alpha\sum_{m=1}^{D}p(d_{m,t})\left[C_t(u,0,0)+C_t(d_{m,t}-u,0,0)+f_{t+1}(1,u,1,d_{m,t}-u)\right] >$$

$$\alpha\sum_{m=1}^{D}p(d_{m,t})\left[C_t(u_1,i,j)+C_t(d_{m,t}-u_1,0,0)+f_{t+1}(i+1,j+u_1,1,d_{m,t}-u_1)\right]. \tag{4.4}$$

Note that the utilization values of u and u_1 in Equations (4.3) and (4.4) are the same. Starting in either state (i,j,I,J) or (i,j,i',j') and making a RR decision results in the same state. This is also true for a KR decision from these two states. Once these keep-replace decisions have been made, there is only one optimal utilization level, by the principle of optimality.

As Equation (4.4) must hold, Equation (4.3) cannot hold and RR cannot be optimal if $(i,j) \in A$, by contradiction. Therefore, the optimal decision is to keep an asset of age i and cumulative utilization j if $(i,j) \in A$. $\square$

Theorem 2 *If state $(i, j) \in C$, then the optimal decision to DP is to replace an asset with age i and cumulative utilization j regardless of the state of the other asset.*

Proof. The proof follows similarly to that of Theorem 1. $\square$

The importance of Theorems 1 and 2 is that they allow the entire solution to be captured from one dynamic programming solution and the examination of only KR solutions. For state dependent decisions (set B), the solution must be examined more closely. For example, let us examine state $(7, 12)$. Figure 4.7 shows the optimal decision for each state $(7, 12, i_2, j_2)$.

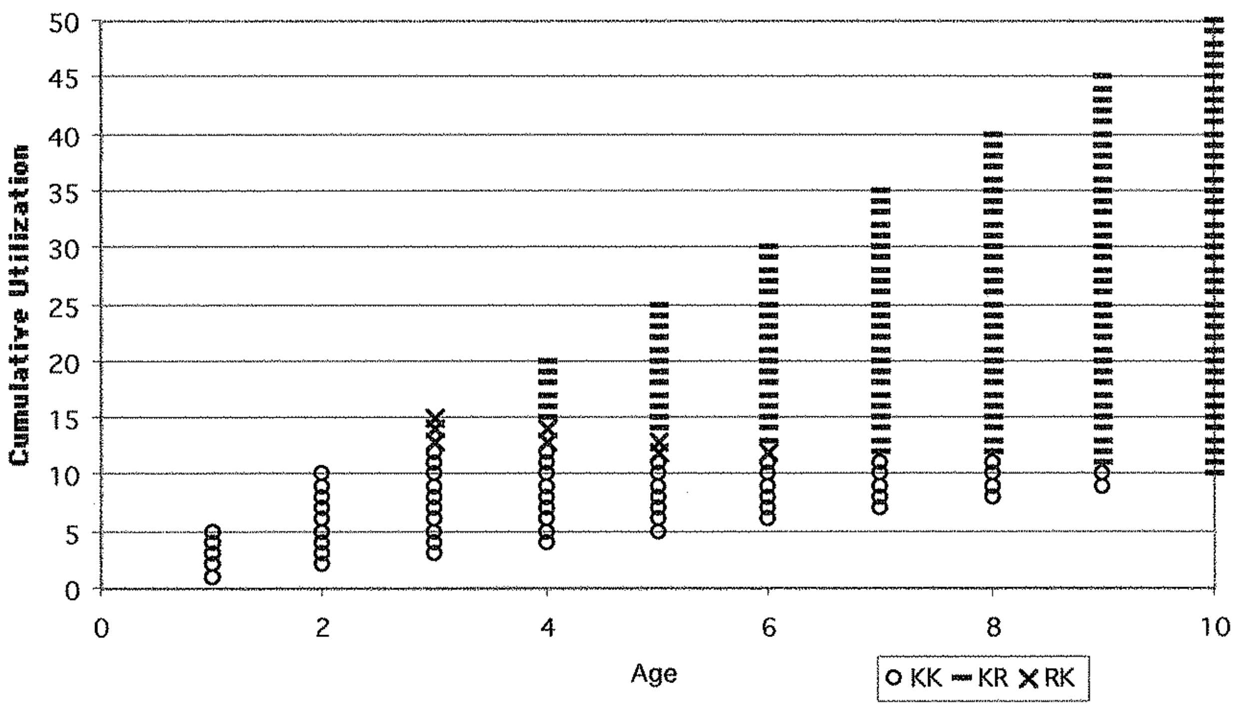

Figure 4.7: Keep and replace decisions for all $(7, 12, i, j)$ combinations for Trial 8.

In this figure, three types of decisions are shown. The circle represents KK decisions, where both assets are retained, the dash represents KR decisions where only the (7,12) asset is retained and finally an 'x' represents RK decisions where the (7,12) asset is salvaged. There are no optimal RR decisions with state (7,12). As the state (7,12) is an element of set B, it is expected that it has both optimal keep and replace decisions.

If we include a fixed charge for asset purchases into Trial 8, such that $K = 10,000$, then the resulting solutions (all KR states plotted) are given in Figure 4.8. Note that there are no asset states where keep and replace decisions intersect (sets Y and Z are disjoint and thus set $A = Y$ and $C = Z$). These states define set B and the results from Theorems 1 and 2 follow.

Examining this solution with economies of scale provides insights into larger problems. The difficulty with using dynamic programming is the exponential growth in size due to an increase in the state space. Obviously, an additional asset adds two dimensions to the state space. An increase in the number of assets also dramatically increases the number of possible replacement combinations. As shown in the previous section, the utilization levels of assets are generally tractable in terms of state space because they form a lattice structure for each asset. The difficulty lies in the combination of asset replacements. However, as shown in Figure 4.8, sets A and C become smaller with economies of scale. This is expected as a high fixed cost associated with purchases promotes group purchases. Therefore, there exists a fixed charge such that the sets A and C defined in this paper are empty and the only optimal decisions are to either keep or replace all assets (similar to results from Jones et al. [14]). This dramatically reduces the number of possible decisions at each state, which may help in the solution of larger problems.

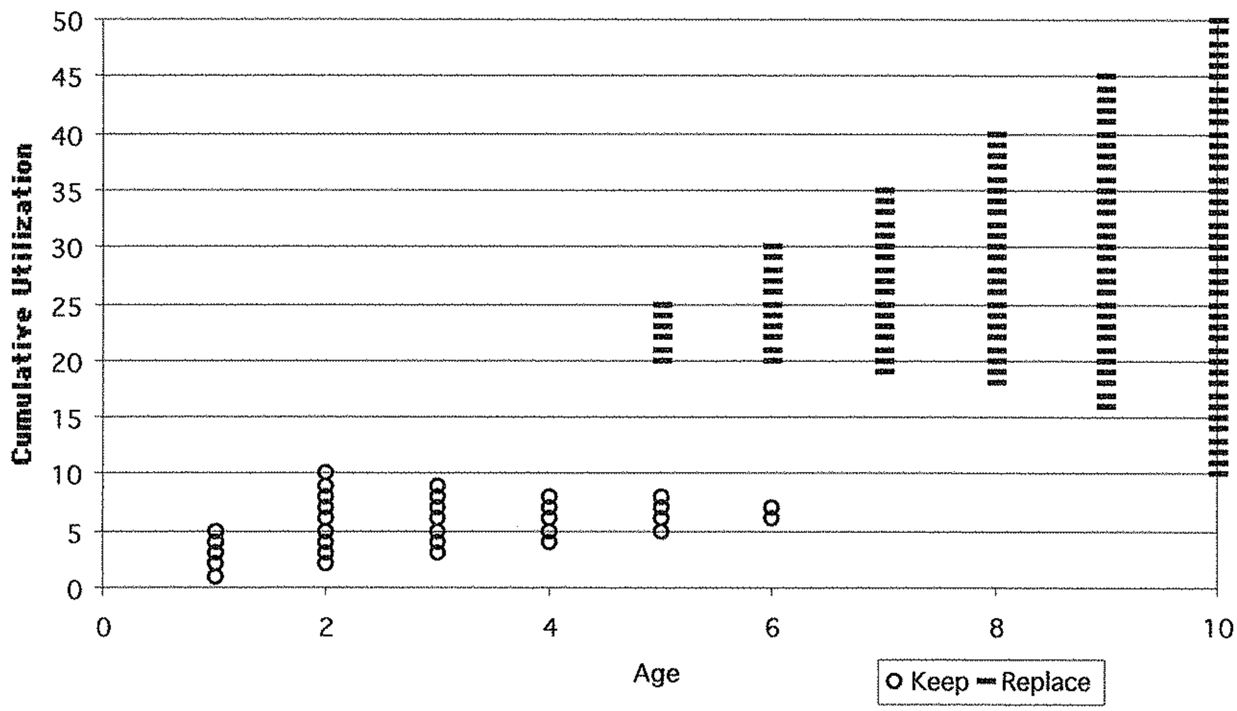

Figure 4.8: Individual asset states and decisions for KR solutions to Trial 8 with a fixed charge.

The previous examples have illustrated time-invariant costs and stationary demand examples. This allows for the determination of stationary replacement policies which can be found with one dynamic programming run, provided the finite horizon is long enough to find infinite horizon solutions. (While important, determining the correct horizon is beyond the scope of this paper. See, for example, Bean et al. [1] [2], Chand and Sethi [6] and Sethi and Chand [21] for a discussion of infinite horizon replacement solutions for equipment replacement models.) The model may also be used to examine time-variant costs and non-stationary demands, as in the following section.

4.2 Time-Variant Examples

Although stationary replacement policies can be determined from the analysis of time-invariant economics problems, most replacement studies require the incorporation of technological change and/or inflation. The same parameters were used in this example as before, with the following new time dependent cost functions:

$$\begin{aligned}
P_t &= 15,000(1.05)^t \\
C_t(u,i,j) &= \left(500 + 300i + 50j\left((j+u)^{1.2} - (j)^{1.2}\right)\right)(1.03)^t \\
S_t(i,j) &= 0.75P_{t-i} - 200i - 150j^{0.8} \\
K_t &= 0 \quad \forall t
\end{aligned}$$

It is assumed that at time zero, assets 1 and 2 are ages 2 and 4, respectively, with cumulative utilization levels of 8 and 15, respectively. Table 3 provides the solutions to 11 different trial runs involving these demands and costs. Again, each trial varies according to the probabilities of each demand level with Trials 1-5 representing the deterministic cases while the remaining trials have stochastic demand. Note that non-stationary demand problems may also be examined.

Table 3: Time invariant economic example solutions with differing non-stationary demand probabilities.

Trial	$p(d_{1,t})$	$p(d_{2,t})$	$p(d_{3,t})$	$p(d_{4,t})$	$p(d_{5,t})$	NPV Cost	Time Zero Decision					
							K or R	$u_{1,0}, u_{2,0}$				
1	1	0	0	0	0	\$108,530.78	RR	5,1				
2	0	1	0	0	0	\$118,155.60	RR	5,2				
3	0	0	1	0	0	\$127,847.16	RR	5,3				
4	0	0	0	1	0	\$136,731.01	RR	5,4				
5	0	0	0	0	1	\$144,481.77	RR	5,5				
6	.50	.125	.125	.125	.125	\$120,032.17	RR	5,1	5,2	5,3	5,4	5,5
7	.125	.50	.125	.125	.125	\$123,792.89	RR	5,1	5,2	5,3	5,4	5,5
8	.125	.125	.50	.125	.125	\$127,403.84	RR	5,1	5,2	5,3	5,4	5,5
9	.125	.125	.125	.50	.125	\$130,657.54	RR	5,1	5,2	5,3	5,4	5,5
10	.125	.125	.125	.125	.50	\$133,689.55	RR	5,1	5,2	5,3	5,4	5,5
11	.20	.20	.20	.20	.20	\$127,103.60	RR	5,1	5,2	5,3	5,4	5,5

As the table illustrates, the optimal decision for these cost assumptions in each of the demand cases is to replace both assets and operate one of the new assets at its maximum capacity. Theorems 1 and 2 were presented earlier to illustrate that examining KR decisions provides information about when individual asset replacement decisions can be made independently of the other asset. These results also hold here, but as costs are time-variant, they only hold for each period. That is, a graph as in Figure 4.6 can be constructed for each time period. As one is generally concerned with the optimal time zero decision (as information may be updated later for a subsequent solution of the model), an examination of the time zero plot of Figure 4.6 would be sufficient.

5 Conclusions and Directions for Future Research

The optimal time to replace an asset is highly dependent on how the asset it utilized over its lifetime. Traditionally, solutions in replacement analysis assume a given or fixed level of utilization in each period. However, if multiple assets are available to meet demand and the assets must not continually operate at their maximum capacity, then one may influence the individual utilization patterns by allocating work among the assets. This, in turn, effects the optimal replacement time of the asset.

This paper presents an efficient stochastic dynamic programming solution to determine the optimal replacement schedules and utilization levels for two assets that operate in parallel over a finite horizon under stochastic demand. Replacement decisions are based on the age and cumulative utilization of the individual assets. Despite four state variables in the dynamic program (age and cumulative utilization of each asset), it is shown that the solution space can be divided into three sets such that in set A, the optimal decision is to keep the asset, regardless of the state of the other asset and in set C, the optimal decision is to replace an asset regardless of the state of the other asset. For those states in set B, a closer examination of the solution is required as replacement decisions for an asset are dependent on the state of the other asset. The benefit of these definitions is that only a limited amount of information must be examined from one solution of the dynamic program (for the time-invariant cost and demand case). More importantly, these sets are defined by examining one type of solution (keep one asset and replace the other asset).

As replacement decisions are dependent on asset utilization, it is clear that optimal replacement strategies must incorporate utilization decisions. This paper has provided an introductory examination of this asset replacement/utilization problem. Future research will examine this problem more deeply in the case of numerous assets. This poses a problem with the use of dynamic programming, as an increase in the number of assets greatly increases the state space size and the number of possible replacement combinations in each period. Additionally, more elaborate search procedures may be incorporated such that utilization is treated as a continuous variable.

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